CSE 332: Data Structures & Parallelism

Lecture 24: Topological Sort

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Winter 2025

Today

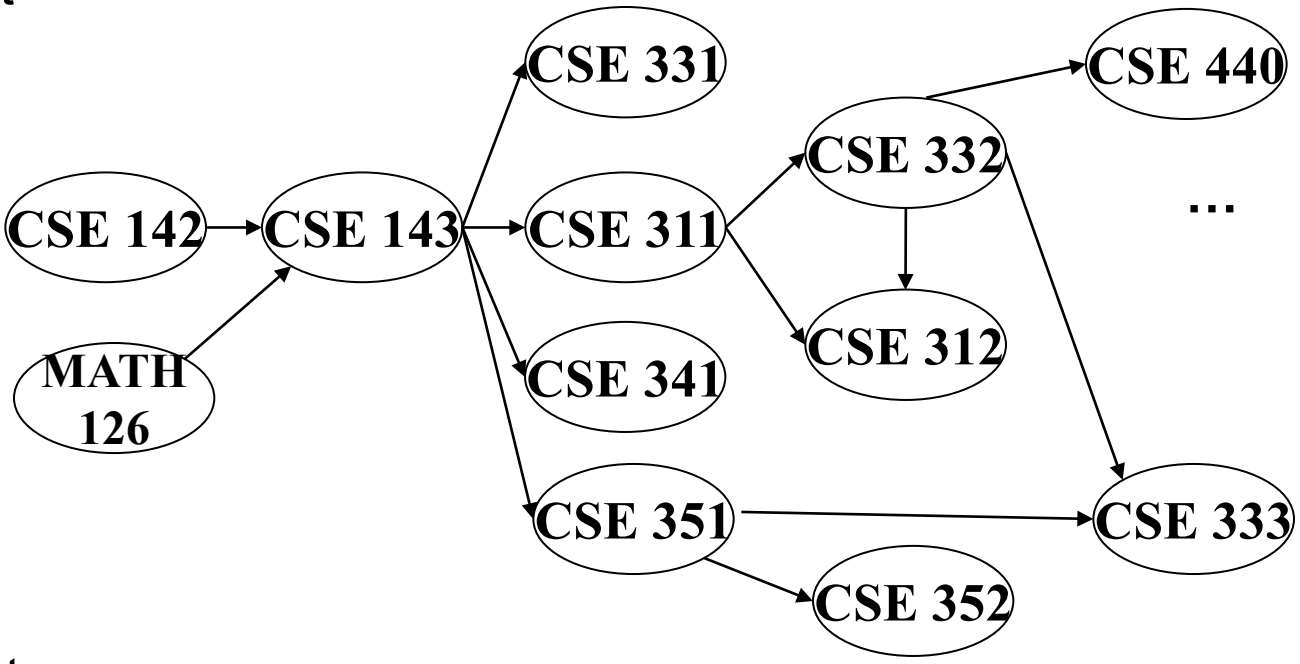
- Graphs
  - Topological Sort

Disclaimer: Do not use for official advising purposes!
(e.g. Implies that CSE 332 is a pre-req for CSE 312 – not true)

Topological Sort

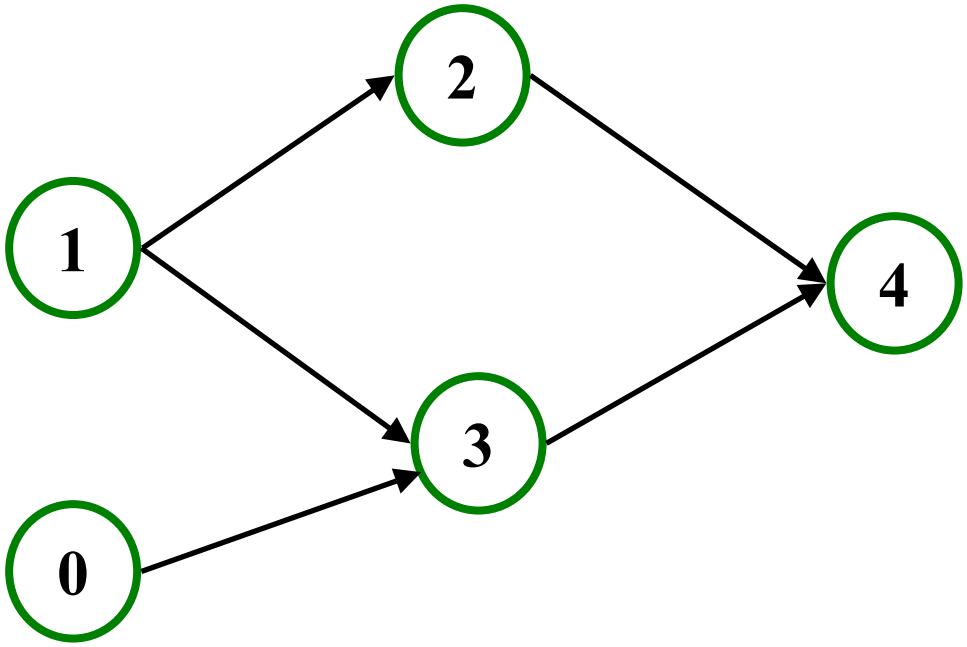
Problem: Given a DAG `G=(V,E)`, output all the vertices in order such that if no vertex appears before any other vertex that has an edge to it

Example input:



Example output:

142, 126, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352



Valid Topological Sorts:

Questions and comments

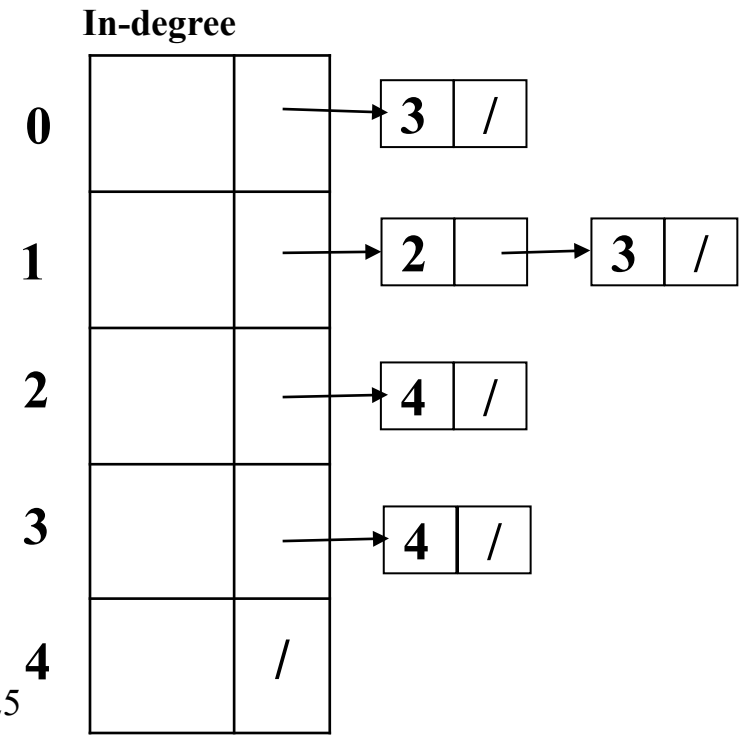
- Why do we perform topological sorts only on DAGs?
- Is there always a unique answer?
- What DAGs have exactly 1 answer?
- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

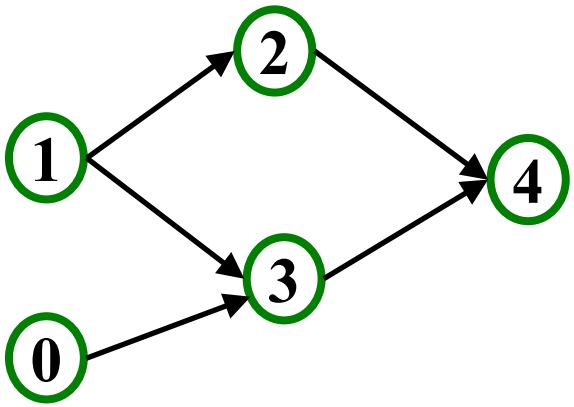
Topological Sort Uses

- Scheduling instructions/tasks
- Figuring out how to finish your degree
- Computing the order in which to recompute cells in a spreadsheet
- Determining the order to compile files using a Makefile
- In general, taking a dependency graph and coming up with an order of execution

A First Algorithm for Topological Sort

1. Label ("mark") each vertex with its in-degree
   - Think "write in a field in the vertex"
   - Could also do this via a data structure (e.g., array) on the side
2. While there are vertices not yet output:
   a) Choose a vertex **v** labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **w** adjacent to **v** (i.e. **w** such that (**v**,**w**) in **E**), decrement the in-degree of **w**





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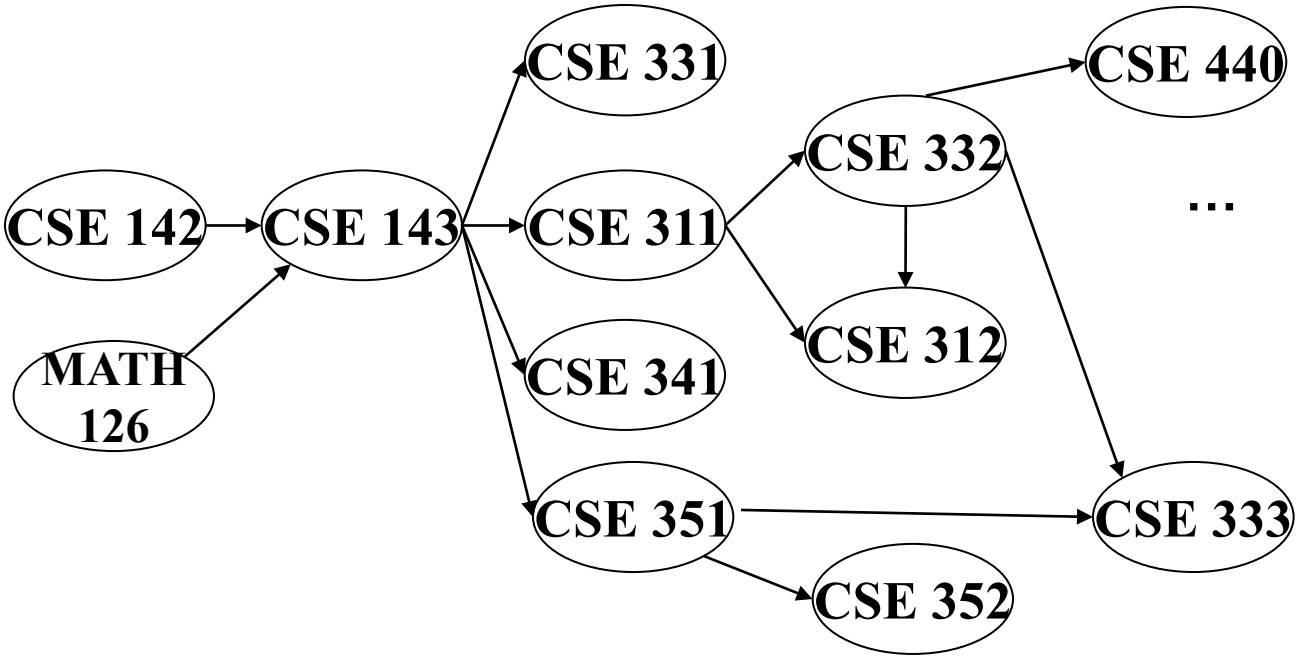
Example

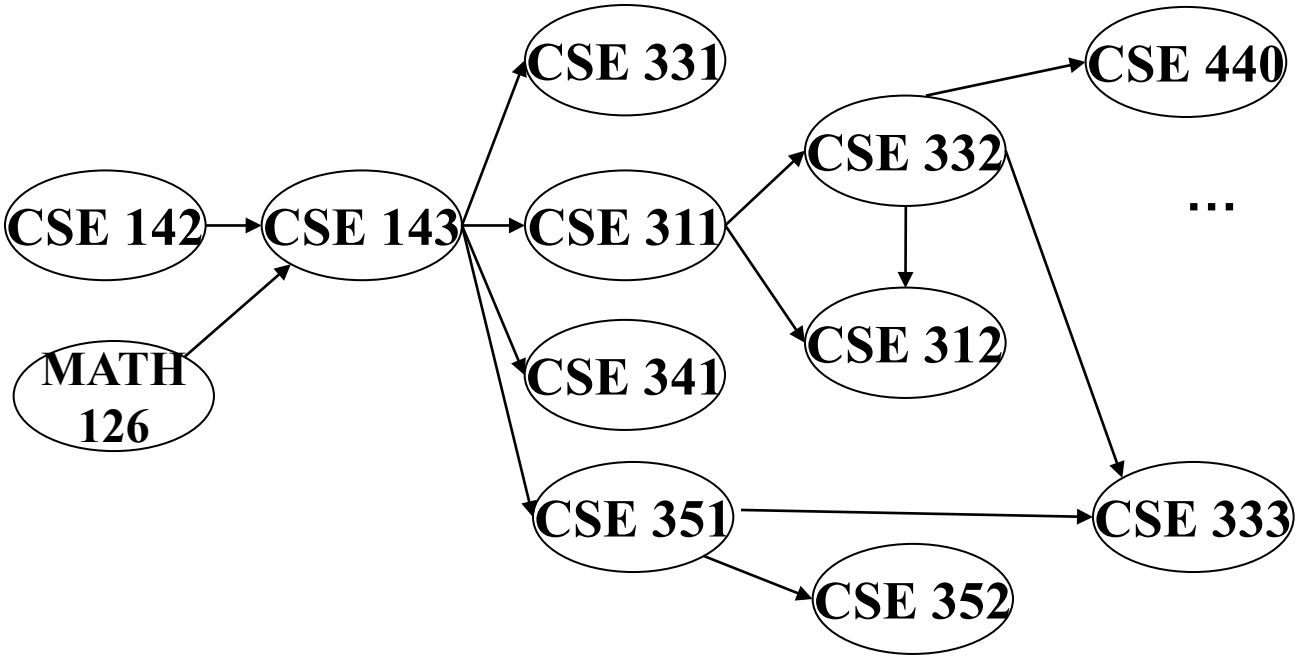
Output:



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?												
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1

Example

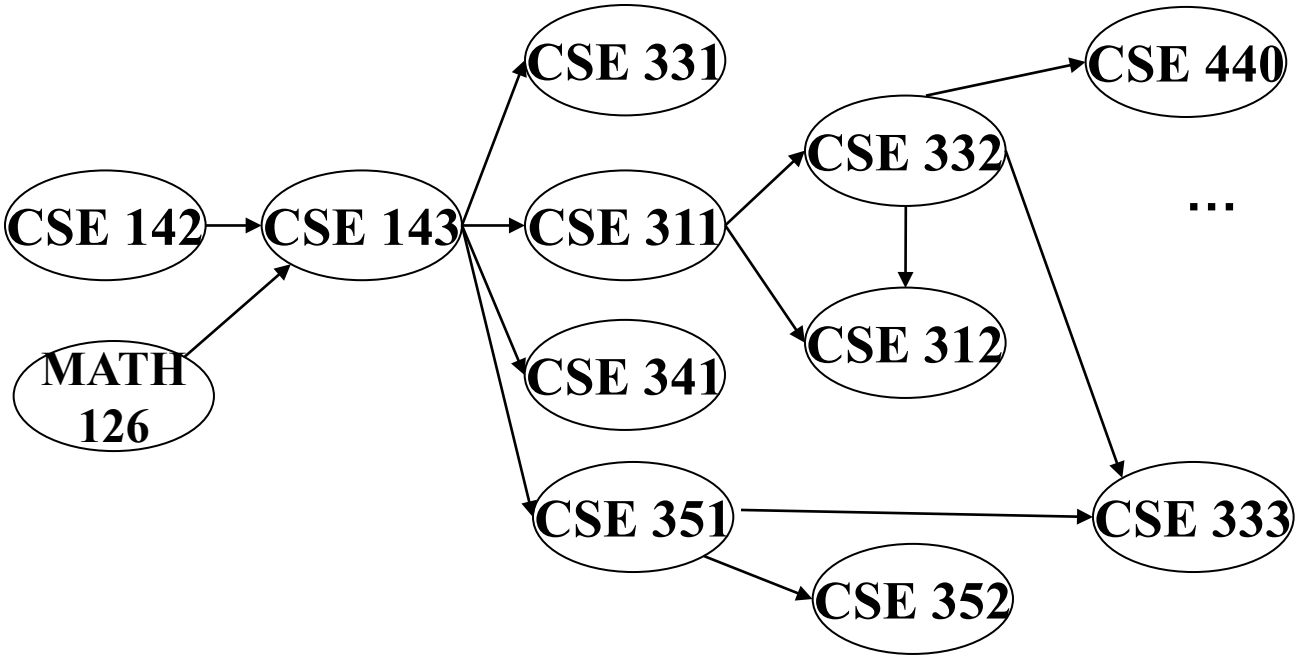
Output: 126



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x											
In-degree:	0	0	2 1	1	2	1	1	2	1	1	1	1

Example

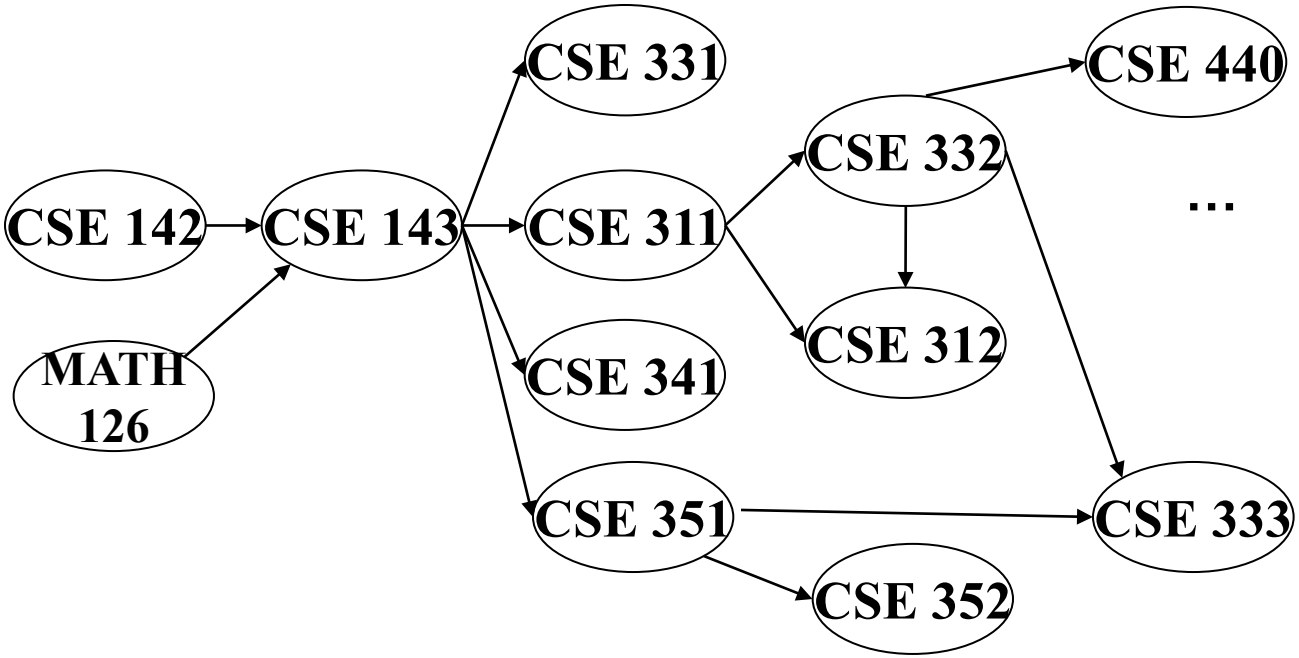
Output: 126
142



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x										
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1									
			0									

Example

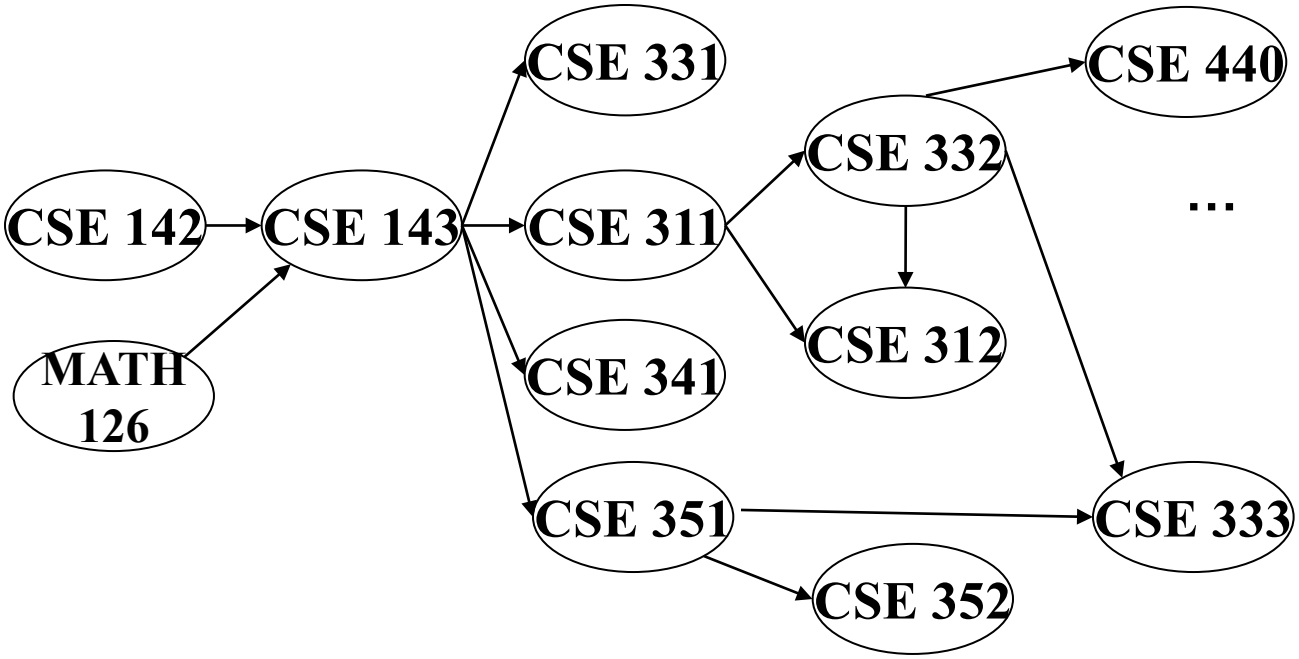
Output: 126
142
143



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x									
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0		0			0	0		
			0									

Example

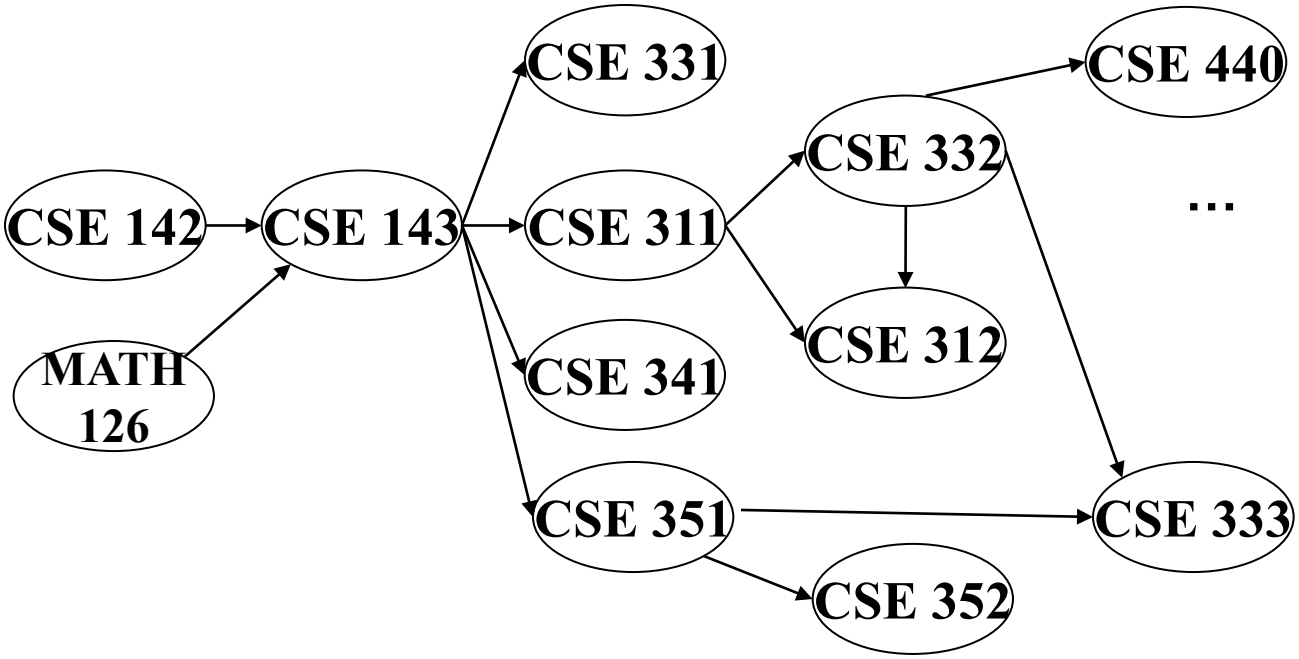
Output: 126
142
143
311



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x								
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

Example

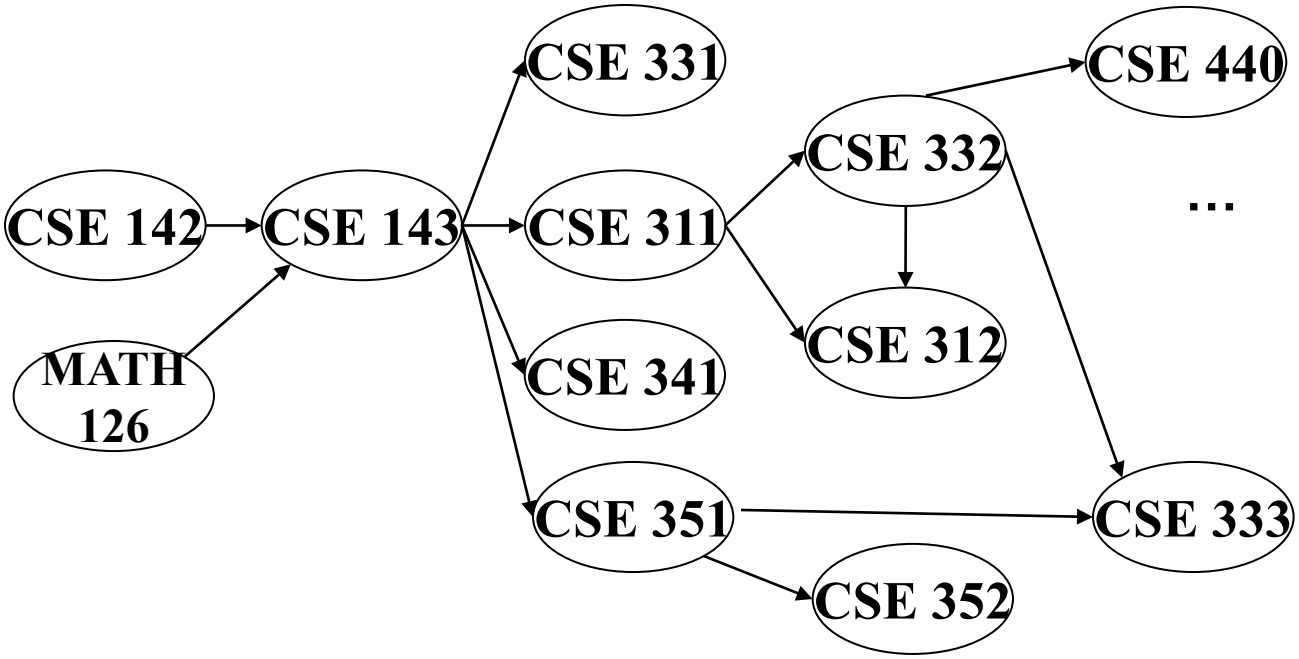
Output: 126
142
143
311
331
…



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x						
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

Example

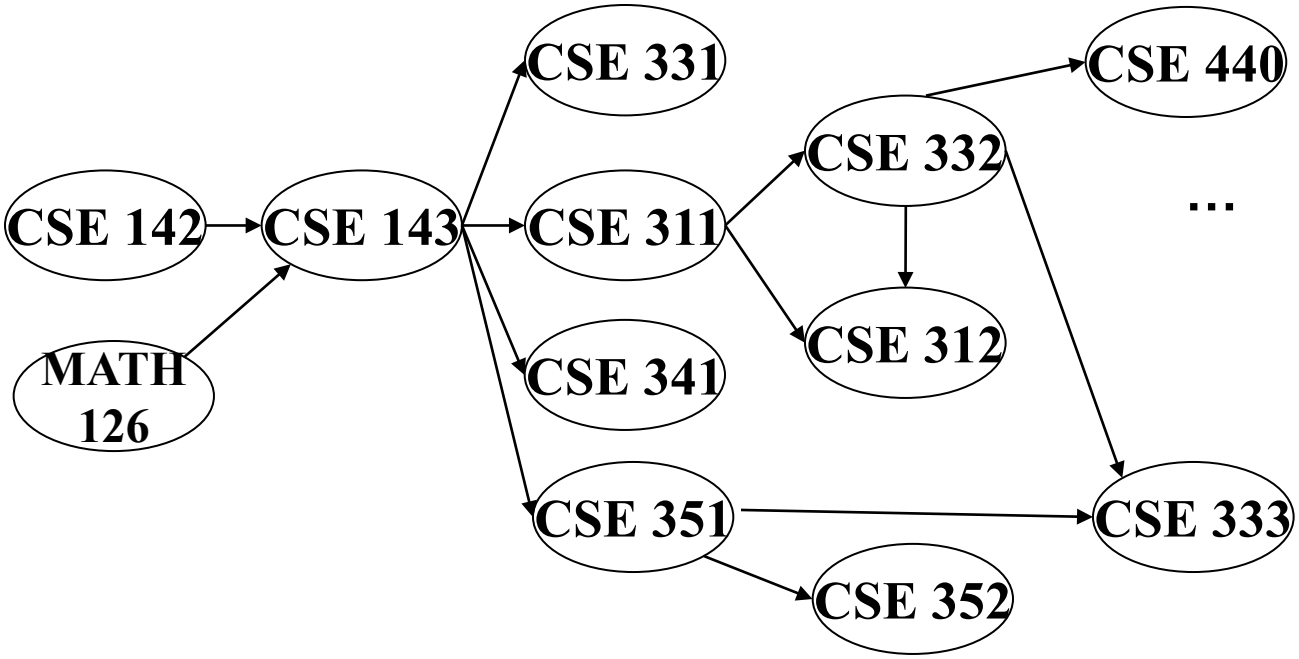
Output: 126
142
143
311
331
332



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x	x					
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

Example

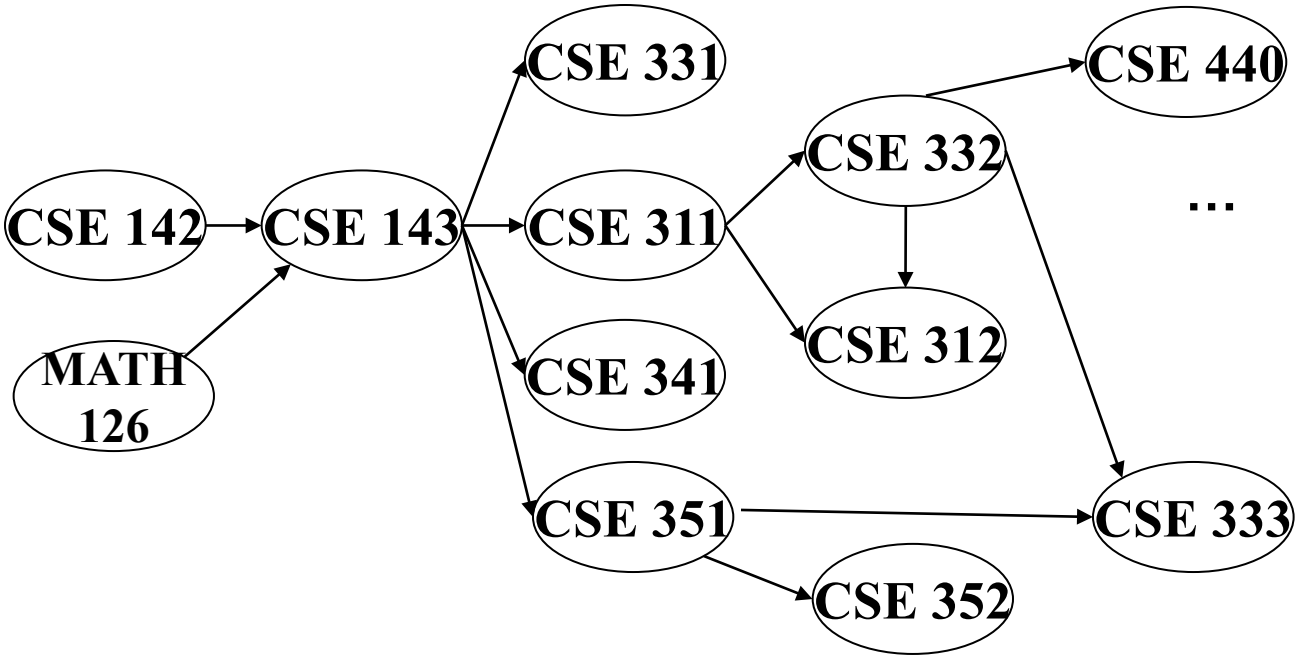
Output: 126
142
143
311
331
332
312



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x					
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

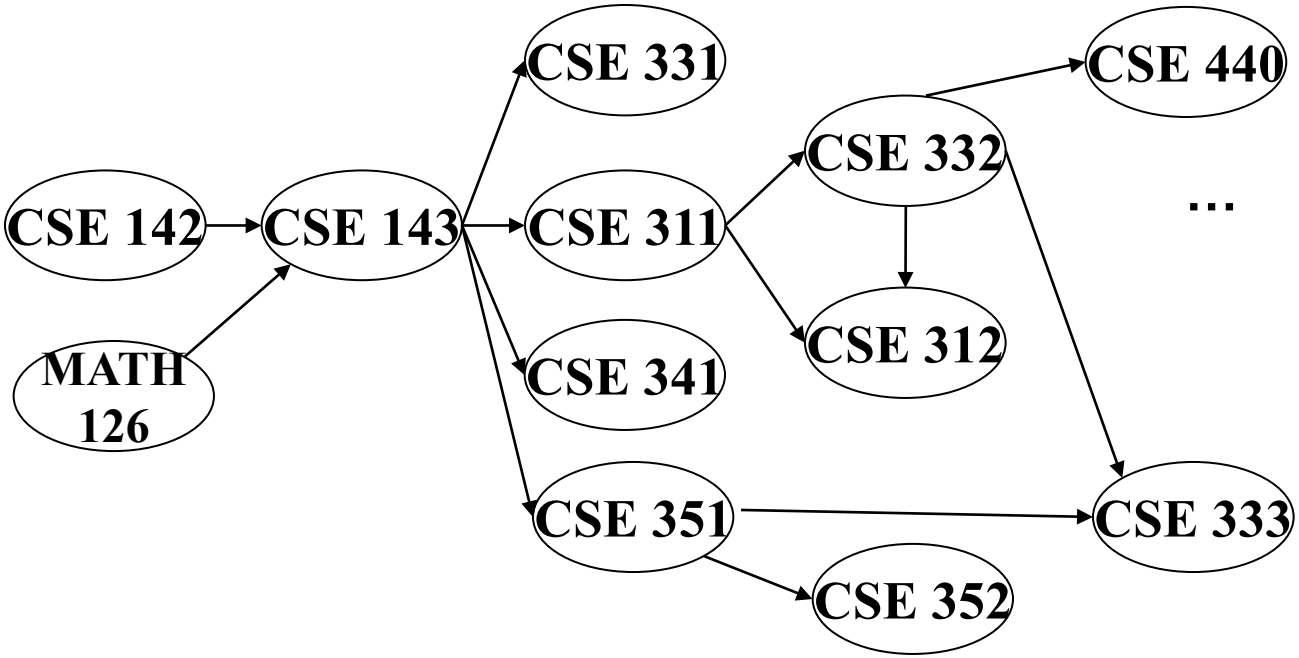
Example

Output: 126
142
143
311
331
332
312
341



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x			
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

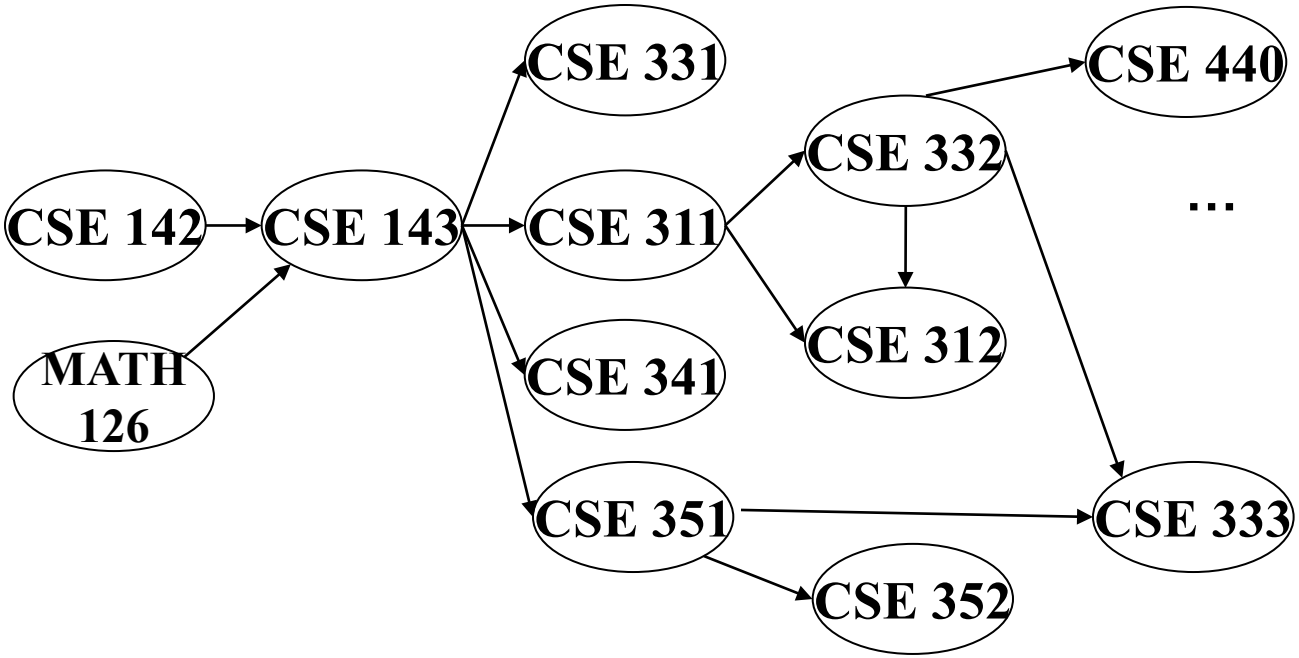
Example



Output: 126
142
143
311
331
332
312
341
351

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x	x		
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example



Output: 126
142
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440

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

A couple of things to note

- Needed a vertex with in-degree of 0 to start
  - No cycles
- Ties between vertices with in-degrees of 0 can be broken arbitrarily
  - Potentially many different correct orders

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
   for each w adjacent to v
     w.indegree--;
}
```

Doing better

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, box, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **w** adjacent to **v** (i.e. **w** such that (**v**,**w**) in **`E`**), decrement the in-degree of **w**, if new degree is 0, enqueue it

Topological Sort(optimized): Running time?

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
   for each w adjacent to v {
     w.indegree--;
     if(w.indegree==0)
       enqueue(w);
  }
}
```