Comparisons Sorts

CSE 332 Spring 2025
Lecture 13

Announcements

Midterm on Wednesday

- Look for an Ed announcement tomorrow about which room
- 6-7:20, we're writing an exam that should take 50-60 minutes, but giving you a buffer time (i.e., 80 minutes to work). We'll try to start as close to 6 as possible.
- Bring your Husky card.

No lecture on Wednesday (but I'll be here to answer last-second questions)

There is section on Thursday

- Sorting AND an optional dictionary implementation that's nice to know.

Don't forget: Exercise 5 (hashing, programming) due Friday.

Three goals

Three things you might want in a sorting algorithm:

In-Place

- Only use $O(1)$ extra heap memory.
- Sorted array given back in the input array.

Stable

- If a appears before b in the initial array and a.compareTo(b) == 0, then a appears before b in the final array.
- Why? Imagine you sort an array by first name, then sort by last name. With a stable sort you get a list sorted by full name! (With an unstable sort the "Smiths" could go in any order).

Fast

Insertion Sort

How you sort a hand of cards.

Maintain a sorted subarray at the front.

Start with one element.

While(your subarray is not the full array)

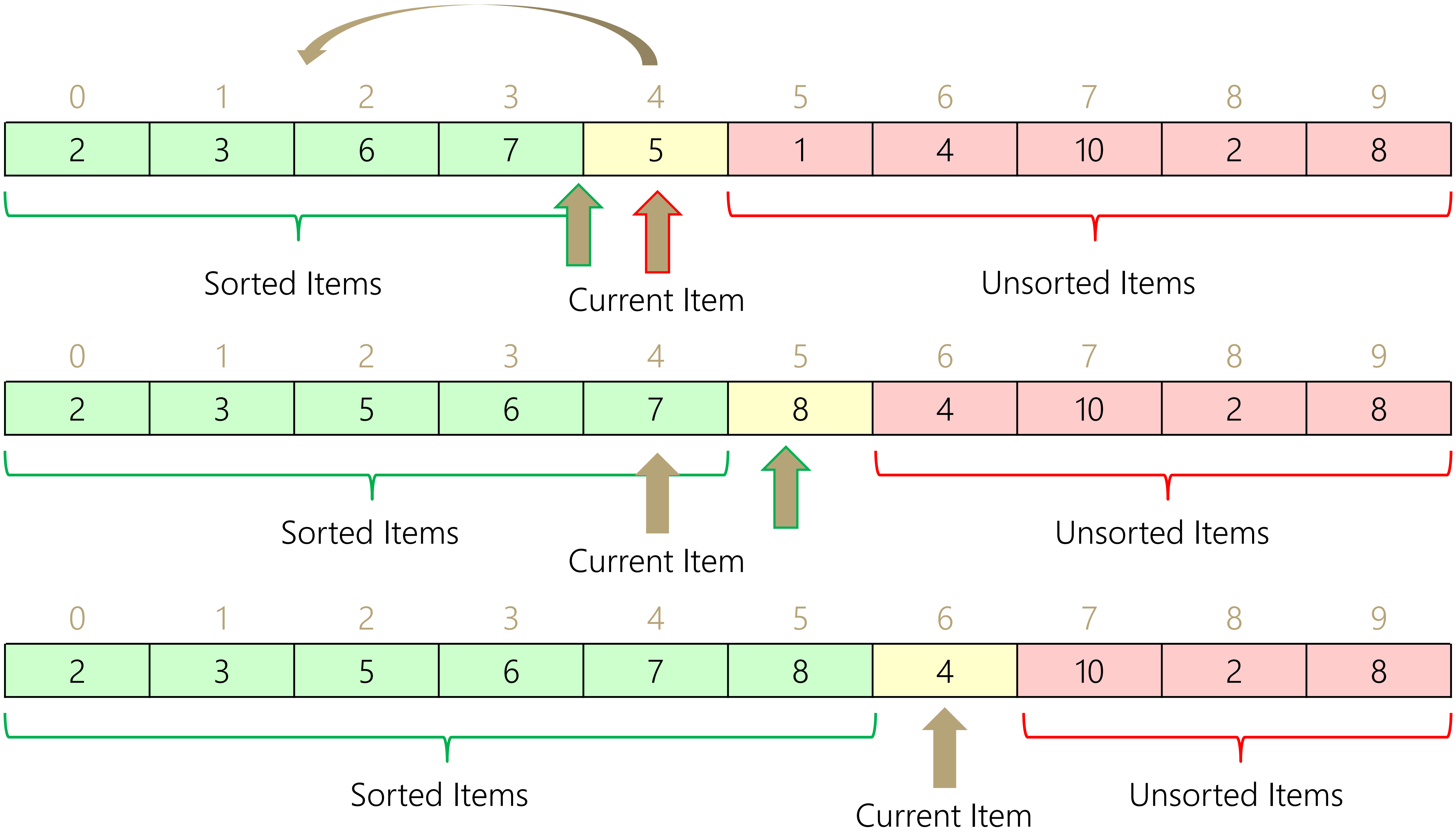
- Take the next element not in your subarray
- Insert it into the sorted subarray

Insertion Sort

```
for(i from 1 to n-1){
    int index = i
    while(a[index-1] > a[index]){
        swap(a[index-1], a[index])
        index = index-1
    }
}
```

Insertion Sort

https://www.youtube.com/watch?v=ROalU379l3U



Insertion Sort Analysis

Stable? Yes! (If you’re careful)

In Place Yes!

Running time:

- Best Case: $O(n)$
- Worst Case: $O(n^2)$
- Average Case: $O(n^2)$

___________ Sort

Here’s another idea for a sorting algorithm:

Maintain a sorted subarray

While(subarray is not full array)

    Find the smallest element remaining in the unsorted part.

    Insert it at the end of the sorted part.

Selection Sort

Here's another idea for a sorting algorithm:

Maintain a sorted subarray

While(subarray is not full array)

Find the smallest element remaining in the unsorted part.

- By scanning through the remaining array

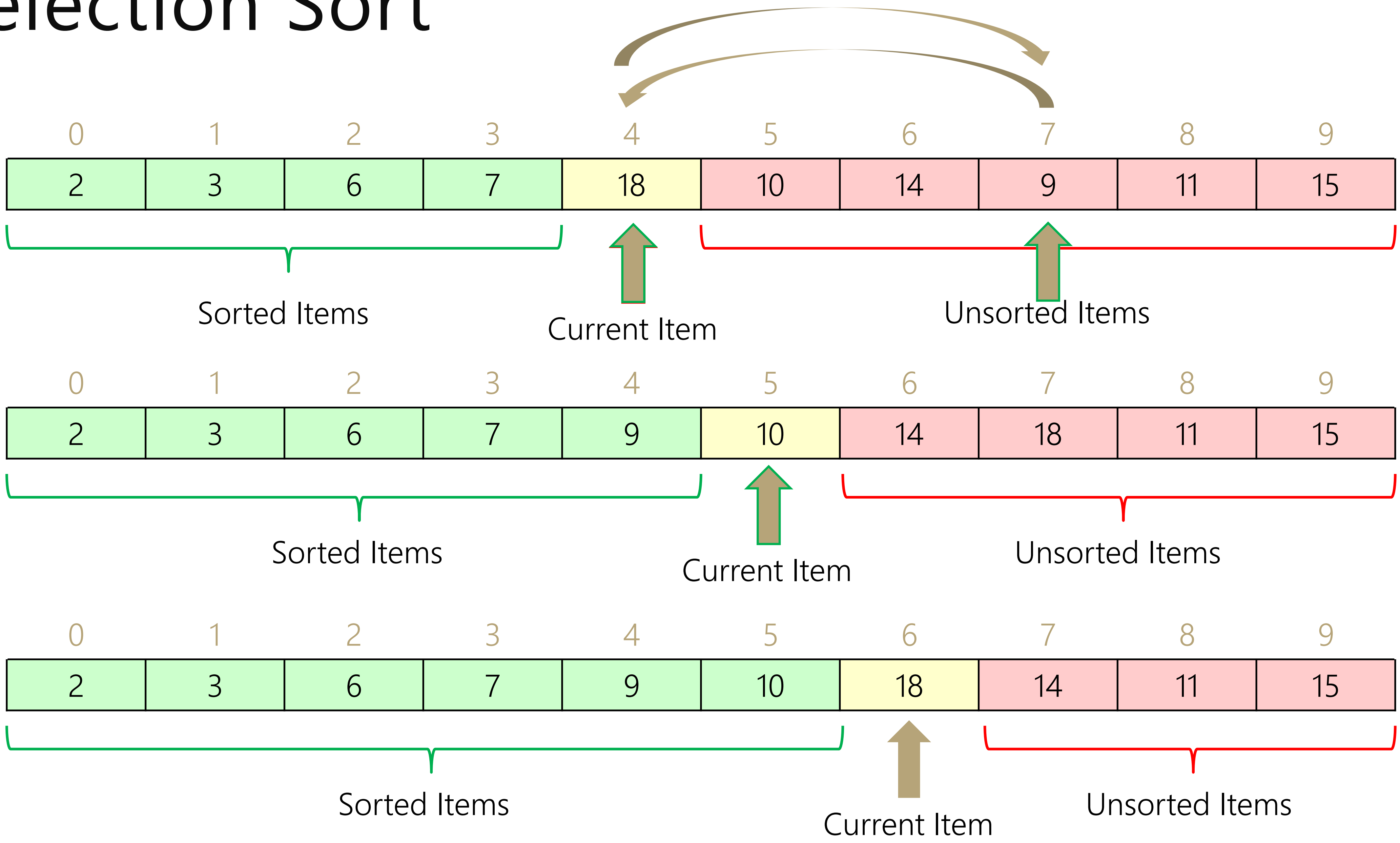
Insert it at the end of the sorted part.

Running time $O(n^2)$

In-Place: Yes; Stable: Yes.

https://www.youtube.com/watch?v=Ns4TPTC8whw

Selection Sort



Selection Sort

Here's another idea for a sorting algorithm:

Maintain a sorted subarray

While(subarray is not full array)

- Find the smallest element remaining in the unsorted part.
  - By scanning through the remaining array
- Insert it at the end of the sorted part.

Running time $O(n^2)$

Can we do better? With a data structure?

Heap Sort

Here's another idea for a sorting algorithm:

Maintain a sorted subarray; **Make the unsorted part a min-heap**

While(subarray is not full array)

Find the smallest element remaining in the unsorted part.

- By calling `removeMin` on the heap

Insert it at the end of the sorted part.

Running time $O(n \log n)$

Heap Sort

https://www.youtube.com/watch?v=Xw2D9aJRBY4

0	1	2	3	4	5	6	7	8	9
1	4	2	14	15	18	16	17	20	22

Current Item

Heap

Sorted Items

0	1	2	3	4	5	6	7	8	9
22	4	2	14	15	18	16	17	20	1

percolateDown(22)

Current Item

Heap

Sorted Items

0	1	2	3	4	5	6	7	8	9
2	4	16	14	15	18	22	17	20	1

Current Item

Heap

Sorted Items

Heap Sort (Better)

We're sorting in the wrong order!

- Could reverse at the end.

Our heap implementation will implicitly assume that the heap is on the left of the array.

Switch to a max-heap, and keep the sorted stuff on the right.

What's our running time? $O(n \log n)$

Heap Sort

Our first step is to make a heap. Does using `buildHeap` instead of inserts improve the running time?

Not in a big-O sense (though we did by a constant factor).

In place: Yes

Stable: No; You'll prove that in your next exercise.

A Different Idea

So far our sorting algorithms:

- Start with an (empty) sorted array
- Add something to it.

Different idea: Divide And Conquer:

Split up array (somehow)

Sort the pieces (recursively)

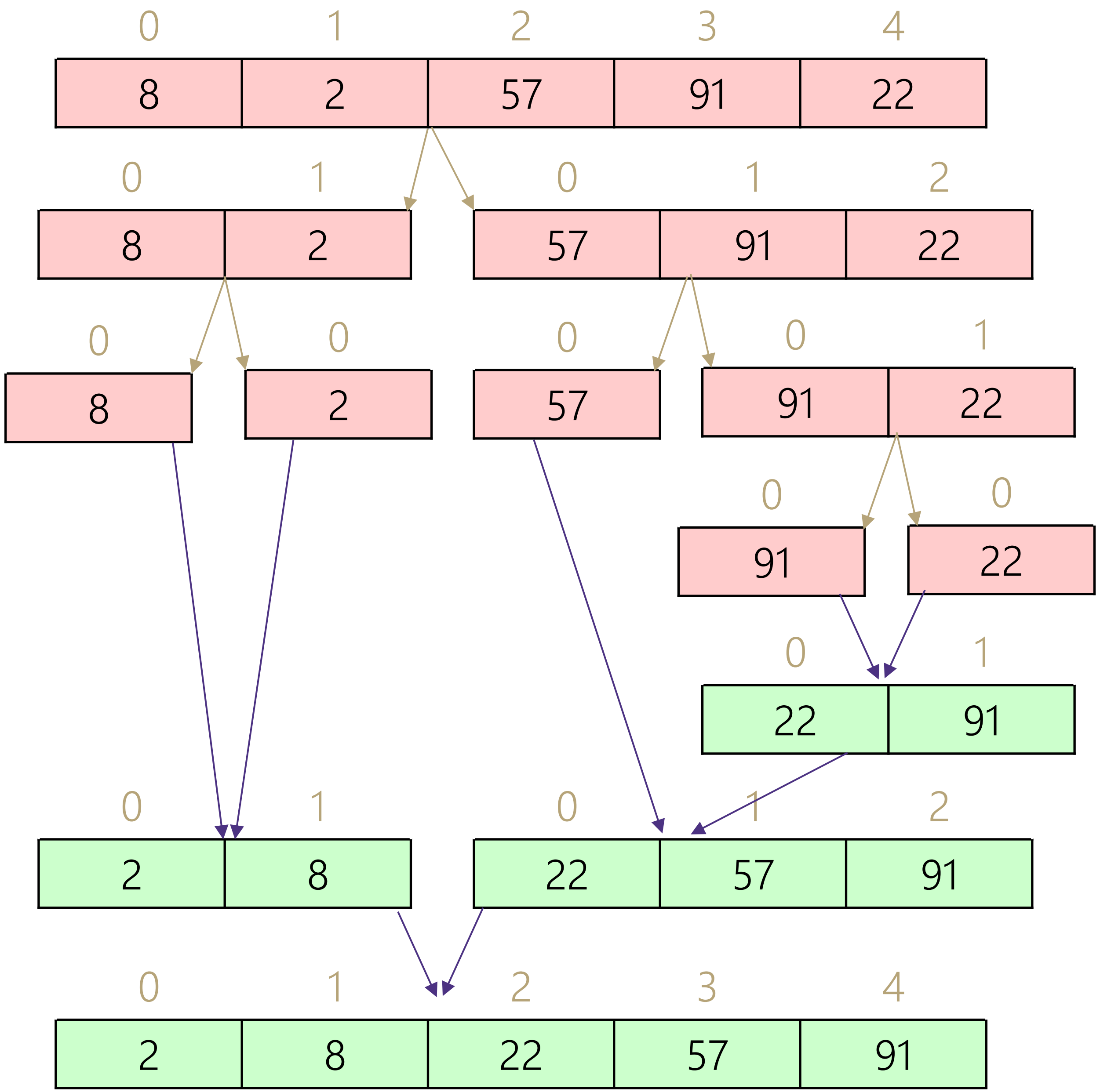
Combine the pieces

Merge Sort

Split array in the middle

Sort the two halves

Merge them together



Merge Sort Pseudocode

https://www.youtube.com/watch?v=XaqR3G_NVoo

```
mergeSort(input) {
   if (input.length == 1)
      return
   else
      smallerHalf = mergeSort(new [0, ..., mid])
      largerHalf = mergeSort(new [mid + 1, ...])
      return merge(smallerHalf, largerHalf)
}
```

How Do We Merge?

Turn two sorted lists into one sorted list:

Start from the small end of each list.

Copy the smaller into the combined list

Move that pointer one spot to the right.

3	15	27

5	12	30

3	5	12	15	27	30

Merge Sort Analysis

Running Time:

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + c_1 n & \text{if } n \geq 1 \\ c_2 & \text{otherwise} \end{cases}$$

This is a closed form you will have memorized by the end of the quarter.

The closed form is $\Theta(n \log n)$.

Stable: yes! (if you merge correctly)

In place: no.

Some Optimizations

We need extra memory to do the merge

It's inefficient to make a new array every time

Instead have a single auxiliary array

- Keep reusing it as the merging space

Even better: make a single auxiliary array

- Have the original array and the auxiliary array "alternate" being the list and the merging space.

Quick Sort

Still Divide and Conquer, but a different idea:

Let’s divide the array into “big” values and “small” values

- And recursively sort those

What’s “big”?

- Choose an element (“the pivot”) anything bigger than that.

How do we pick the pivot?

For now, let’s just take the first thing in the array:

Swapping

How do we divide the array into "bigger than the pivot" and "less than the pivot?"

1. Swap the pivot to the far left.
2. Make a pointer i on the left, and j on the right
3. Until i, j meet
   - While $A[i] < \mathbf{pivot}$ move i left
   - While $A[j] > \mathbf{pivot}$ move j right
   - Swap $A[i], A[j]$
4. Swap A[i] or A[i-1] with pivot.

Swapping

0	1	2	3	4	5	6	7	8	9
8	3	5	6	9	1	4	7	2	10

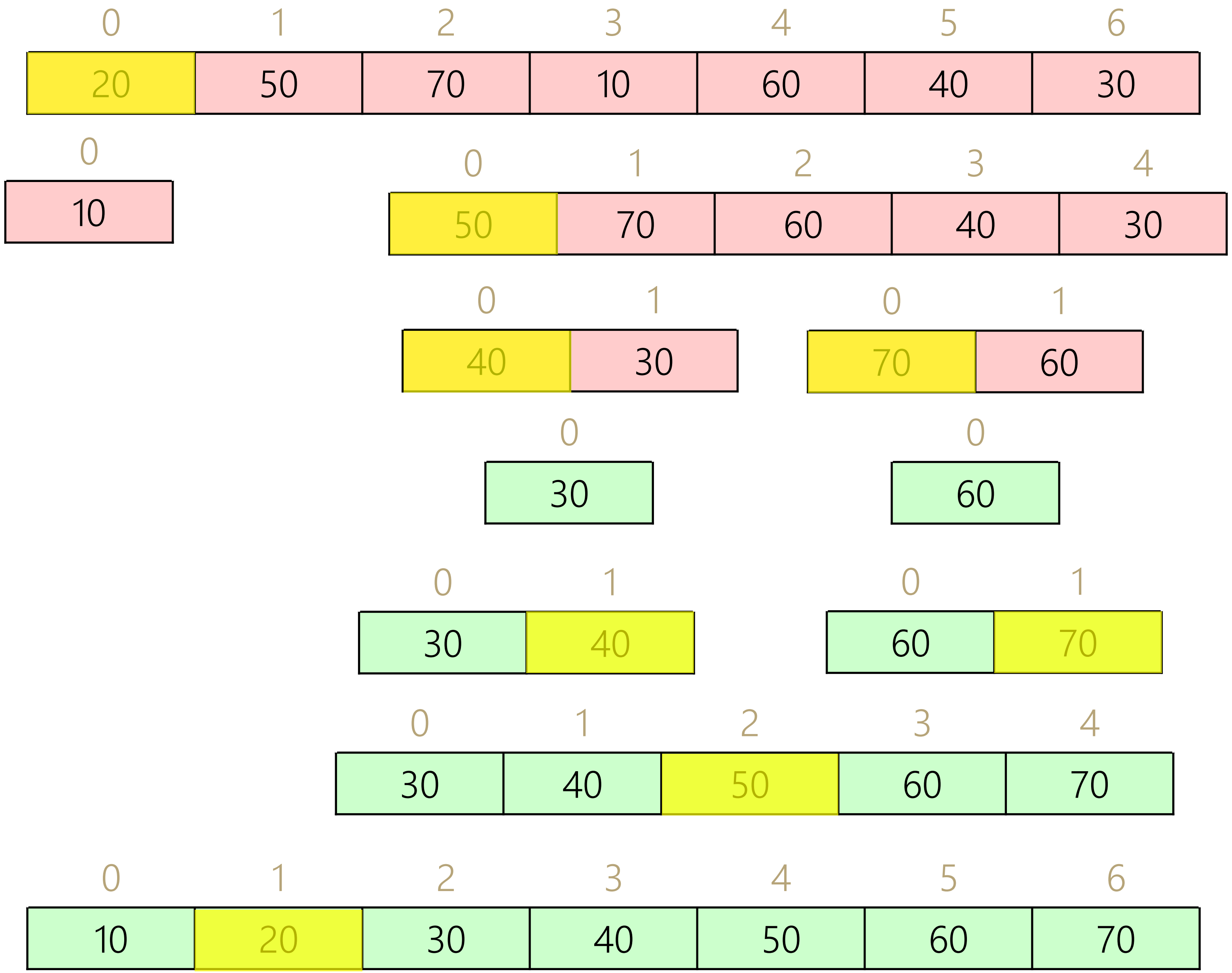
0	1	2	3	4	5	6	7	8	9
8	3	5	6	2	1	4	7	9	10

i, j met. $A[i]$ is larger than the pivot, so it belongs on the right, but $A[i-1]$ belongs on the left. Swap pivot and $A[i-1]$.

0	1	2	3	4	5	6	7	8	9
7	3	5	6	2	1	4	8	9	10

Quick Sort

https://www.youtube.com/watch?v=ywWBy6J5gz8



Quick Sort Analysis (Take 1)

What is the best case and worst case for a pivot?

- Best case:
- Worst case:

Recurrences:

Best:

Worst:

Running times:

- Best:
- Worst:

Quick Sort Analysis (Take 1)

What is the best case and worst case for a pivot?

- Best case: Picking the median
- Worst case: Picking the smallest or largest element

Recurrences:

Best:

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + c_1 n & \text{if } n \geq 2 \\ c_2 & \text{otherwise} \end{cases}$$

Worst:

$$T(n) = \begin{cases} T(n-1) + c_1 n & \text{if } n \geq 2 \\ c_2 & \text{otherwise} \end{cases}$$

Running times:

- Best: $O(n \log n)$
- Worst: $O(n^2)$

Choosing a Pivot

Average case behavior depends on a good pivot.

Pivot ideas:

Just take the first element

- Simple. But an already sorted (or reversed) list will give you a bad time.

Pick an element uniformly at random.

- $O(n \log n)$ running time with probability at least $1 - 1/n^2$.
- Regardless of input!
- Probably too slow in practice :(

Find the actual median!

- You can actually do this in linear time
- Definitely not efficient in practice

Choosing a Pivot

Median of Three

- Take the median of the first, last, and midpoint as the pivot.
- Fast!
- Unlikely to get bad behavior (but definitely still possible)
- Reasonable default choice.

Quick Sort Analysis

Running Time:

- Worst $O(n^2)$
- Best $O(n \log n)$
- Average $O(n \log n)$ (not responsible for the proof, talk to Robbie if you're curious)

In place: Yes

Stable: No.

Lower Bound

We keep hitting $O(n \log n)$ in the worst case.

Can we do better?

Or is this $O(n \log n)$ pattern a fundamental barrier?

Without more information about our data set, we can do no better.

Comparison Sorting Lower Bound

Any sorting algorithm which only interacts with its input by comparing elements must take $\Omega(\mathrm{n} \log \mathrm{n})$ time.

We'll prove this theorem on Friday!

Non-Comparison Sorts

Avoiding the Lower Bound

Can we avoid using comparisons?

In general, probably not.

- If you're trying to write the most general code, definitely not.

But what if we know that all of our data points are small integers?

Bucket Sort (aka Bin Sort)

4	3	1	2	1	1	2	3	4	2

1	2	3	4
3	3	2	2

1	1	1	2	2	2	3	3	4	4

Bucket Sort

Running time?

If we have m possible values and an array of size n?
$O(m+n)$.

How are we beating the lower bound?

When we place an element, we implicitly compare it to all the others in $O(1)$ time!

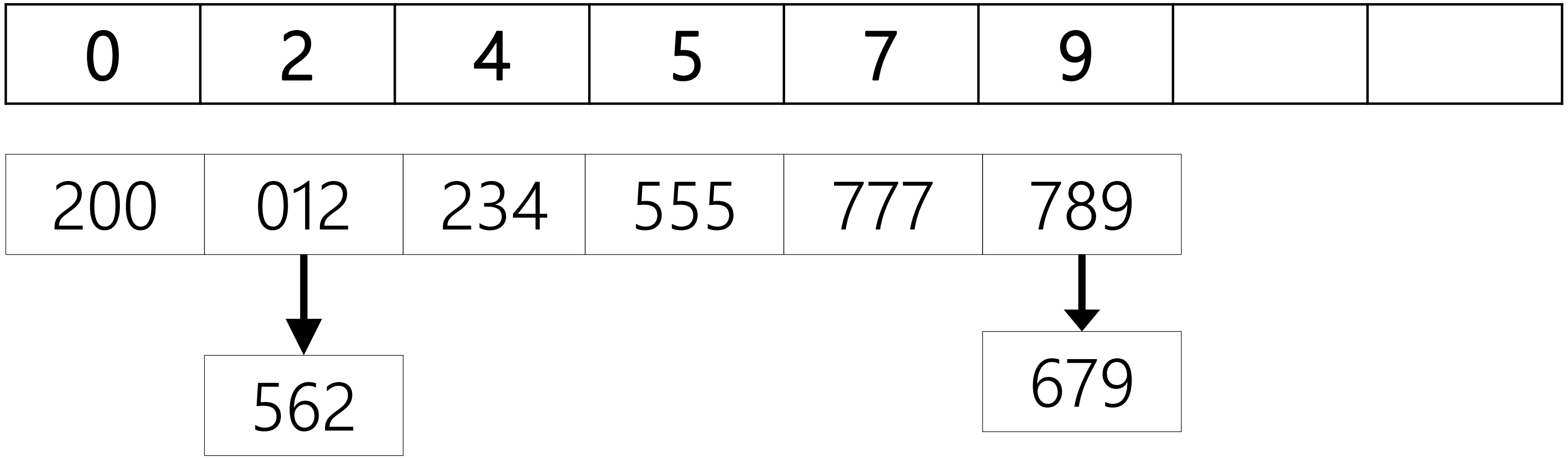
Radix Sort

For each digit (starting at the ones place)

- Run a “bucket sort” with respect to that digit
- Keep the sort stable!

Radix Sort: Ones Place

012	234	789	555	679	200	777	562



Radix Sort by Ones Place
Keep Stable: Later elements go at <u>end</u> of linked list

200	012	562	234	555	777	789	679

Copy back in new order.

Radix Sort: Tens Place

200	012	562	234	555	777	789	679

0	1	2	3	5	6	7	8
200	012		234	555	562	777	789
						679	

Radix Sort by Tens Place
Keep Stable: Later elements go at **end** of linked list

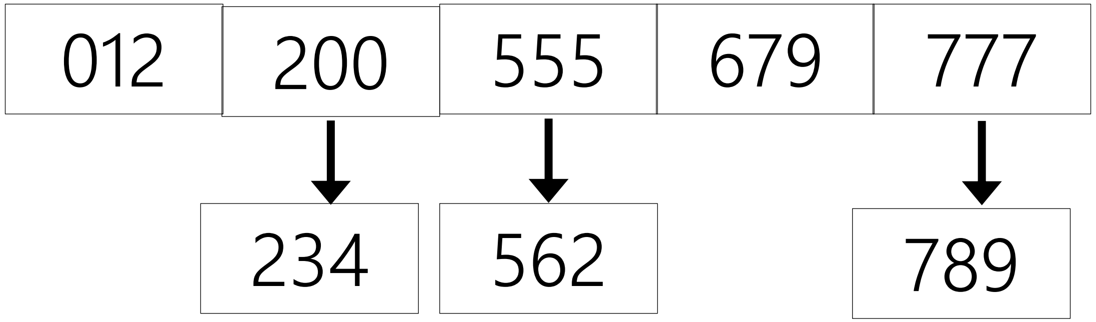
200	012	234	555	562	777	679	789

Copy back in new order. Sorted by tens, then ones.

Radix Sort: Hundreds Place

200	012	234	555	562	777	679	789

0	2	5	6	7			



012	200	234	555	562	679	777	789

Radix Sort

Key idea: by keeping the sorts stable, when we sort by the hundreds place, ties are broken by tens place (then by ones place).

Running time? $O((n+r)d)$

Where d is number of digits in each entry,

r is the radix, i.e. the base of the number system.

How do we avoid the lower bound?

- Same way as bucket sort, we implicitly get free comparison information when we insert into a bucket.

Radix Sort

When can you use it?

ints and strings. As long as they aren’t too large.

Decision Trees

Decision Trees

Suppose we have a size 3 array to sort.

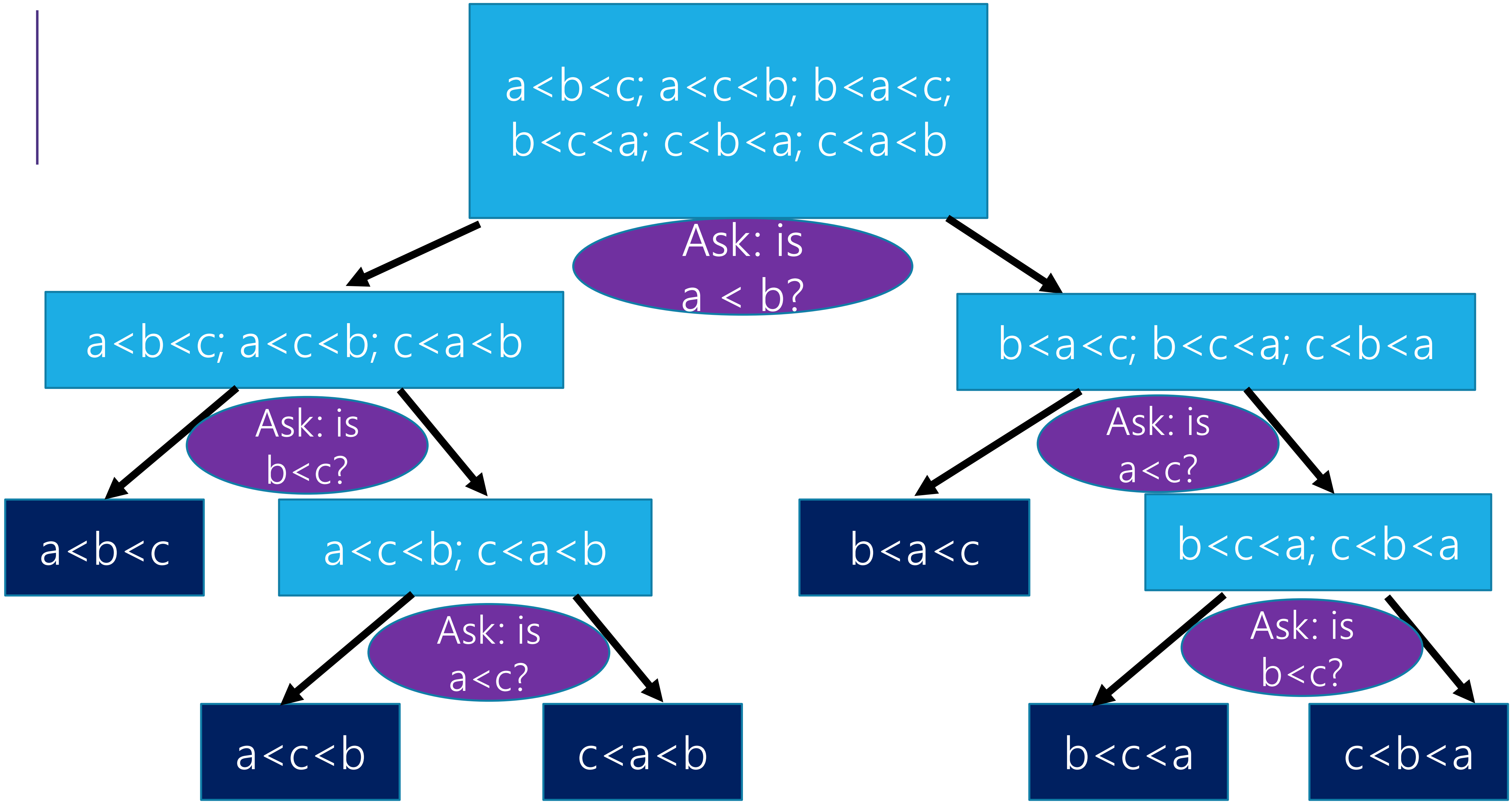
We will figure out which array to return by comparing elements.

When we know what the correct order is, we'll return that array.

In our real algorithm, we're probably moving things around to make the code understandable.

Don't worry about that for the proof.

Whatever tricks we're using to remember what's big and small, it doesn't matter if we don't look first!


a<b<c; a<c<b; b<a<c;
b<c<a; c<b<a; c<a<b
Ask: is
a < b?
a<b<c; a<c<b; c<a<b
b<a<c; b<c<a; c<b<a
Ask: is
b<c?
Ask: is
a<c?
a<b<c
a<c<b; c<a<b
b<a<c
b<c<a; c<b<a
Ask: is
a<c?
Ask: is
b<c?
a<c<b
c<a<b
b<c<a
c<b<a

Complete the Proof

How many operations can we guarantee in the worst case?

How tall is the tree if the array is length n?

What's the simplified $\Omega()$?

Complete the Proof

How many operations can we guarantee in the worst case?
- Equal to the height of the tree.

How tall is the tree if the array is length n?
- One of the children has at least half of the possible inputs.
- What level can we guarantee has an internal node? $\log_2(n!)$

What's the simplified $\Omega()$?

$$\log_2(n!) = \log_2(n) + \log_2(n-1) + \log_2(n-2) + \cdots + \log_2(1)$$

$$\geq \log_2\left(\frac{n}{2}\right) + \log_2\left(\frac{n}{2}\right) + \cdots + \log_2\left(\frac{n}{2}\right) \text{ (only } n/2 \text{ copies)}$$

$$\geq \frac{n}{2}\log_2\left(\frac{n}{2}\right) = n/2(\log_2(n) - 1) = \Omega(n \log n)$$

Takeaways

A tight lower bound like this is **very** rare.

This proof had to argue about every possible algorithm

- that's really hard to do.

We can't come up with a more clever recurrence to sort faster.

This theorem actually says things about data structures, too!

- You'll prove it yourselves in an upcoming exercise.

Unless we make some assumptions about our input.

And get information without doing the comparisons.

Wrap-up

Summary

You have a bunch of data. How do you sort it?

Honestly…use your language's default implementation

- It's been carefully optimized.

Unless you really know something about your data, or the situation your in

- Not a lot of extra memory? Use an in place sort.
- Want to sort repeatedly to break ties? Use a stable sort.
- Know your data all falls into a small range? Bucket (or maybe Radix) sort.