# Lecture 21: Complexity Theory

## Announcements:

- The final approaches fast!
- Three HW's: forkjoin, concurrency, complexity (all due by Wednesday afternoon!)
- Monday: no lecture, but instead time to work on your HW's / study hall.
  - I will hold OH monday instead of Friday.
- Wednesday: no lecture, still more study hall
  - I can go through P/NP section material if people are interested.
- We will try to grade HW's by Wednesday night, so you have feedback in time.
- $\boxed{\text{Complexity concept check due Monday}}$ (instead of friday)
- No section next week (day of final)
- Final thurs 5-7pm Sieg 134

review session Tues 5-7pm

+2 points on final if you fill out the official eval by next Friday.

This quarter we focused on fast algorithms and the data structures behind them.

- We could've used a linked list as our dictionary (slow)
- instead we saw how to use AVL trees or hashing to get faster runtimes!

In general, computer scientists often divide problems into two types

1. "easy" problems: computer can solve in time $O(n^c)$ ($c$ = constant)
2. "hard" problems: computer cannot solve in time $O(n^c)$

| $n$ | $\Theta(n)$ | $\Theta(n \log_2 n)$ | $\Theta(n^2)$ | $\Theta(n^3)$ | $\Theta(1.5^n)$ | $\Theta(2^n)$ | $\Theta(n!)$ |
|---|---|---|---|---|---|---|---|
| 10 | < 1 sec | < 1 sec | < 1 sec | < 1 sec | < 1 sec | < 1 sec | 4 sec |
| 30 | < 1 sec | < 1 sec | < 1 sec | < 1 sec | < 1 sec | 18 min | $10^{25}$ years |
| 50 | < 1 sec | < 1 sec | < 1 sec | < 1 sec | 11 min | 36 years | very long |
| 100 | < 1 sec | < 1 sec | < 1 sec | 1 sec | 12,892 years | $10^{17}$ years | very long |
| 1,000 | < 1 sec | < 1 sec | 1 sec | 18 min | very long | very long | very long |
| 10,000 | < 1 sec | < 1 sec | 2 min | 12 days | very long | very long | very long |
| 100,000 | < 1 sec | 2 sec | 3 hours | 32 years | very long | very long | very long |
| 1,000,000 | 1 sec | 20 sec | 12 days | 31,710 years | very long | very long | very long |

complexity theory's goal: <u>prove</u> that certain problems are <u>hard</u> (for computers)

What is a problem? It is simply a set of inputs/outputs

ex 1| Minimum spanning tree

input = graph  output = any MST

ex2| "tell me if this list is sorted"

input = list $\in \mathbb{R}^n$  output = $\{0,1\}$  ( $\{$yes, no$\}$ )

ex 3| "sort this array"

input = list  output = list.

Def| An **instance** is a specific input to a problem.

Complexity Theorists focus on <u>decision problems</u> (output is "yes" or "no")

so instead of "find shortest path"

ask "is there a path of length $\leq k$ ?"

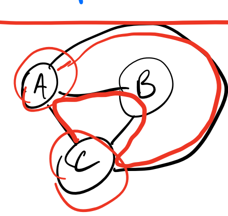
(this is because if you can solve the "decision version" then you can solve the "search version"!)

(search-to-decision reductions)

## Examples!!!

① <u>Eulerian Path</u>: "Does G have a path that uses each edge exactly once?"

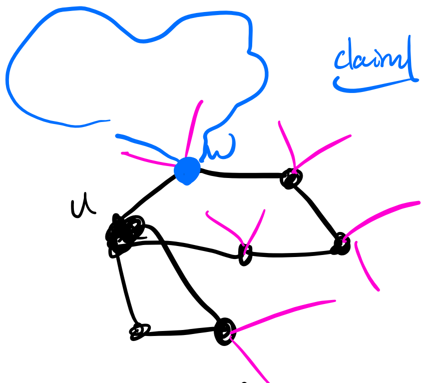
Theorem| $\exists$ Eulerian Path iff exactly 0 or 2 vertices have odd degree



intuition: suppose every vertex is even degree

ALG: starting at u, keep following edges you haven't used yet, until you cannot anymore.

claim: This process stops at u.




every time I pass through a node, I "reduce" # of edges touching it by 2.

$\Rightarrow$ Eulerian Path is EASY: $\exists$ $O(|V|+|E|)$ algo solving it (why?)

ALG: For v in V:

check if $deg(v) == 0 \pmod 2$

if # of odd deg vertices == 0 or 2, output YES.

(2) Hamiltonian Path: "Does G have a path that touches every vertex exactly once?"

This seems like a similar problem!

ALG: for each permutation of the |V| vertices, check if that path exists

runtime? $O(|V|!)$.

better algorithm? $O(2^{c|V|})$ is best known.

... we think Hamiltonian Path is HARD

**Def** **P** is the class of all problems solvable in time $O(n^c)$ where $c$ is constant

"solvable" = $\exists$ an algorithm with this runtime

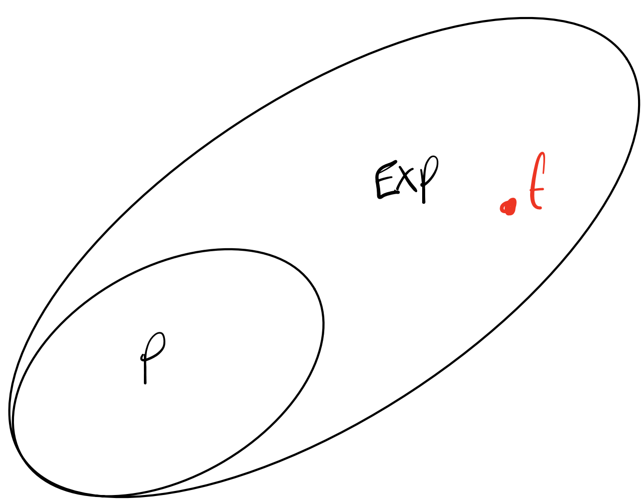
$P$ stands for "polynomial time". It is a <u>set</u> of problems.

**Def** **EXP** is the class of problems that can be solved in $O(2^{n^c})$ time, $c$ = constant.

<u>questions</u>:

1) is Eulerian Path $\in P$ ? YES!
2) is Hamiltonian Path $\in P$ ? ??
3) If $f \in P$ then $f \in EXP$ ?

$\underline{O(n^c)} \subseteq \underline{O(2^{n^c})}$




**Def** **NP** is the class of decision problems for which YES instances have a "proof" or "certificate" that can be verified in polynomial time $O(n^c)$, $c$ = const.

<u>more formally</u>: $f \in NP$ iff $\exists$ algorithm $A(x,y)$ and polynomials $p$ and $q$, s.t.

① $\forall x$ $\cancel{\in f^{-1}(1)}$ , $A(x,y)$ runs in time $p(|x|)$

A = "verifier"
y = "proof/certificate"

③ $\forall x \in f^{-1}(1)$ $\exists y$, $|y| \le q(|x|)$, s.t. $A(x,y) = 1$.
③ $\forall x \in f^{-1}(0)$ and $\forall y$ of length $\le q(|x|)$, $A(x,y) = 0$.

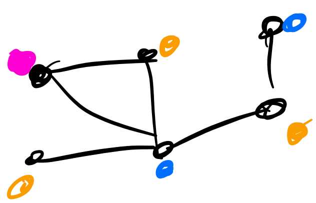
examples: Hamiltonian path $\in NP$ because if G has a Hamiltonian path, then the "proof" can be the path itself

the "verifier" checks that the path its given

① is a valid path
② touches each vertex once

3-COLOR: "can I color vertices of G with $\le 3$ colors such that all neighbors are colored differently?"

3-COLOR $\in NP$ : proof = assignment of colors to vertices



verifier = checks if ① only 3 colors used
② each edge has endpoints colored differently.

2-COLOR: "can I color vertices of G with $\le 2$ colors such that all neighbors are colored differently?"

2-COLOR $\in P$

$\boxed{\text{2-COLOR} \in NP}$

ALG: run BFS. If layer of vertex is even, color 0
If layer is odd, color 1

(if you must color vertex with a different color than it already has, output NO)

Theorem: $P \subseteq NP$

proof: let $f \in P$. So there exists an algorithm that runs in polynomial time deciding $f$.

if $f(x) = 1$, then the "proof" is empty, and the verifier just runs the algorithm to see whether $f(x) = 1$.

$\boxed{NP \overset{?}{\subseteq} P}$

ex) "Is there an MST of weight $\leq k$" $\in NP$

① we could have the "proof" be a MST of weight $\leq k$
and the "verifier" check that the proof a) is a tree, and
b) has cost $\leq k$.

② we could also have the "proof" be nothing at all,
and let the verifier run Prim's / Kruskal's algo.

$\Big\}$ use fact that "Light MST" $\in P$

$$n! \leq n^n = 2^{n \log n} \leq 2^{n^2}$$

**Theorem** $NP \subseteq EXP$

proof) Let $f \in NP$. given $x$, consider the following algorithm:

ALG: ① write down all possible "proofs" of length $\leq n^c$ that $f(x) = 1$

② run verifier on all these proofs

if verifier succeeds on any, output 1. Otherwise, output 0.

runtime of ALG?

how many possible proofs are there? $O(2^{n^c})$

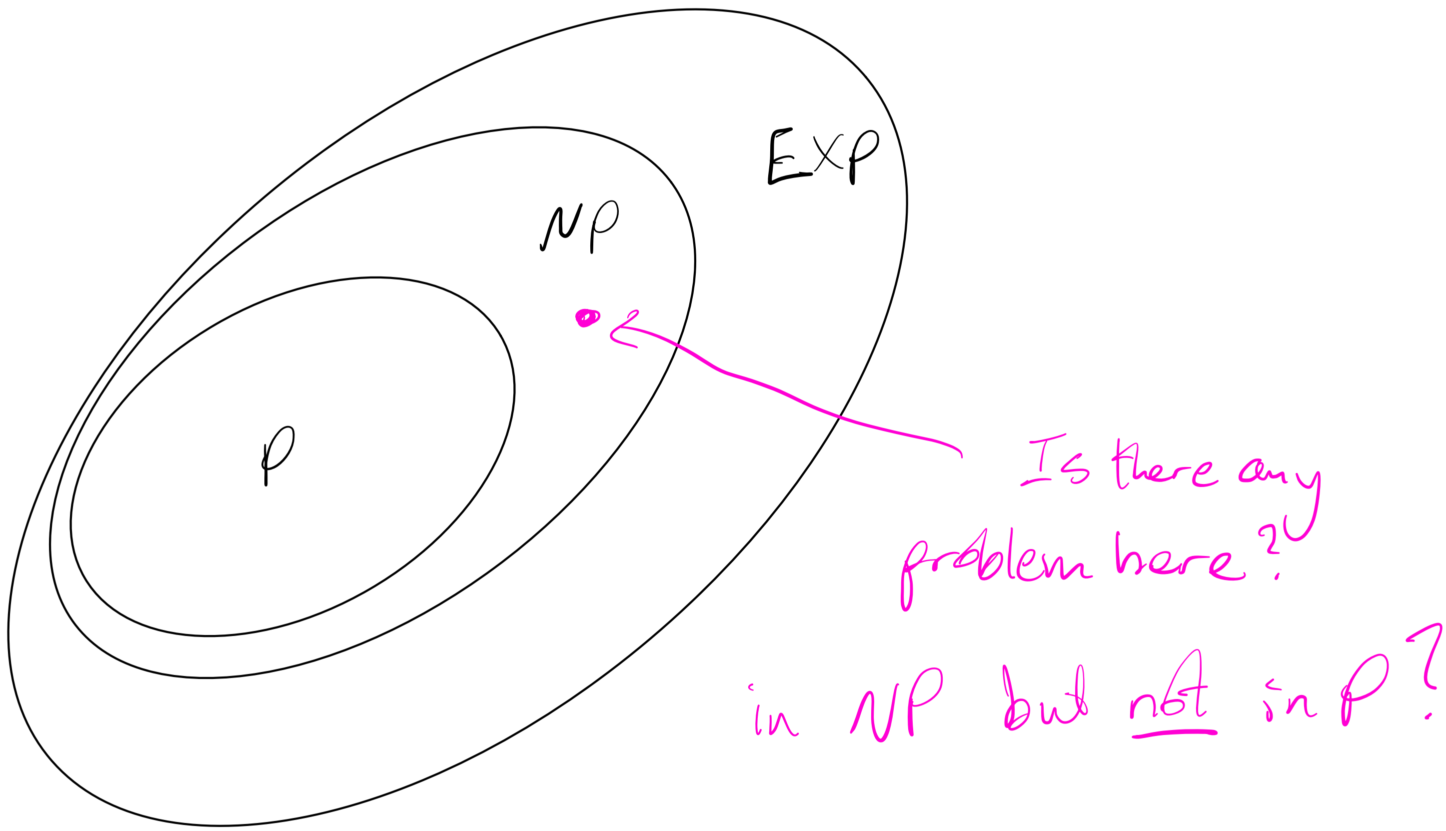
$$O\left(\text{poly}(n) \cdot 2^{\text{poly}(n)}\right) \subseteq O\left(2^{\text{poly}(n)}\right)$$

$\Rightarrow f \in EXP$ ∎

## BIG COMPLEXITY THEORY QUESTION: $P \stackrel{?}{=} NP$

intuitively: If I can "verify" a good solution quickly, can I use this to come up with a fast algorithm myself?

... most people think the answer is no.




One more fun example

"is there MST of weight $\leq k$" $\in P$

"is there a walk that visits each vertex exactly once, and returns to start, and has weight $\leq k$" $\in NP$ traveling salesman problem

electric company can optimize its grid, but Amazon doesn't know how to optimally route it's drivers!! (without solving P vs. NP, that is)

# REDUCTIONS

Def| We say that problem A is polynomial time reducible to B if there is an algorithm which, using a (hypothetical) polynomial time algorithm for B, solves A in polynomial time.

notation: $A \leq_p B$

intuition: A is no harder to solve than B, because if we can solve B, then the reduction tells us we can also solve A!

warning: Don't get it backwards!

Silly example: suppose A = I have spiders in my room, I want them gone.

B = I blow up my house, and rebuild it.

Then $A \leq_p B$, because I can run my algo for B, and it gets rid of the spiders.

A is no harder than B.

another example: 3-COLOR $\leq_p$ 4-COLOR

ALG: on input $G$, add a new vertex, and connect it to all other vertices

call this graph $G'$

return 4-COLOR($G'$)

need to check: 1) if $G$ 3-colorable, then $G'$ 4-colorable

2) if $G$ not 3-colorable, then $G'$ not 4-colorable

$\Big\}$ correctness

3) ALG runs in polynomial time, assuming 4-COLOR does ] poly-time

**Def** $A$ is NP-complete iff ① $\forall B \in NP$, $B \leq_p A$

② and $A \in NP$.

motivation: these are the "hardest" problems in $NP$

so if we want to show $P \neq NP$, these would be the best candidates! (most likely not to have a poly time algo)

**Def** $A$ is NP-hard iff $\forall B \in NP$, $B \leq_p A$.

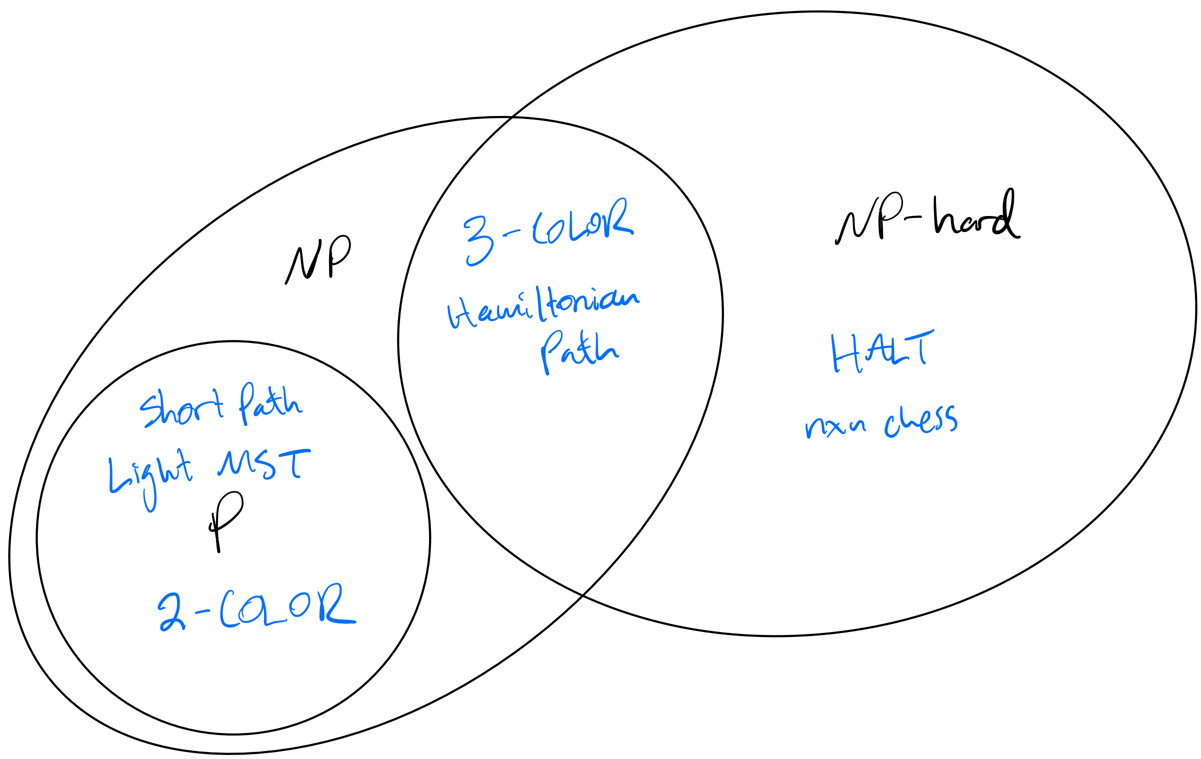
**Remark:** **Many** problems are NP hard, but not in NP

ex] halting problem is NP-hard (but certainly not NP-complete!)

nxn chess: is there an optimal strategy for white?

not easy to verify!

What we *think* the world looks like




if P=NP, then P "grows" and "consumes" all of NP.

# History / Why You Should Care