

# CSE 332 Spring 2026

## Lecture 6: Recurrences

Nathan Brunelle

<http://www.cs.uw.edu/332>

# Recursive Binary Search

5	8	13	42	75	79	88	90	95	99
0	1	2	3	4	5	6	7	8	9

```
public static boolean binarySearch(List<Integer> lst, int k){
    return binarySearch(lst, k, 0, lst.size());
}
private static boolean binarySearch(List<Integer> lst, int k, int start, int end){
    if(start == end)
        return false;
    int mid = start + (end-start)/2;
    if(lst.get(mid) == k){
        return true;
    } else if(lst.get(mid) > k){
        return binarySearch(lst, k, start, mid);
    } else{
        return binarySearch(lst, k, mid+1, end);
    }
}
```

# Analysis of Recursive Algorithms

- Overall structure of recursion:
  - Do some non-recursive “work”
  - Do one or more recursive calls on some portion of your input
  - Do some more non-recursive “work”
  - Repeat until you reach a base case
- Running time:  $T(n) = T(p_1) + T(p_2) + \dots + T(p_x) + f(n)$ 
  - The time it takes to run the algorithm on an input of size  $n$  is:
  - The sum of how long it takes to run the same algorithm on each smaller input
  - Plus the total amount of non-recursive work done in that stack frame
- Usually:
  - $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$ 
    - Called “divide and conquer”
  - $T(n) = T(n - c) + f(n)$ 
    - Called “chip and conquer”

# How Efficient Is It?

- $T(n) = 1 + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$
- Base case:  $T(1) = 1$

$T(n)$  = “cost” of running the entire algorithm on an array of length  $n$

# Let's Solve the Recurrence!

$$T(1) = 1$$

$$T(n) = 1 + \cancel{T(n/2)}$$

$$1 + \cancel{T(n/4)}$$

$$1 + \cancel{T(n/8)}$$

...

1

Substitute until  $T(1)$   
So  $\log_2 n$  steps

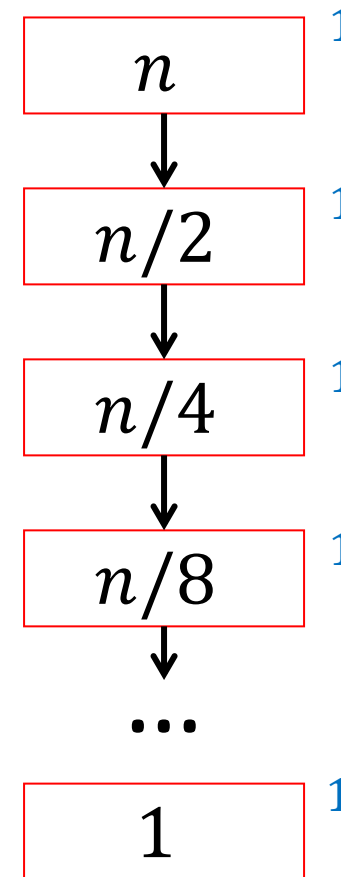
$$T(n) = \sum_{i=0}^{\log_2 n} 1 = \log_2 n$$

$$T(n) \in \Theta(\log n)$$

# Make our process “prettier”

- Draw a picture of the recursion
- Identify the work done per stack frame
- Add up all the work!
  - Sum is the answer!
  - In this case  $\Theta(\log_2 n)$

$$T(n) = T\left(\frac{n}{2}\right) + 1$$



$\log_2 n$  levels  
of recursion

## The “Tree Method”

# Recursive Linear Search

5	8	13	42	75	79	88	90	95	99
0	1	2	3	4	5	6	7	8	9

```
public static boolean linearSearch(List<Integer> lst, int k){
    return linearSearch(lst, k, 0, lst.size());
}

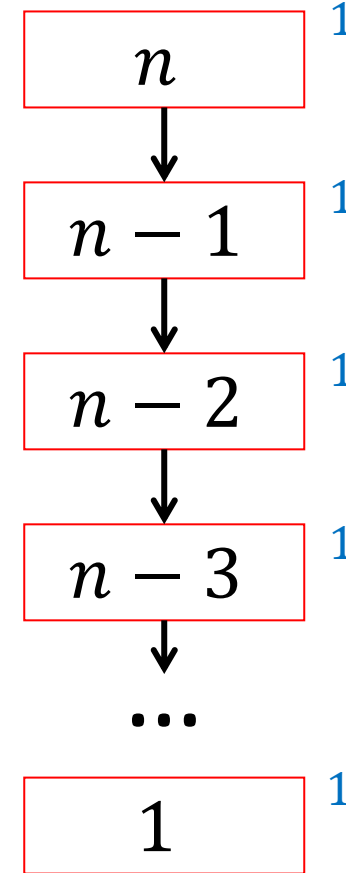
private static boolean linearSearch(List<Integer> lst, int k, int start, int end){
    if(start == end){
        return false;
    } else if(lst.get(start) == k){
        return true;
    } else{
        return linearSearch(lst, k, start+1, end);
    }
}
```

# Prettier List Sum Substitution

- Draw a picture of the recursion
- Identify the work done per stack frame
- Add up all the work!

Running time:  $\Theta(n)$

$$T(n) = T(n - 1) + 1$$



$n$  levels  
of recursion

# Recursive List Summation Code

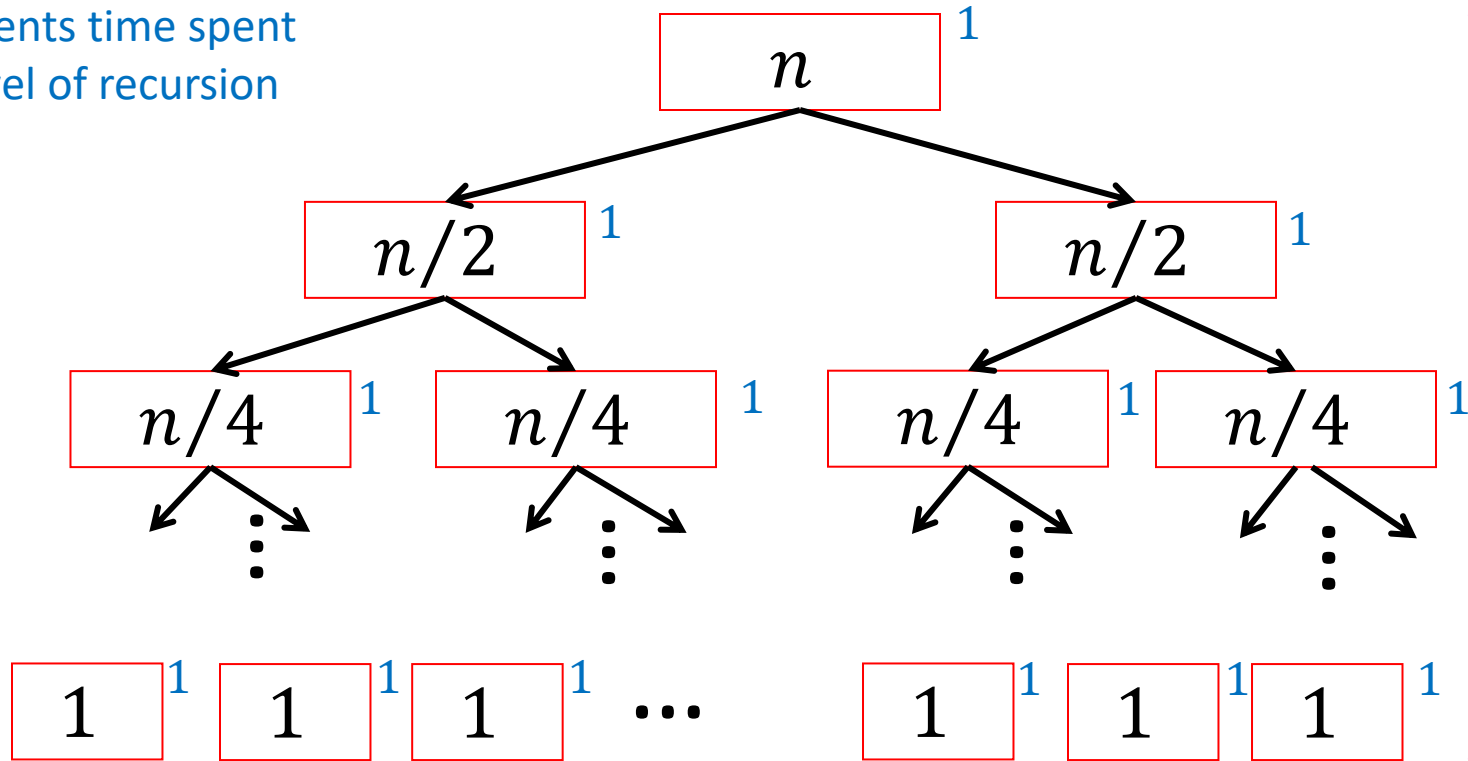
```
public int sum(int[] list){
    return sum_helper(list, 0, list.size);
}
private int sum_helper(int[] list, int low, int high){
    if (low == high){ return 0; }
    if (low == high-1){ return list[low]; }
    int middle = (high+low)/2;
    return sum_helper(list, low, middle) + sum_helper(list, middle, high);
}
```

# Tree Method for Recursive Summation

Red box represents a problem instance

Blue value next to each box represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$



$\Rightarrow 2^i$  work per level

$\log_2 n$  levels of recursion

$$T(n) = \sum_{i=0}^{\log_2 n} 2^i$$

# Recursive List Summation – Solving the Series

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log_2 n} 2^i \\ &= \left( \frac{2^{1+\log_2 n} - 1}{2 - 1} \right) \\ &= 2 \cdot 2^{\log_2 n} - 1 \\ &= 2n - 1 = \Theta(n) \end{aligned}$$

# Tree Method Summary: Chip and Conquer

- Recurrence looks like  $T(n) = aT(n - b) + f(n)$
- Use the recurrence to draw a tree
  - $a$  is the branching factor of the tree (e.g. if  $a = 2$  then it's a binary tree)
  - Subtract  $b$  from the parent's input size to get children's input size
  - Work done per node is given by applying  $f(n)$  to that node's input size
  - Height of the tree is  $\frac{n}{b}$ 
    - Because that is the number of times we must subtract  $b$  until reaching a base case
    - Answer to the question "how many times must we subtract  $b$  until we reach 0?"
      - Any base case is a constant, so to reach a larger value would just be a constant change
- Use the tree to express running time as a series
  - Adding work done for each node level-by-level
  - Identify a pattern to express work done at level  $i$  as a function of  $i$
  - Write a series using  $i = 0$  up to  $\frac{n}{b}$
- Solve the series

# Tree Method Summary: Divide and Conquer

- Recurrence looks like  $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
- Use the recurrence to draw a tree
  - $a$  is the branching factor of the tree (e.g. if  $a = 2$  then it's a binary tree)
  - Divide the parent's input size by  $b$  to get children's input size
  - Work done per node is given by applying  $f(n)$  to that node's input size
  - Height of the tree is  $\log_b n$ 
    - Because that is the number of times we must divide by  $b$  until reaching a base case
    - Answer to the question "how many times must we divide by  $b$  until we reach 1?"
      - Any base case is a constant, so to reach a larger value would just be a constant change
- Use the tree to express running time as a series
  - Adding work done for each node level-by-level
  - Identify a pattern to express work done at level  $i$  as a function of  $i$
  - Write a series using  $i = 0$  up to  $\log_b n$
- Solve the series

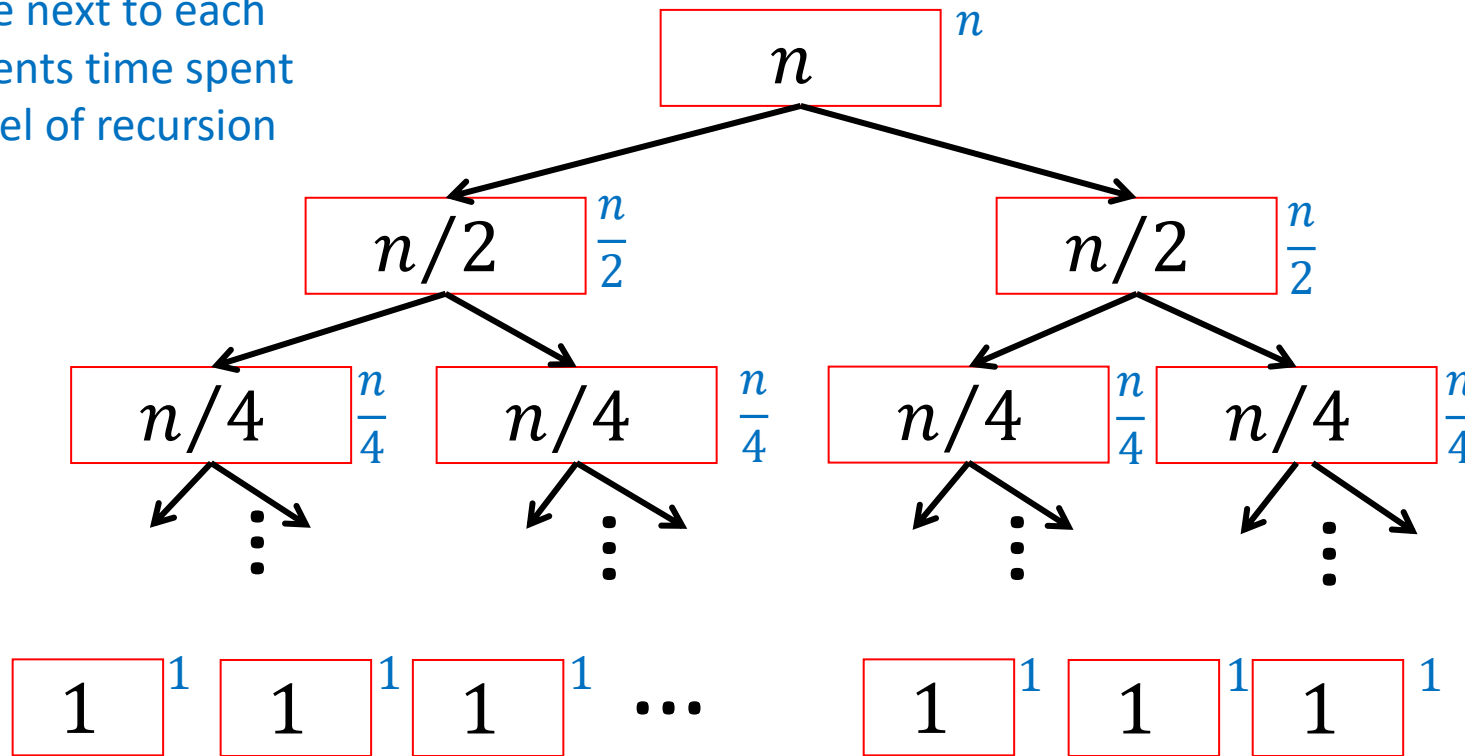
# Let's do some more!

- For each, assume the base case is  $n = 1$  and  $T(1) = 1$
- $T(n) = 2T\left(\frac{n}{2}\right) + n$
- $T(n) = 2T\left(\frac{n}{2}\right) + n^2$
- $T(n) = 2T\left(\frac{n}{8}\right) + 1$

# Tree Method $T(n) = 2T\left(\frac{n}{2}\right) + n$

Red box represents a problem instance

Blue value next to each box represents time spent at that level of recursion



$\Rightarrow n$  work per level

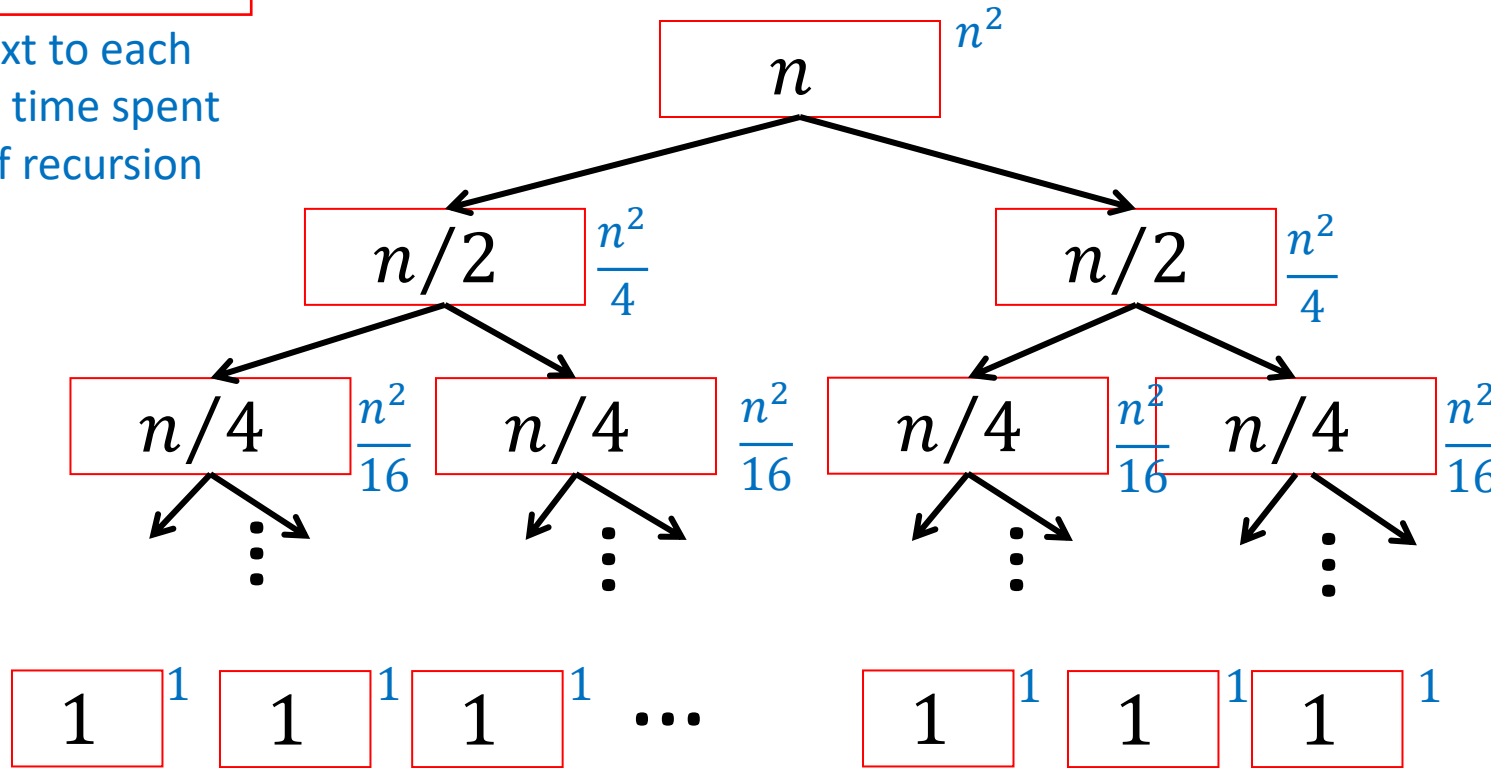
$\log_2 n$  levels of recursion

$$T(n) = \sum_{i=0}^{\log_2 n} n$$

# Tree Method $T(n) = 2T\left(\frac{n}{2}\right) + n^2$

Red box represents a problem instance

Blue value next to each box represents time spent at that level of recursion



$\Rightarrow$  ?? work per level

$\log_2 n$  levels of recursion

$$T(n) = \sum_{i=0}^{\log_2 n} ??$$

$$\text{Solving } T(n) = 2T\left(\frac{n}{2}\right) + n^2$$

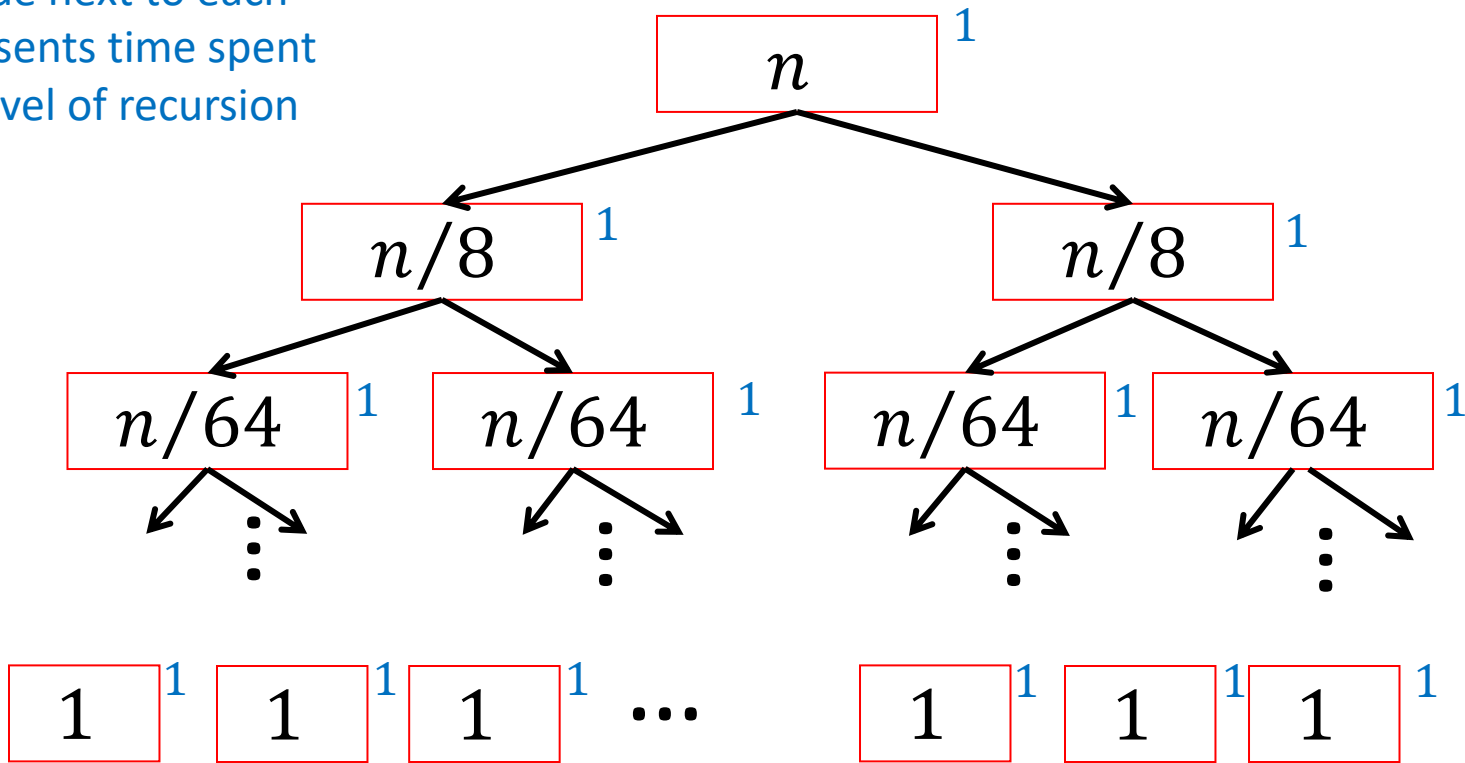
$$T(n) = \sum_{i=0}^{\log_2 n} \frac{n^2}{2^i}$$

$$= n^2 \cdot \sum_{i=0}^{\log_2 n} \left(\frac{1}{2}\right)^i$$

# Tree Method $T(n) = 2T\left(\frac{n}{8}\right) + 1$

Red box represents a problem instance

Blue value next to each box represents time spent at that level of recursion



$\Rightarrow 2^i$  work per level

$\log_8 n$  levels of recursion

$$T(n) = \sum_{i=0}^{\log_8 n} 2^i$$

$$\text{Solving } T(n) = 2T\left(\frac{n}{8}\right) + 1$$

$$T(n) = \sum_{i=0}^{\log_8 n} 2^i$$

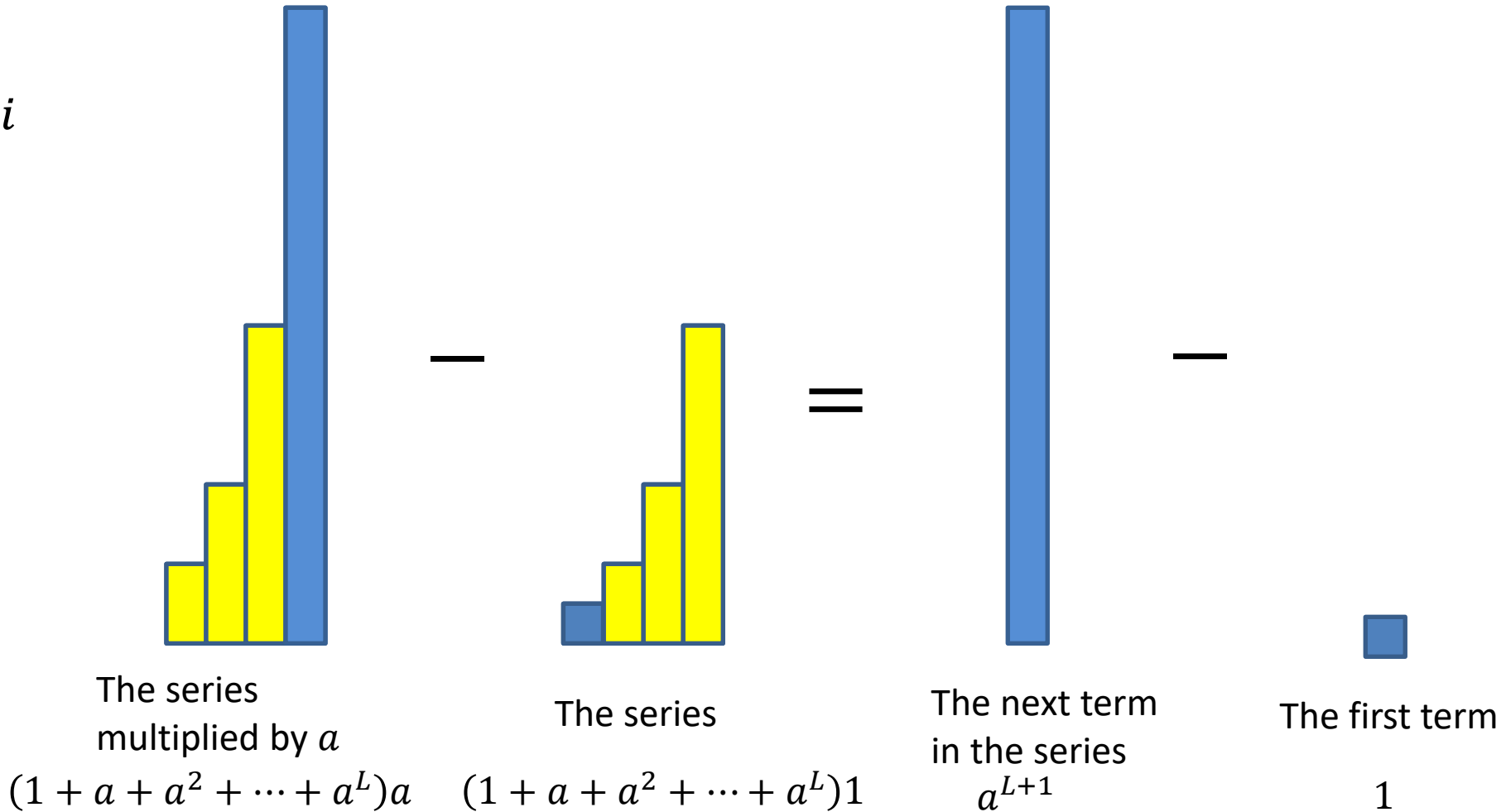
$$= \left( \frac{2^{1+\log_8 n} - 1}{2 - 1} \right)$$

$$= 2 \cdot 2^{\log_8 n} - 1$$

$$= 2n^{\log_8 2} - 1 = \Theta\left(n^{\frac{1}{3}}\right)$$

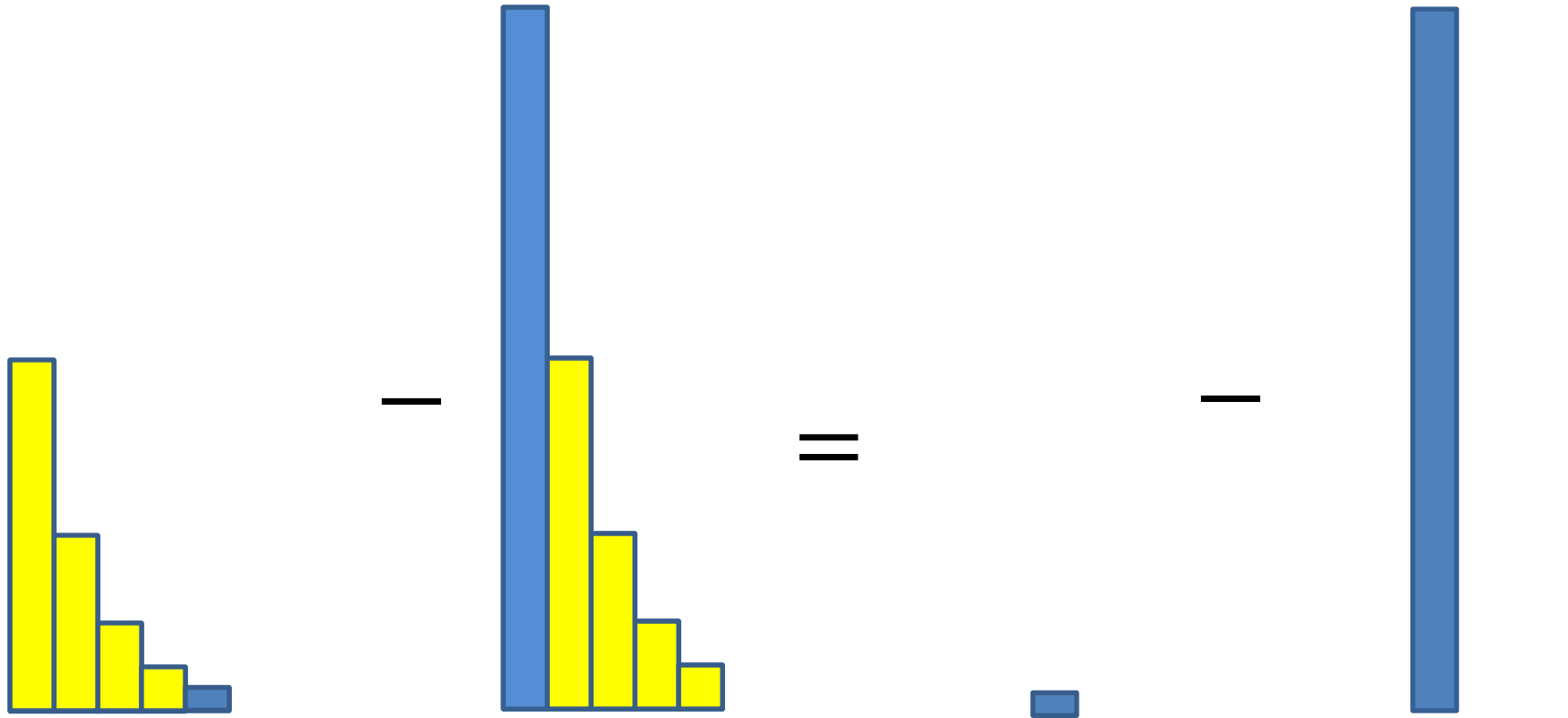
# Finite Geometric Series ( $a > 1$ )

$$\sum_{i=0}^L a^i$$



# Finite Geometric Series ( $a < 1$ )

$$\sum_{i=0}^L a^i$$



The series multiplied by  $a$

The series

The next term in the series

The first term

$$(1 + a + a^2 + \dots + a^L)a$$

$$(1 + a + a^2 + \dots + a^L)1$$

$$a^{L+1}$$

$$1$$

Solve for the series