CSE 332 Summer 2026
Lecture 16: MST Wrap/Parallelism

Michael Whitmeyer

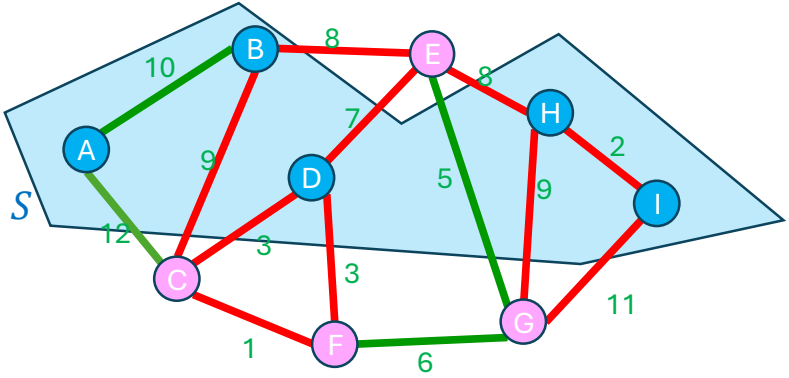
http://www.cs.uw.edu/332

Announcements

- Graph exercises are out!
  - Ex07 cycle detection
  - Ex08 Dijkstra
- Fill out feedback form for +2 points on midterm!
  - By next Wednesday

Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, S and $V - S$



Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. (A, C)

A set of edges R Respects a cut if no edges cross the cut e.g. $R = \{(A, B), (E, G), (F, G)\}$

Cut Theorem

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.

Cut Theorem

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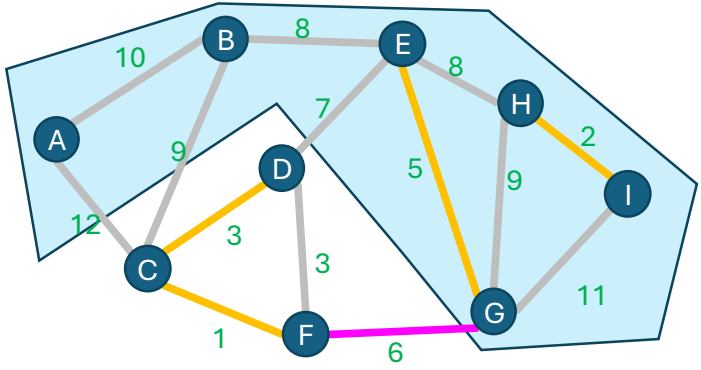
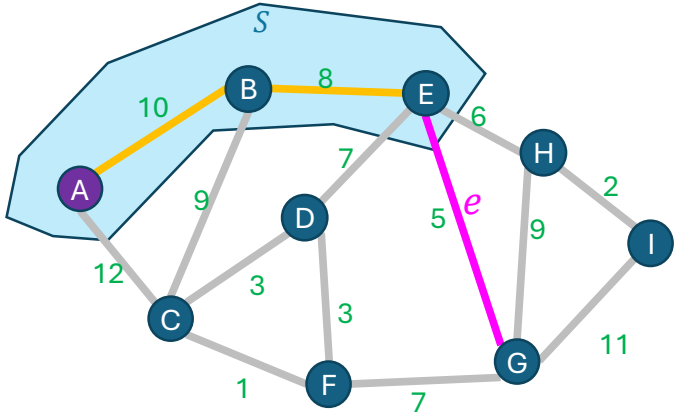


Proof of Prim's Algorithm

Start with an empty tree A

Repeat $V - 1$ times:

Add the min-weight edge that connects to a node not currently in the tree



Proof: By Induction

Suppose we have some arbitrary set of edges A that Prims's has already selected to include in the MST. $e = (E, G)$ is the edge Prims's selects to add next

Consider the cut:

- S = Nodes that have been removed from the priority queue (vertices touched by A)

e is the minimum cost edge that crosses this cut, so by the Cut Theorem, $e \in T$!

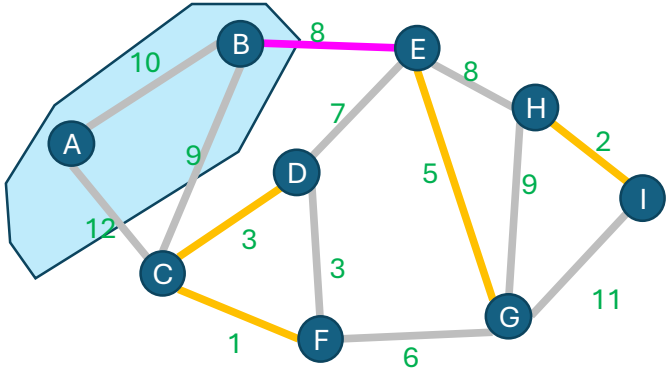
General MST Algorithm

Start with an empty tree A

Repeat $V - 1$ times:

Pick a cut $(S, V - S)$ which A respects

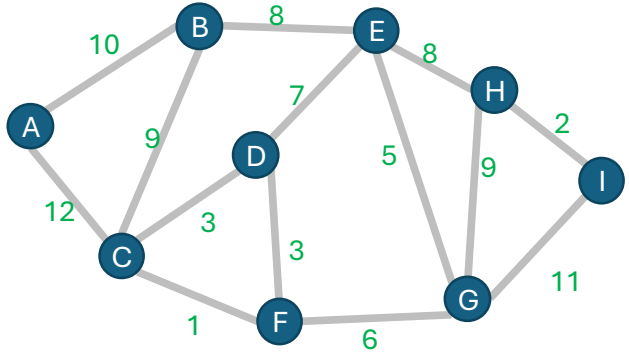
Add the min-weight edge which crosses $(S, V - S)$



Kruskal's Algorithm

Start with an empty tree A

Add to A the lowest-weight edge that does not create a cycle



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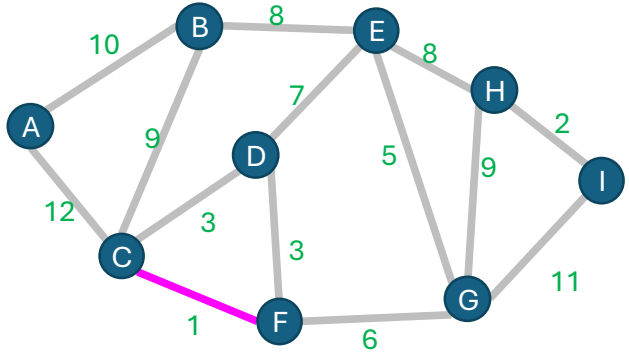

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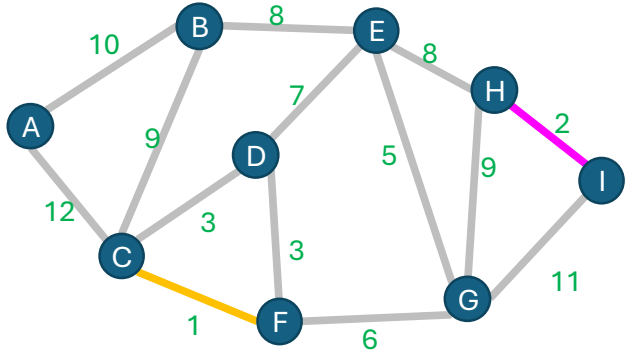

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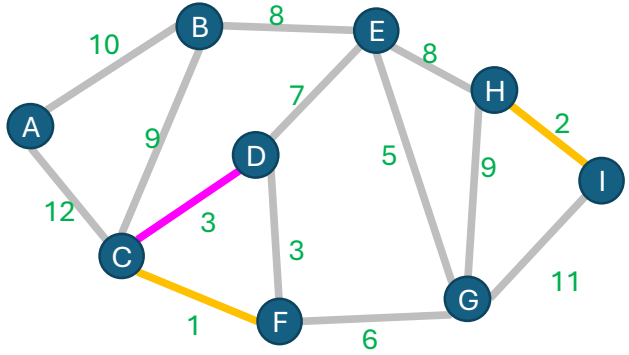

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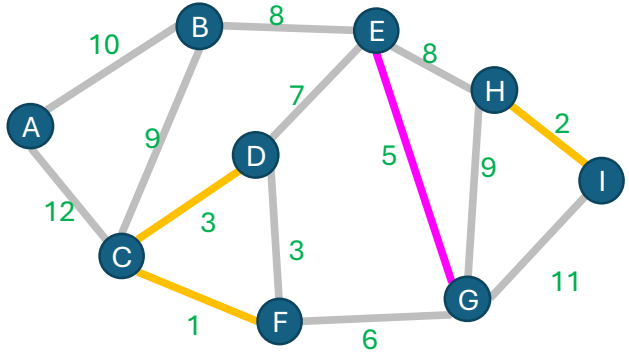

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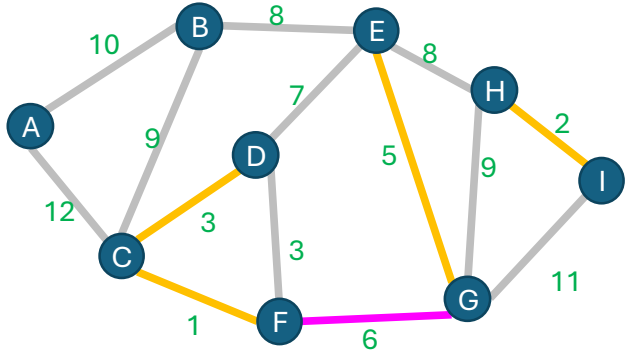


Correctness of Kruskal's Algorithm

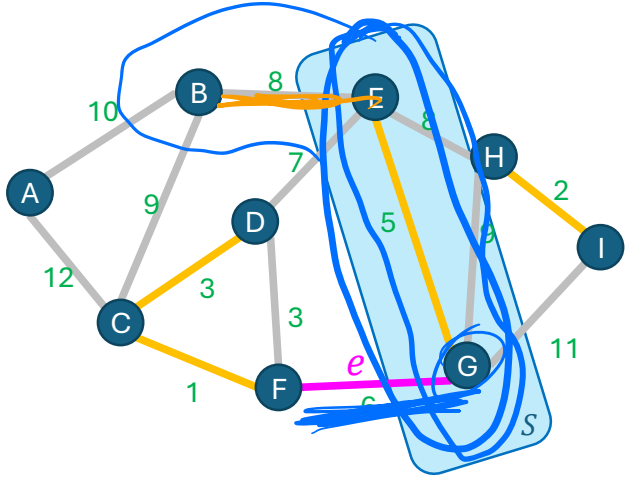
- It's sufficient to just show that it follows the template of our "General MST Algorithm"
  - Show that for every edge chosen, it is the least-weight edge which crosses some cut that respects all already-chosen edges.

Proof of Kruskal's Algorithm

Start with an empty tree A

Repeat $V - 1$ times:

Add the min-weight edge that doesn't cause a cycle



Proof: By induction, edges in A that Kruskal's has already selected are in T. $e = (F, G)$ is the edge Kruskal's selects to add next

Cannot exist path from F to G using only edges in A (why?) Algorithm picks e that does not create a cycle!

Consider the cut defined by S = connected component containing G in A. A respects this cut (why?)

e is the minimum cost edge that crosses this cut, so by the Cut Theorem, $A \cup \{e\} \in T$!

Parallelism

Parallelism Vs. Concurrency (with Potatoes)

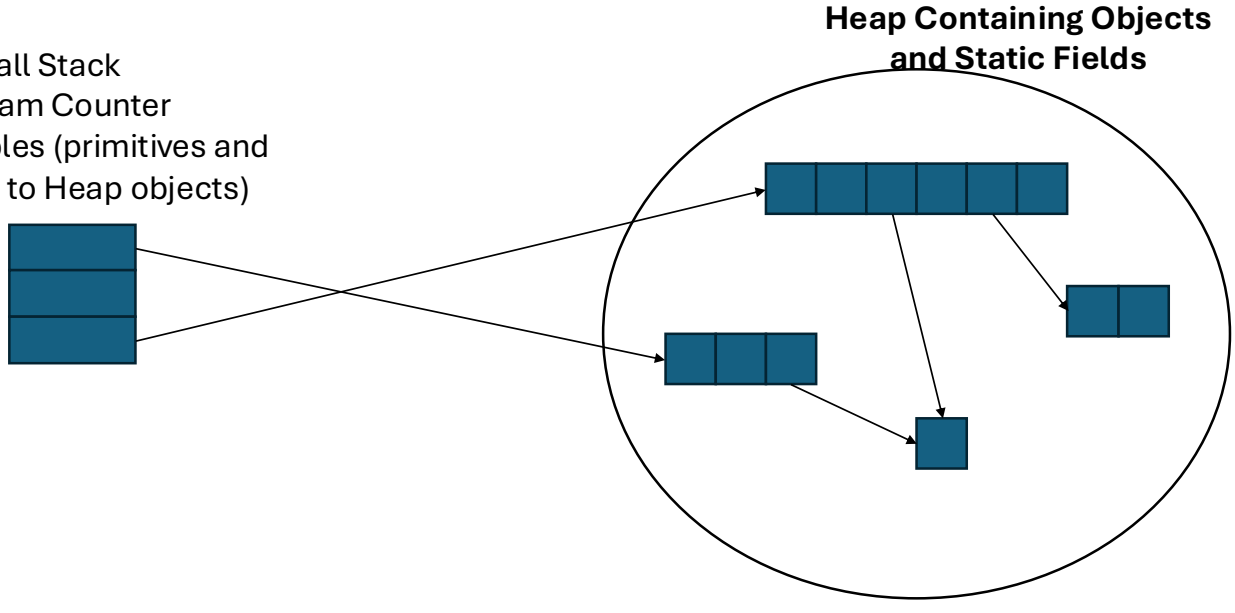
- Sequential:
  - The task is completed by just **one processor** doing one thing at a time
  - There is one cook who peels all the potatoes
- Parallelism:
  - One task being completed by **many threads/processors**
  - Recruit several cooks to peel a lot of potatoes faster
- Concurrency:
  - Parallel tasks using a **shared resource**
  - Several cooks are making their own recipes, but there is only 1 oven

New Story of Code Execution

- Old Story:
  - One **program counter** (current line of code)
  - One **call stack** (with each stack frame holding local variables)
  - **Objects in the heap** created by memory allocation (i.e., **new** ___)
    - (nothing to do with data structure called a heap)
- New Story:
  - Collection of threads each with its own:
    - Program Counter
    - Call Stack
    - Local Variables
    - References to objects in a shared heap

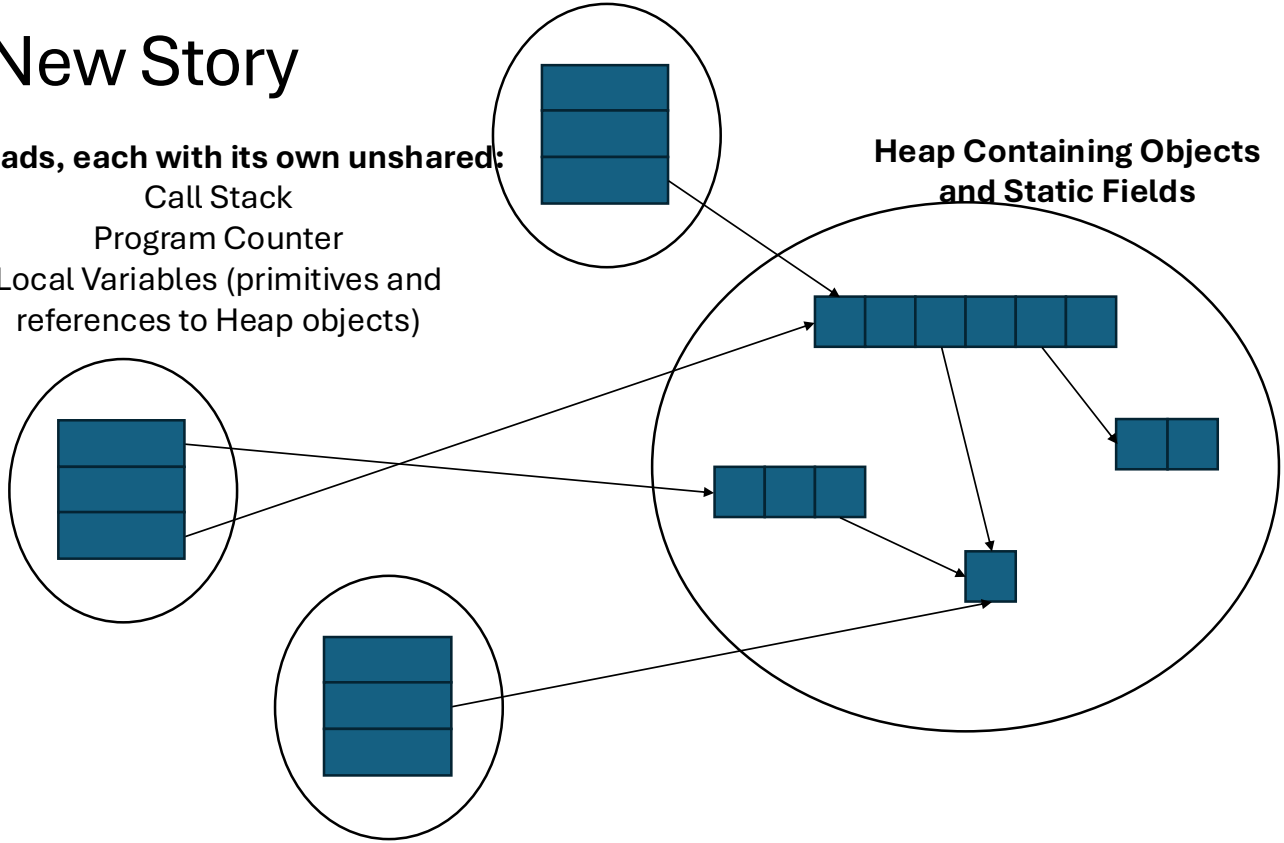
Old Story

Call Stack
Program Counter
Local Variables (primitives and references to Heap objects)



New Story

Threads, each with its own unshared:
Call Stack
Program Counter
Local Variables (primitives and references to Heap objects)



Running Example: Summing an Array

- **Goal**: Find the sum of an array
- **Idea**: Split array into 4 pieces, sum those pieces in parallel, and then sum the results.

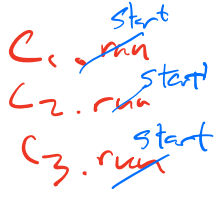
Input: array (arr) of integers

1. **In parallel**: split array into 4 equal pieces, sum up those four pieces.
   1. res[i] = sum arr[i*len/4, (i+1)*len/4]
2. **Return** res[0] + res[1] + res[2] + res[3]

- **Needs from Programming language**
  - Way to **create threads**, and run them in parallel
  - Way to access shared memory (refs to common objects)
  - Way to **synchronize** threads (**wait** until finished)
    - Cannot sum res[0] + … + res[3] until each thread has finished!

Accomplishing this in Java (**Java.lang.Thread**)

1. Create class C extending **java.lang.Thread**
   1. C must have a **run** method (acts as **main**)
2. Call C's **start** method to
   1. create a new thread (i.e. parallel task, not Thread object in java)
   2. execute **run** in that separate, parallel thread



- Calling "**run**" directly causes the program to execute "**run**" sequentially

First Attempt (part 1, Defining Thread Object)

```
class SumThread extends java.lang.Thread {
    int lo;   int hi;   int[] arr;  int ans = 0;
    SumThread(int[] a, int l, int h) {
        lo=l; hi=h; arr=a;
    }
    public void run() { //override, must have this signature
                        // no arguments and no return allowed
                        // must use the object's fields for input/output
        for(int i=lo; i < hi; i++)
            ans += arr[i];
    }
```

First Attempt (part 2, Creating Thread Objects)

```
static int parallelSum(int[] arr){ // this method could be anywhere
        int len = arr.length;
        int ans = 0;
        SumThread[] threads = new SumThread[4];
        for(int i=0; i < 4; i++) // create threads
                threads[i] = new SumThread(arr, i*len/4, (i+1)*len/4);
        // more stuff to follow
                threads[i].start()
}
```

```
class SumThread extends java.lang.Thread
{
        int lo, int hi, int[] arr; int ans = 0;
        SumThread(int[] a, int l, int h) { ... }
        public void run(){ ... } // override
}
```

First Attempt (part 3, Running Thread Objects)

```
static int parallelSum(int[] arr){ // this method could be anywhere
        int len = arr.length;
        int ans = 0;
        SumThread[] threads = new SumThread[4];
        for(int i=0; i < 4; i++){ // create threads, do parallel computations
                threads[i] = new SumThread(arr,i*len/4,(i+1)*len/4);
                threads[i].start(); // start not run
        }
        for(int i=0; i < 4; i++) // combine results
                ans += threads[i].ans;
        return ans;
}
```

```
class SumThread extends java.lang.Thread
{
        int lo, int hi, int[] arr; int ans = 0;
        SumThread(int[] a, int l, int h) { ... }
        public void run(){ ... } // override
}
```

First Attempt (part 4, Synchronizing)

```
static int parallelSum(int[] arr){ // this method could be anywhere
        int len = arr.length;
        int ans = 0;
        SumThread[] threads = new SumThread[4];
        for(int i=0; i < 4; i++){ // do parallel computations
                threads[i] = new SumThread(arr,i*len/4,(i+1)*len/4);
                threads[i].start(); // start not run
        }
        for(int i=0; i < 4; i++){ // combine results
                threads[i].join(); // wait for thread to finish!
                ans += threads[i].ans;
        }
        return ans;
}
```

```
class SumThread extends java.lang.Thread
{
        int lo, int hi, int[] arr; int ans = 0;
        SumThread(int[] a, int l, int h) { ... }
        public void run(){ ... } // override
}
```

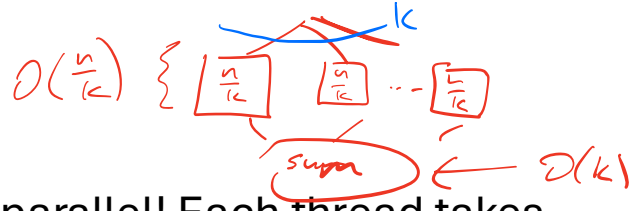
Join

- Causes program to pause until the other thread completes its **run** method
- Avoids a **race condition**
  - Without join the other thread's **ans** field may not have its final answer yet

Recap so far

- Way to **create threads**, and run them in parallel
  - C extends java.lang.Thread
  - C.start()
- Way to **wait for threads to finish**
  - C.join()

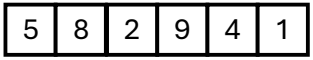
More Threads?



- **Issue:** our parallel algorithm is not that parallel! Each thread takes $O\left(\frac{n}{4}\right) = O(n)$ time.
- **Idea 1:** make # of threads (# of array chunks) a parameter. Each thread runs in $O\left(\frac{n}{k}\right)$ time, combining takes $O(k)$ time.
  - **Pros**: Different machines have different # processors. More efficient/reusable across machines
  - **Cons:** OS ultimately in charge of how processors get used, and perhaps not all subproblems take same amount of time
    - **Runtime barrier**: cannot do better than $O(\sqrt{n})$

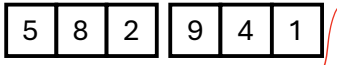
A Better Solution: Divide and Conquer!

- **Idea**: Each thread checks its input size. If smaller than ℓ, sum sequentially. Otherwise, split the problem in half across two separate threads.
  - ℓ is the "sequential cutoff" (typical range: 500 to 5000)
  - Creating threads takes some time, at some point it's more efficient to do things sequentially!

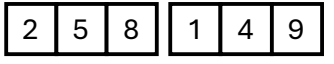


Merge Sort

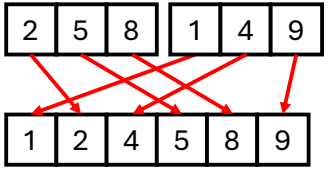
- **Base Case**:
  - If the list is of length 1 or 0, it's already sorted, so just return it



- **Divide**:
  - Split the list into two "sublists" of (roughly) equal length



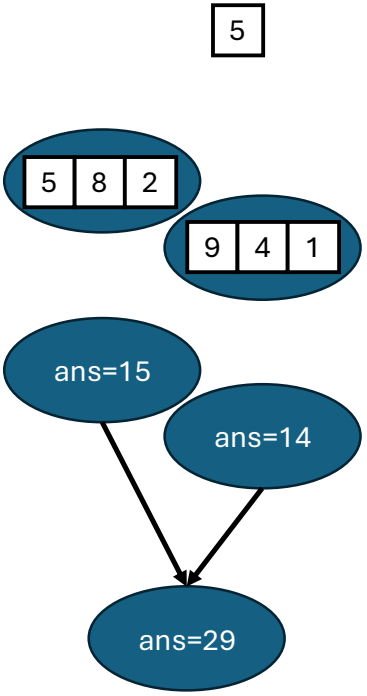
- **Conquer**:
  - Sort both lists recursively



- **Combine**:
  - **Merge** sorted sublists into one sorted list

Parallel Sum

5	8	2	9	4	1



- **Base Case**:
  - If the list's length is smaller than the Sequential Cutoff, find the sum sequentially
- **Divide**:
  - Split the list into two "sublists" of (roughly) equal length, create a thread to sum each sublist.
- **Conquer**:
  - Call **start()** for each thread
- **Combine**:
  - Sum together the answers from each thread