Lecture 15 : Graphs 4

Single Source Shortest Path

Task: given start node s in a weighted graph with non-negative edge weights, find the shortest path from s to all other vertices (or a specific node)

Dijkstra's Algorithm!

intuition: in BFS, we visited nodes in order based on their layer/depth
in Dijkstra, we visit nodes in order based on their distance

Dijkstra's Pseudocode

① extract node curr from priority queue
this makes curr "done"

② for each not done neighbor of curr:
- update its distance if we have found a better path.

Seen = added to PQ

done = removed from PQ

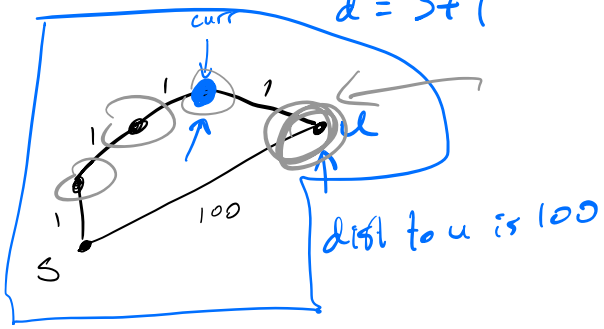
BFS Pseudocode

① extract node curr

② for each not visited neighbor of curr:
add it to queue
depth of neighbor = depth(curr) + 1

Dijkstra Pseudocode

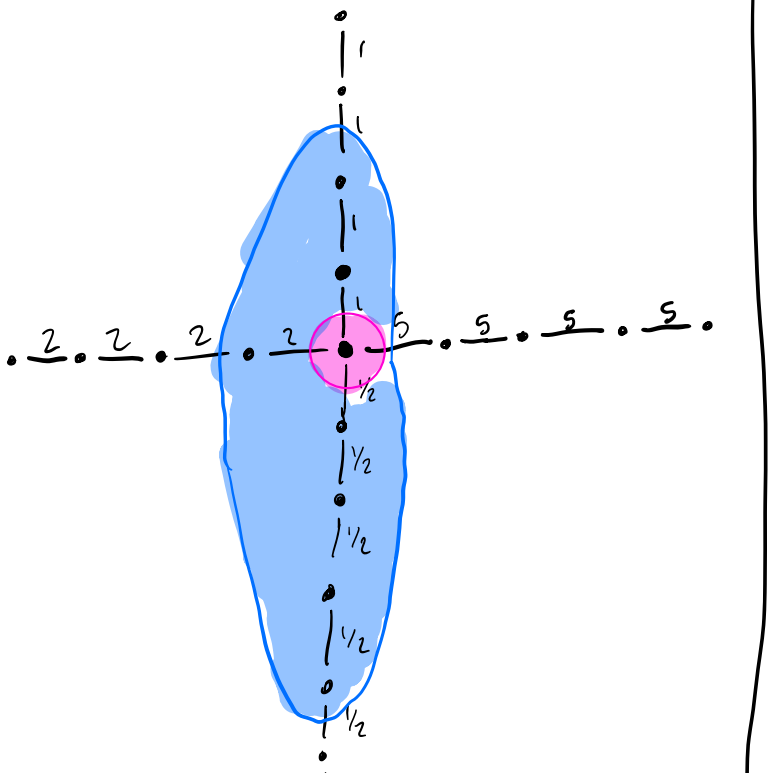
```
dijkstra(G, s):
    PQ = new priority queue
    add s to PQ with priority 0
    mark s "seen"
    while PQ not empty:
        curr = PQ.extract()
        mark curr as done
        for u in neighbors(curr):
            d = dist to curr + w(curr, u)
            if u not "seen":
                mark u seen
                PQ.add(u, d)
            if u not done AND d < dist to u:    (better path!)
                distance to u = d
                PQ.decreaseKey(u, d)
```



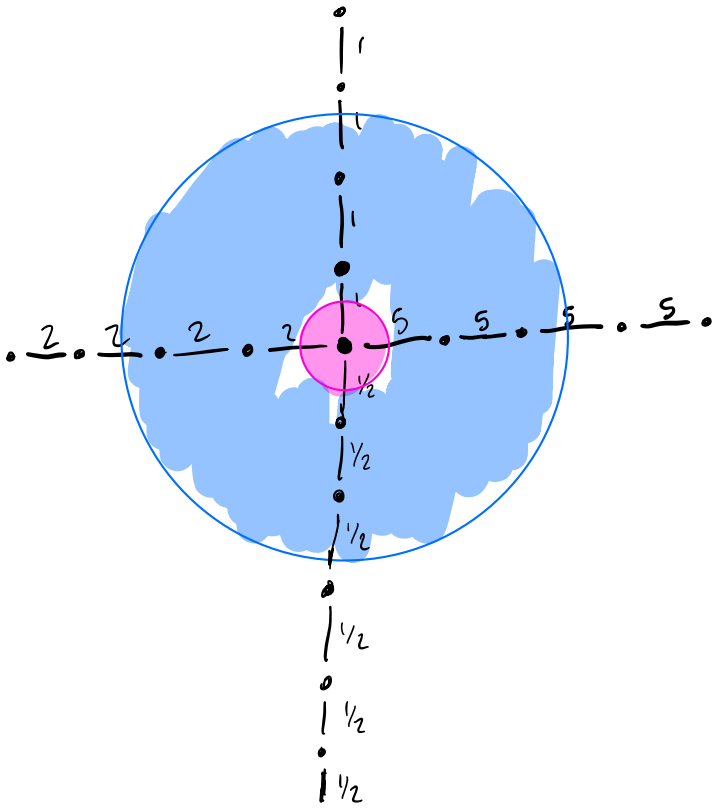
Runtime?

$\Theta((|E| + |V|)\log|V|)$

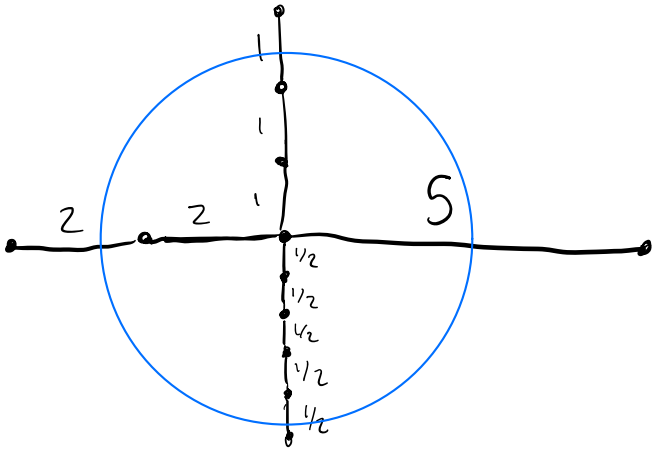
Dijkstra



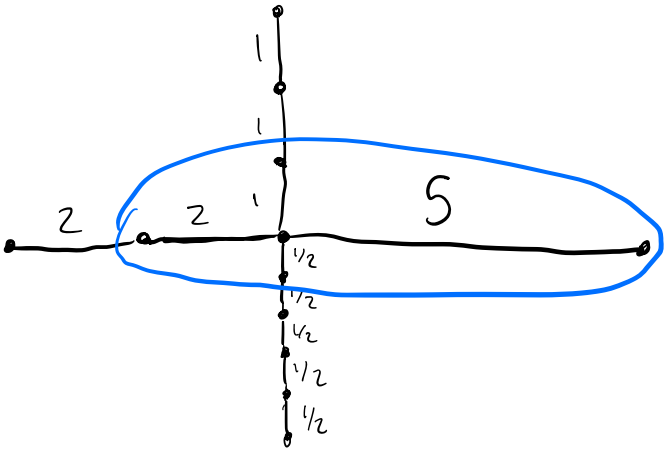
BFS



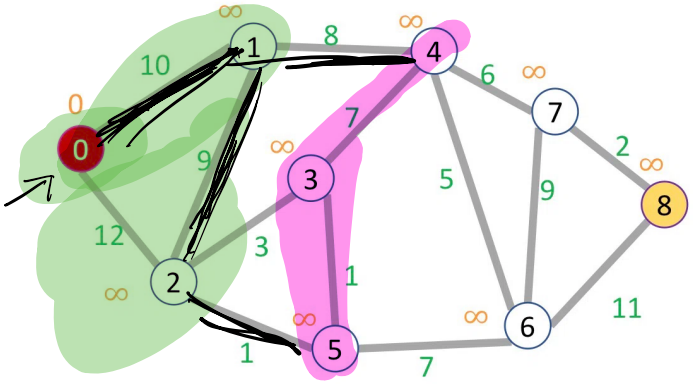
Dijkstra



BFS



Ex!



PQ:

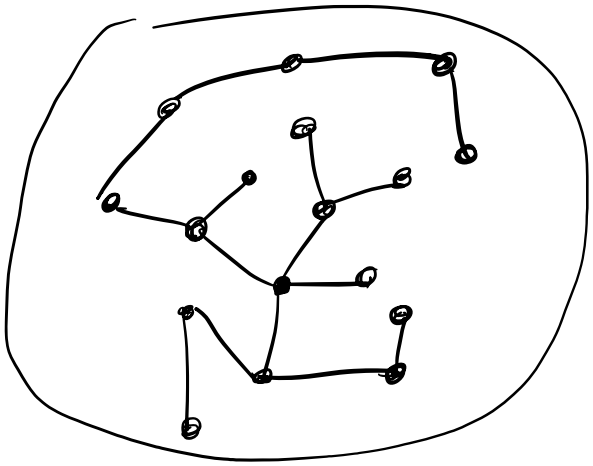
~~1~~	~~2~~	4
0	12	18

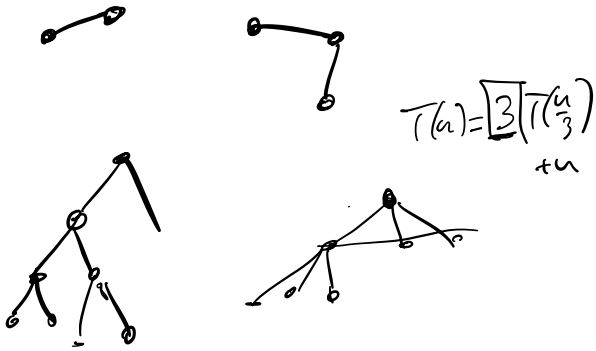
node	seen?	done?	distance
0	True	T	0
1	T	T	10
2	T	T	12
3	T		15
4	T		18
5	T		13
6			∞
7			∞
8			∞

Minimum Spanning Trees

recall trees are (connected) graphs with no cycles

question: how many edges in a tree? $|V|-1$





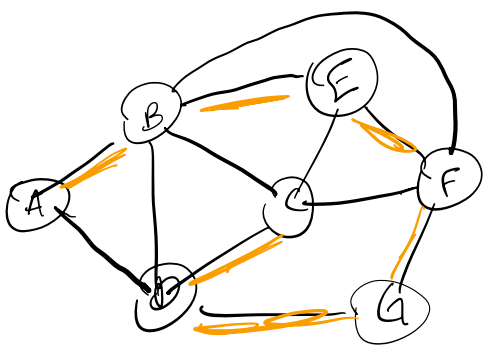
$T(n) = 3T(\frac{n}{3}) + n$

Def (spanning tree) $T = (V_T, E_T)$ is a spanning tree for $G = (V, E)$ iff $V_T = V$, $E_T \subseteq E$, and T is a tree.

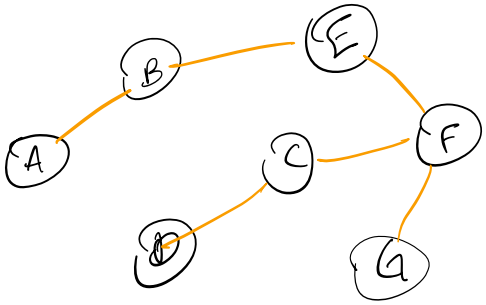
assume T is connected.

Ex

G



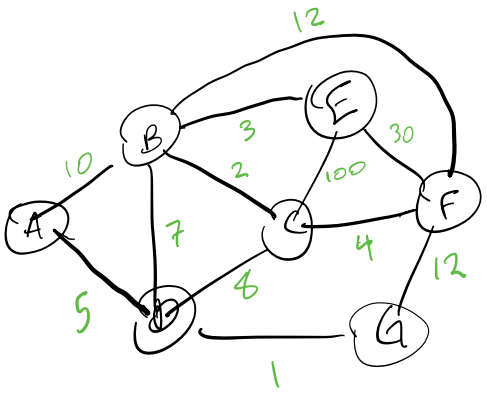
T



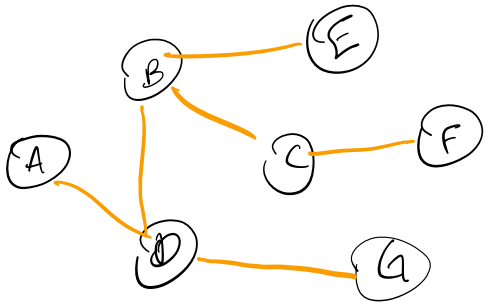
Aside: The number of spanning trees for a complete graph on n vertices is n^{n-2}

Def (Minimum Spanning Tree) Given a (weighted) graph $G = (V, E)$ and weight function $w: E \to \mathbb{R}$, $T = (V_T, E_T)$ is a minimum spanning tree if T is a spanning tree and has minimum cost

$$\text{cost}(T) = \sum_{e \in E_T} w(e)$$



minimum spanning tree T

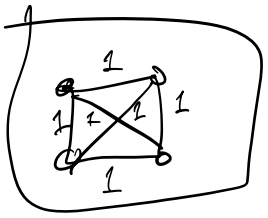


$\text{cost}(T) = 3+2+7+5+1+4 = 22$

$\text{cost}(T) = 22$

intuition: any "trade" only increases cost.

question: is the minimum spanning tree unique? NO



Applications?

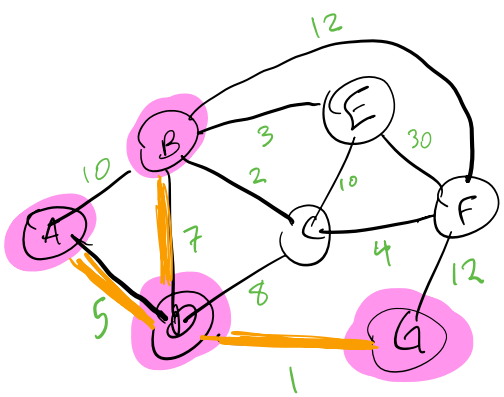
- electric grids
- Transportation
-

Prim's Algorithm

- Start with empty tree T
- Pick starting node S, add to T
- Repeat $|V|-1$ times:

  add the min weight edge with one endpoint in T and one endpoint outside of T to T

Ex



$s = A$

Prim's Pseudocode

```
Prim(G,s):
    PQ = new priority queue
    add s to PQ with priority 0
    mark s "seen"
    while PQ not empty:
        curr = PQ.extract()
        mark curr as done
        for u in neighbors(curr):
            d = w(curr, u)
            if u not "seen":
                mark u seen
                PQ.add(u,d)
            if u not done AND d < dist to u:
                distance to u = d
                PQ.decreaseKey(u,d)
```

current value of u in PQ

Dijkstra Pseudocode

```
dijkstra(G,s):
    PQ = new priority queue
    add s to PQ with priority 0
    mark s "seen"
    while PQ not empty:
        curr = PQ.extract()
        mark curr as done
        for u in neighbors(curr):
            d = dist to curr + w(curr, u)
            if u not "seen":
                mark u seen
                PQ.add(u,d)
            if u not done AND d < dist to u:
                distance to u = d
                PQ.decreaseKey(u,d)
```

Runtime: still $O(|E| \log |V|)$!

caveat: Java does not have decreaseKey built in!

solution 1: add edges to PQ, check if their "destination" is done when extracting

solution 2: allow PQ to have duplicate vertices

both solutions have runtime $O(|E| \log |E|)$

$|E| \leq |V|^2$

$\log |E| \leq 2 \log |V|$

$= O(|E| \log |V|)$ (unchanged!)

Question: why does Prim's work?

(1) Prim's produces a spanning tree (when G connected)

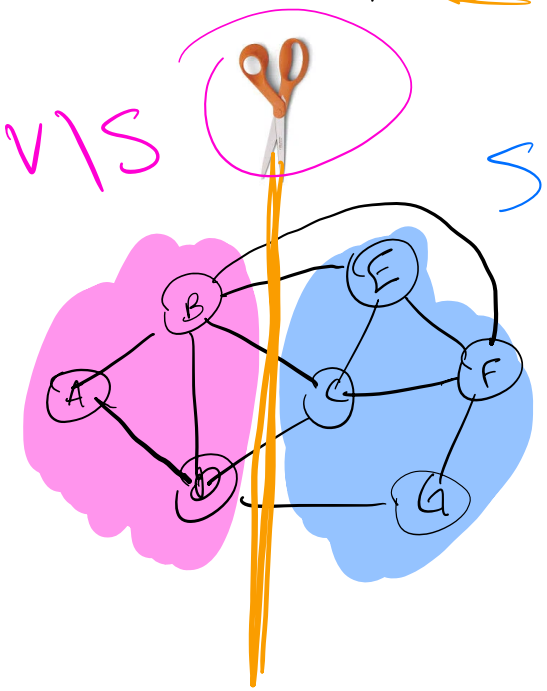
need to show 2 of:

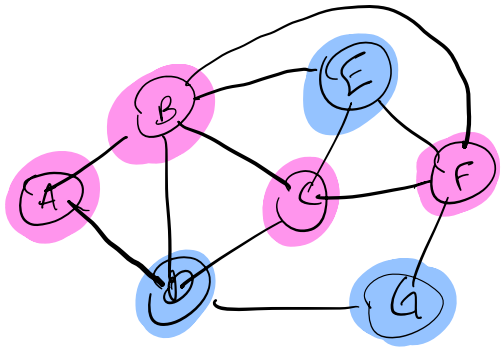
- connected
- acyclic
- $|V|-1$ edges

(2) Prim's produces minimum spanning tree

- uses the "Cut Theorem"

Def (cut) a cut of a graph $G = (V, E)$ is a partition of the vertices into two sets, S and $V \setminus S$

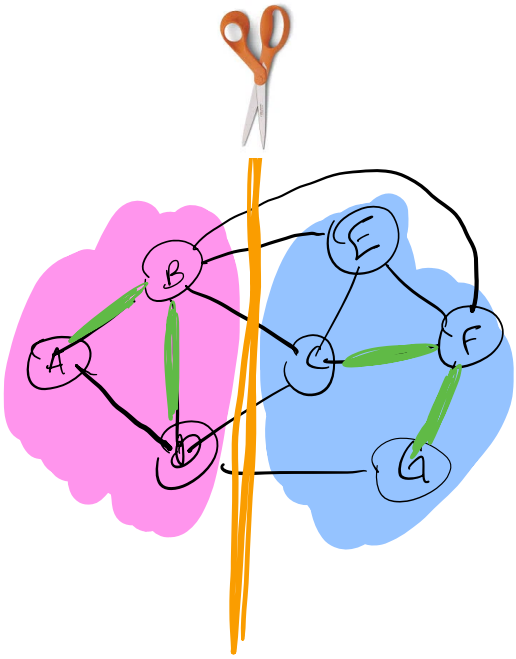




edge $e = (v_1, v_2)$ crosses a cut $(S, V \setminus S)$ if $v_1 \in S$ and $v_2 \in V \setminus S$
or $v_1 \in V \setminus S$ and $v_2 \in S$

A set of edges R "respects a cut" iff

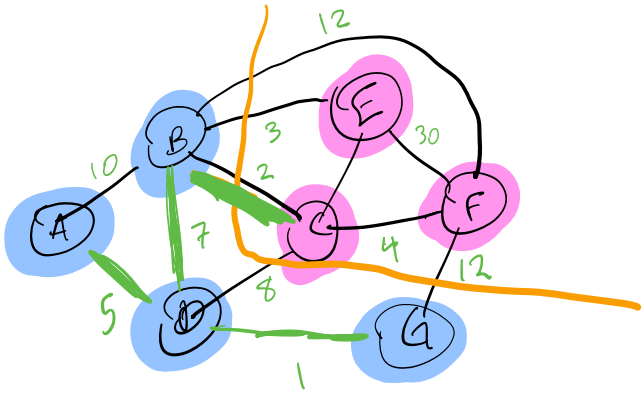
$\forall\, e \in R$, e does not cross the cut



$R = \{(A,B), (B,D), (C,F), (F,G)\}$

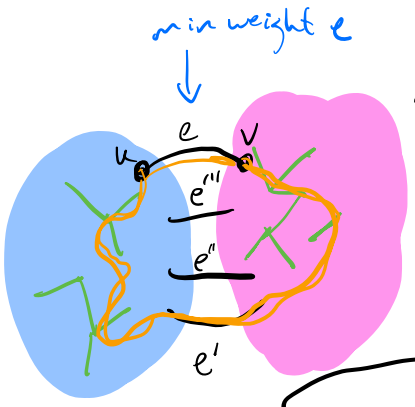
Theorem (Cut Theorem) Suppose $A \subseteq E$ is a subset of a minimum spanning tree T, and let $(S, V \setminus S)$ be any cut which A respects. If e is a smallest-weight edge crossing $(S, V \setminus S)$, then $A \cup \{e\} \subseteq T$.

Ex



(B, C) must be in T

Proof



$w(e) \leq w(e')$

any T must have at least one edge crossing $(S, V \setminus S)$

Suppose towards a contradiction that $e = (u,v) \notin T$

There must exist path from u to v in T

That path must involve some e' that crosses $(S, V \setminus S)$

consider $\tilde{T} = T \setminus \{e'\} \cup \{e\}$ (swap e' with e !)

$\tilde{T}$ is a spanning tree

$$cost(\tilde{T}) = cost(T) - w(e') + w(e)$$

$$< cost(T)$$

contradiction to T being a MST

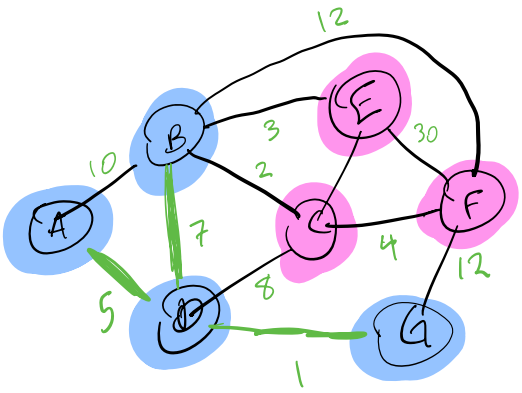
□

So... why does Prim's produce a minimum spanning tree?

- Start with empty tree A
- Pick starting node S, add to A
- Repeat $|V|-1$ times:

  add the min weight edge with one endpoint in A and one endpoint outside of A

This crosses the cut $(A, V \setminus A)$, which is respected by the current subtree!



General MST Algorithm

- Start with empty tree A
- Repeat $|V|-1$ times:
  - Pick a cut $(S, V \setminus S)$ that A respects and add the min-weight edge crossing that cut.

Kruskal's Algorithm

- Start with empty tree A
- repeatedly add lowest weight e that does not create a cycle.

if $e=(u,v)$ S = connected component of A containing u.