CSE 332: Data Structures & Parallelism

Lecture 25: Complexity Classes and Tractability

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Autumn 2025

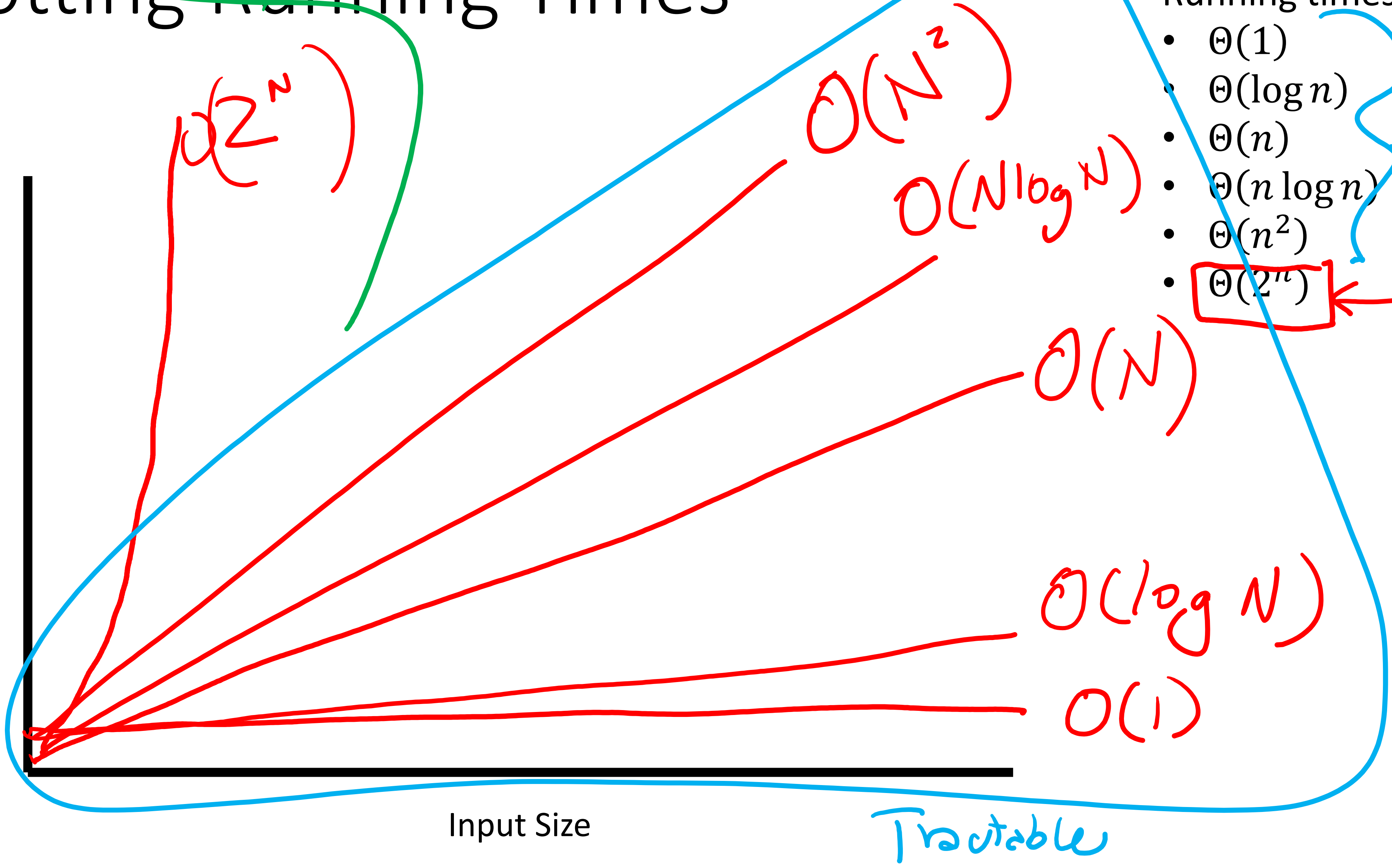
(Slides adapted from Nathan Brunelle)

Administrative

- EX10 – Concurrency, Due TONIGHT **Mon** Nov 24
- EX11 – MSTs, programming, Due **Mon** Dec 1
- EX12 – P/NP, last exercise! Due Fri Dec 5 (last day of class)
- Lecture on Wed 11/26
  - 12:30pm Optional TA Guest lecture: Tries & More Parallelism
  - 3:30pm Class Cancelled
- Resources!
  - **Conceptual Office Hours**: 11:30 Tues (Connor) and 11:30 Wed (Samarth) both in CSE1 006. A space to ask about **course content and topics only** *as opposed to direct help with exercises*.
  - 1-on-1 Meeting Requests - Request a meeting with a staff member if you cannot make it to regularly scheduled office hours, or feel like you have an issue that requires a more in depth discussion.

Intractable

Plotting Running Times



Running times we've seen:

- $\Theta(1)$
- $\Theta(\log n)$
- $\Theta(n)$
- $\Theta(n \log n)$
- $\Theta(n^2)$
- $\boxed{\Theta(2^n)}$

Polynomial

Exponential

Examining Running Times

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

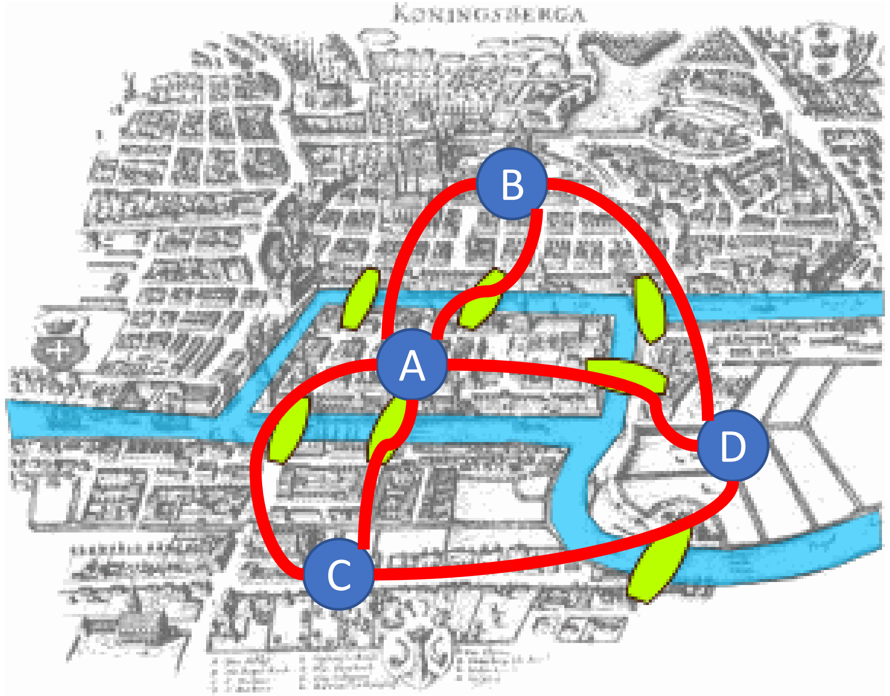
	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	~~32 years~~	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Tractability

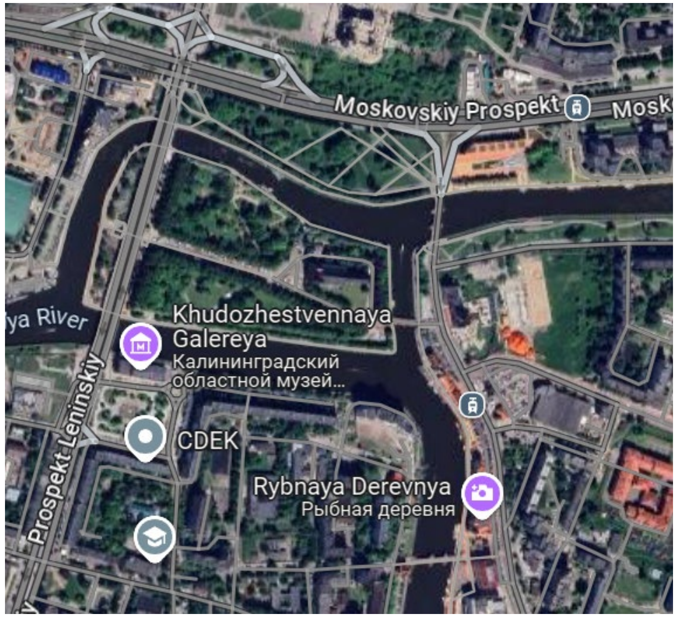
- Tractable:
  - Feasible to solve in the "real world"
- Intractable:
  - Infeasible to solve in the "real world"
- Whether a problem is considered "tractable" or "intractable" depends on the use case
  - For machine learning, big data, etc. tractable might mean $O(n)$ or even $O(\log n)$
  - For most applications it's more like $O(n^3)$ or $O(n^2)$
- A strange pattern:
  - Most "natural" problems are either done in small-degree polynomial (e.g. n^2) or else exponential time (e.g. 2^n)
  - It's rare to have problems which require a running time of n^5, for example

7 Bridges of Königsberg

In 1736



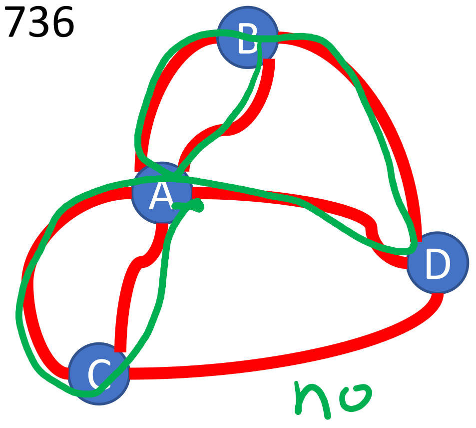
In 2025



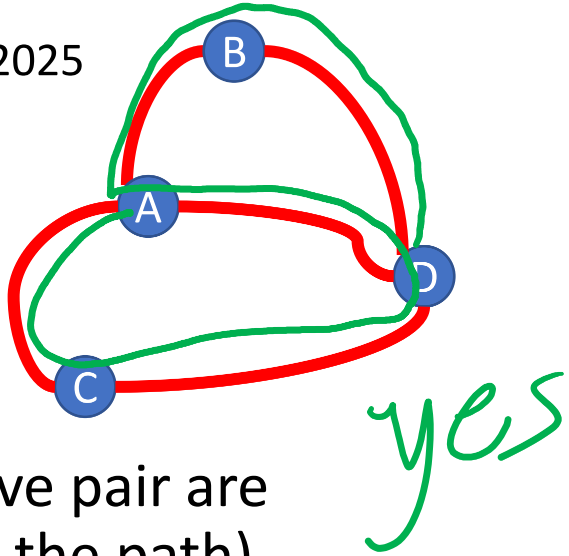
The Pregel River runs through the city of Koenigsberg, creating 2 islands. Among these 2 islands and the 2 sides of the river, there are 7 bridges. Is there any path starting at one landmass which crosses each bridge exactly once?

Euler Path Problem

In 1736

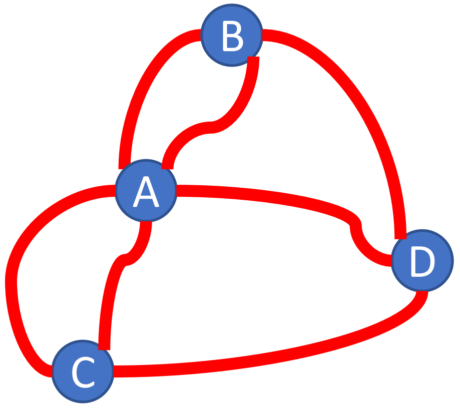


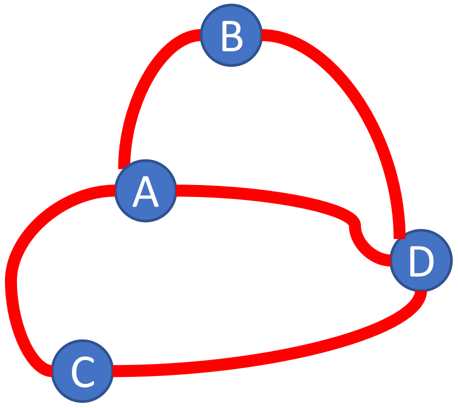
In 2025

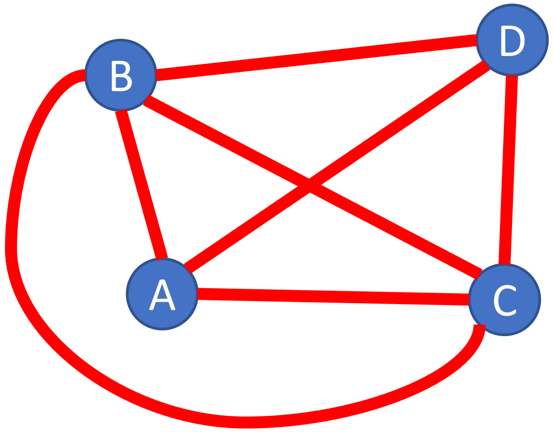


- Path:
  - A sequence of nodes $v_1, v_2, \ldots$ such that for every consecutive pair are connected by an edge (i.e. (v_i, v_{i+1}) is an edge for each i in the path)
- Euler Path:
  - A path such that every edge in the graph appears exactly once
    - If the graph is not simple then some pairs need to appear multiple times!
- Euler path problem:
  - Given an undirected graph $G = (V, E)$, does there exist an Euler path for G?

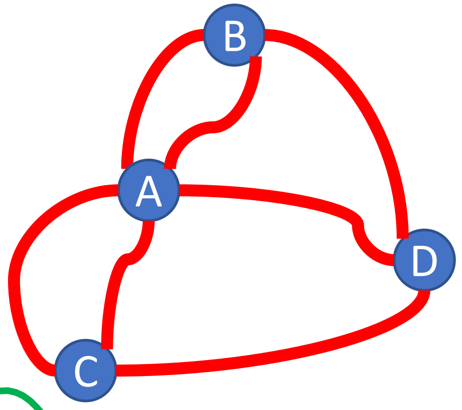
Examples: Which of the graphs below have an Euler path?



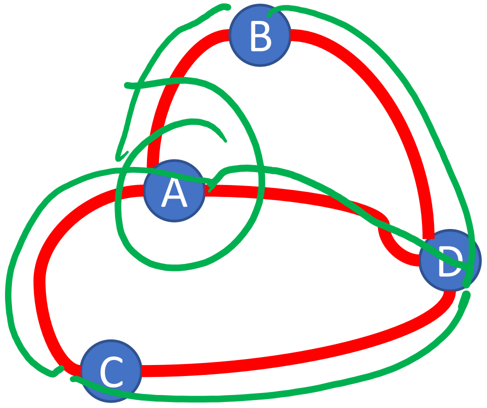




Examples: Which of the graphs below have an Euler path? (Answers)

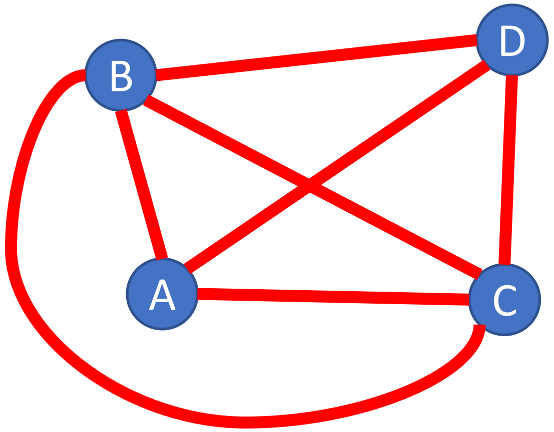


No Euler path exists!



Euler path exists!
A, B, D, A, C, D

Yes

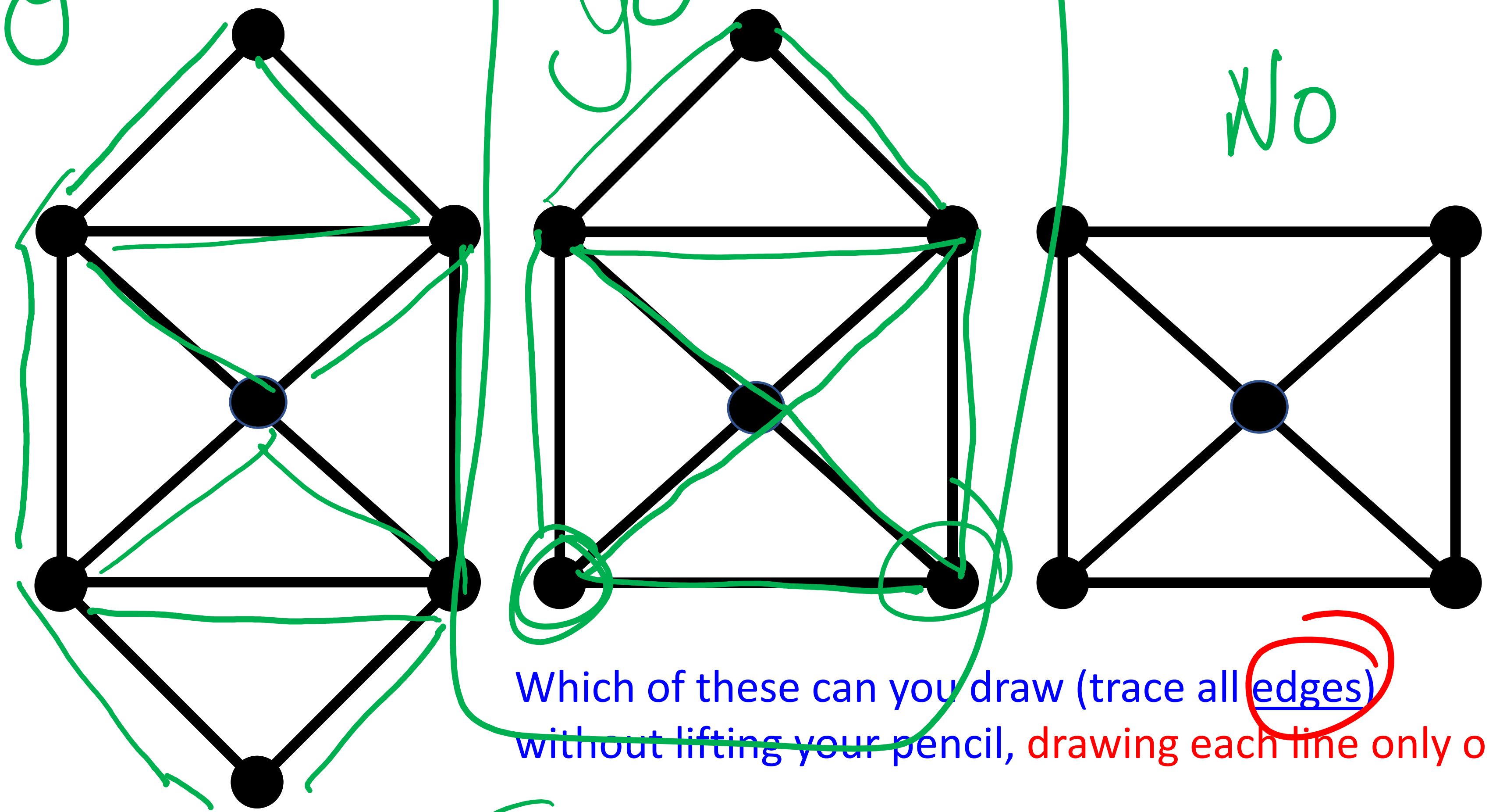


Euler path exists!
A, B, C, D, A, C, B, D

Which of the graphs below have an Euler path?

Edges

yes yes No

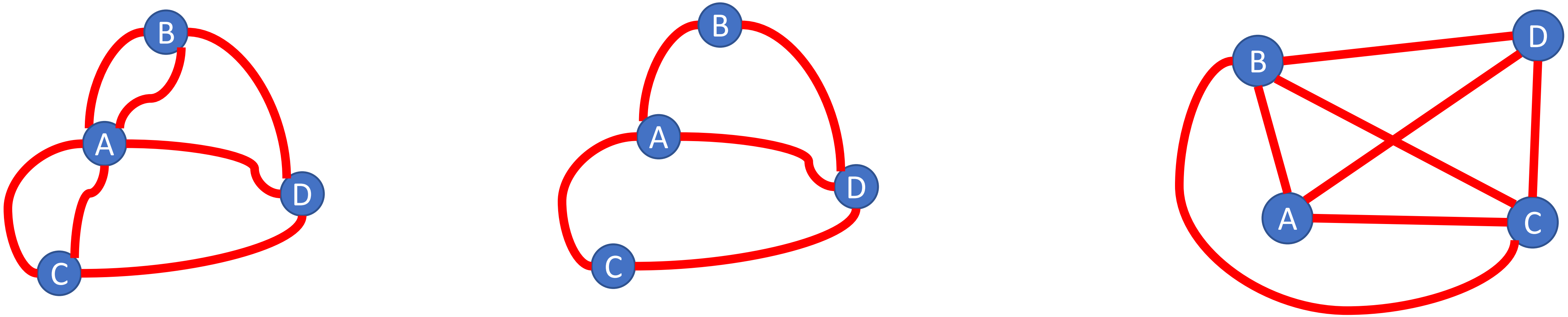


Which of these can you draw (trace all edges) without lifting your pencil, drawing each line only once?

Can you start and end at the same point?

Euler's Theorem

- A graph has an Euler Path if and only if it is connected and has exactly 0 or 2 nodes with odd degree.

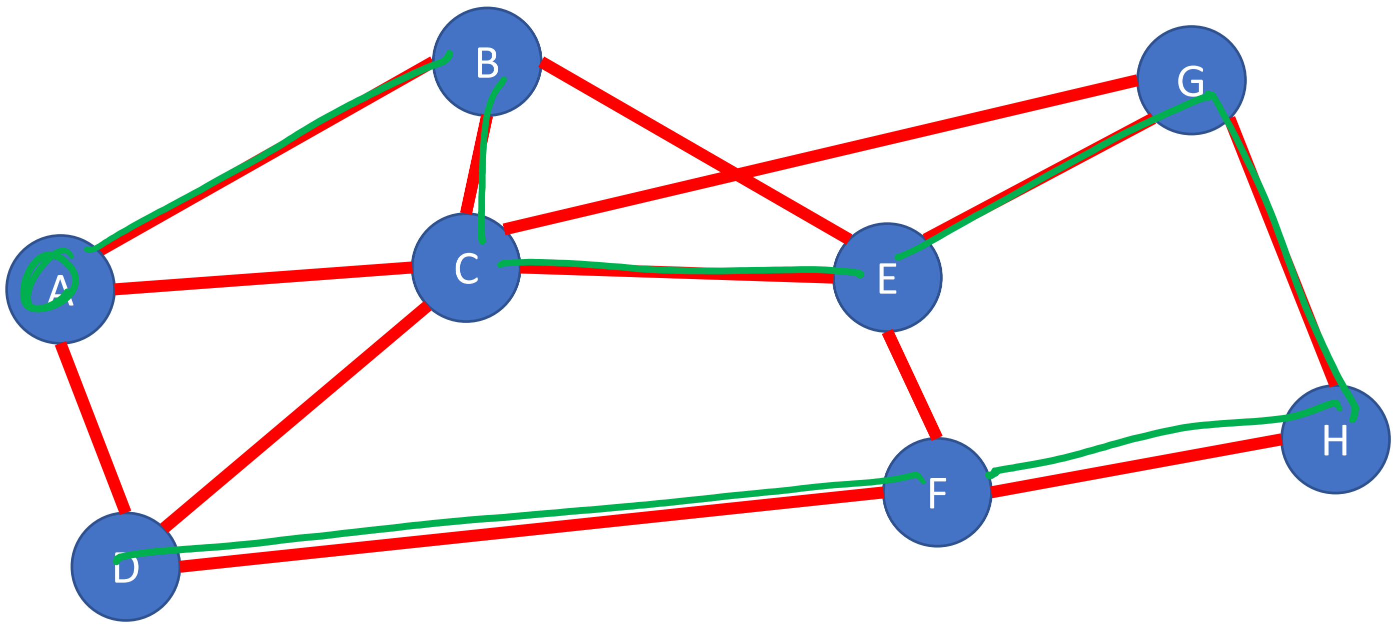


Algorithm for the Euler Path Problem

- Given an undirected graph $G = (V, E)$, does there exist an Euler path for G?
- Algorithm:
  - Check if the graph is connected
  - Check the degree of each node
  - If the number of nodes with odd degree is 0 or 2, return true
  - Otherwise return false
- Running time? $O(E+V)$

A Seemingly Similar Problem

- Hamiltonian Path:
  - A path that includes every node in the graph exactly once
- Hamiltonian Path Problem:
  - Given a graph $G = (V, E)$, does that graph have a Hamiltonian Path?



True!
A, B, C, E, G, H, F, D

Algorithms for the Hamiltonian Path Problem

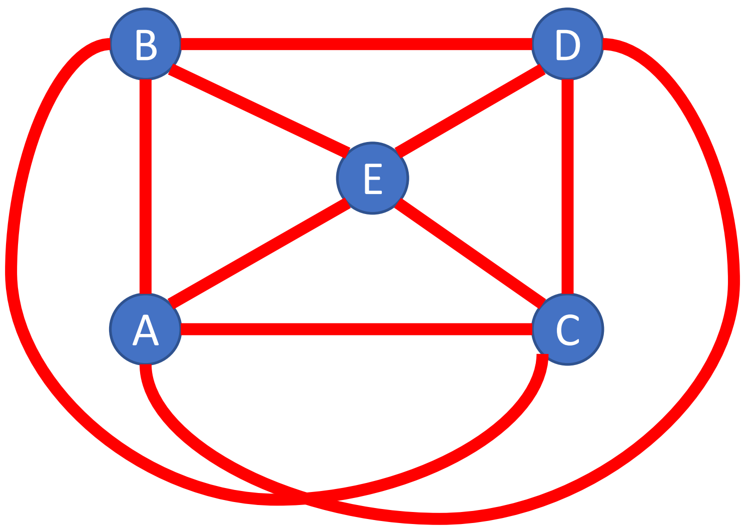
- Option 1:
    - Explore all possible simple paths through the graph
    - Check to see if any of those are length V
- Option 2:
    - Write down every sequence of nodes
    - Check to see if any of those are a path
- Both options are examples of an **Exhaustive Search ("Brute Force") algorithm**

Option 2: List all sequences, look for a path

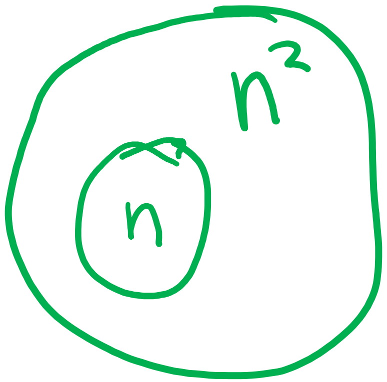
- Running time:
  - $G = (V, E)$
  - Number of permutations of V is $|V|!$
    - $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$
  - How does $n!$ compare with 2^n?
    - $n! \in \Omega(2^n)$
  - Exponential running time!

Option 1: Explore all simple paths, check for one of length V

- Running time:
  - $G = (V, E)$
  - Number of paths
    - Pick a first node ($|V|$ choices)
    - Pick a neighbor (up to $|V| - 1$ choices)
    - Pick a neighbor (up to $|V| - 2$ choices)
    - …. Repeat $|V| - 1$ total times
    - Overall: $|V|!$ paths
  - Exponential running time



Complexity Classes



- A Complexity Class is a set of problems (e.g. sorting, Euler path, Hamiltonian path)
  - The problems included in a complexity class are those whose most efficient algorithm has a specific upper bound on its running time (or memory use, or…)
- Examples:
  - The set of all problems that can be solved by an algorithm with running time $O(n)$
    - Contains: Finding the minimum of a list, finding the maximum of a list, buildheap, summing a list, etc.
  - The set of all problems that can be solved by an algorithm with running time $O(n^2)$
    - Contains: everything above as well as comparison based sorting, Euler path
  - The set of all problems that can be solved by an algorithm with running time $O(n!)$
    - Contains: everything we've seen in this class so far

Complexity Classes and Tractability

- To explore what problems are and are not tractable, we give some complexity classes special names:
- Complexity Class P:
  - Stands for "Polynomial"
  - The set of problems which have an algorithm whose running time is $O(n^p)$ for some choice of $p \in \mathbb{R}$.
  - We say all problems belonging to P are "Tractable"
- Complexity Class EXP:
  - Stands for "Exponential"
  - The set of problems which have an algorithm whose running time is $O\left(2^{n^p}\right)$ for some choice of $p \in \mathbb{R}$
  - We say all problems belonging to $EXP - P$ are "Intractable"
    - Disclaimer: Really it's all problems outside of P, and there are problems which do not belong to EXP, but we're not going to worry about those in this class

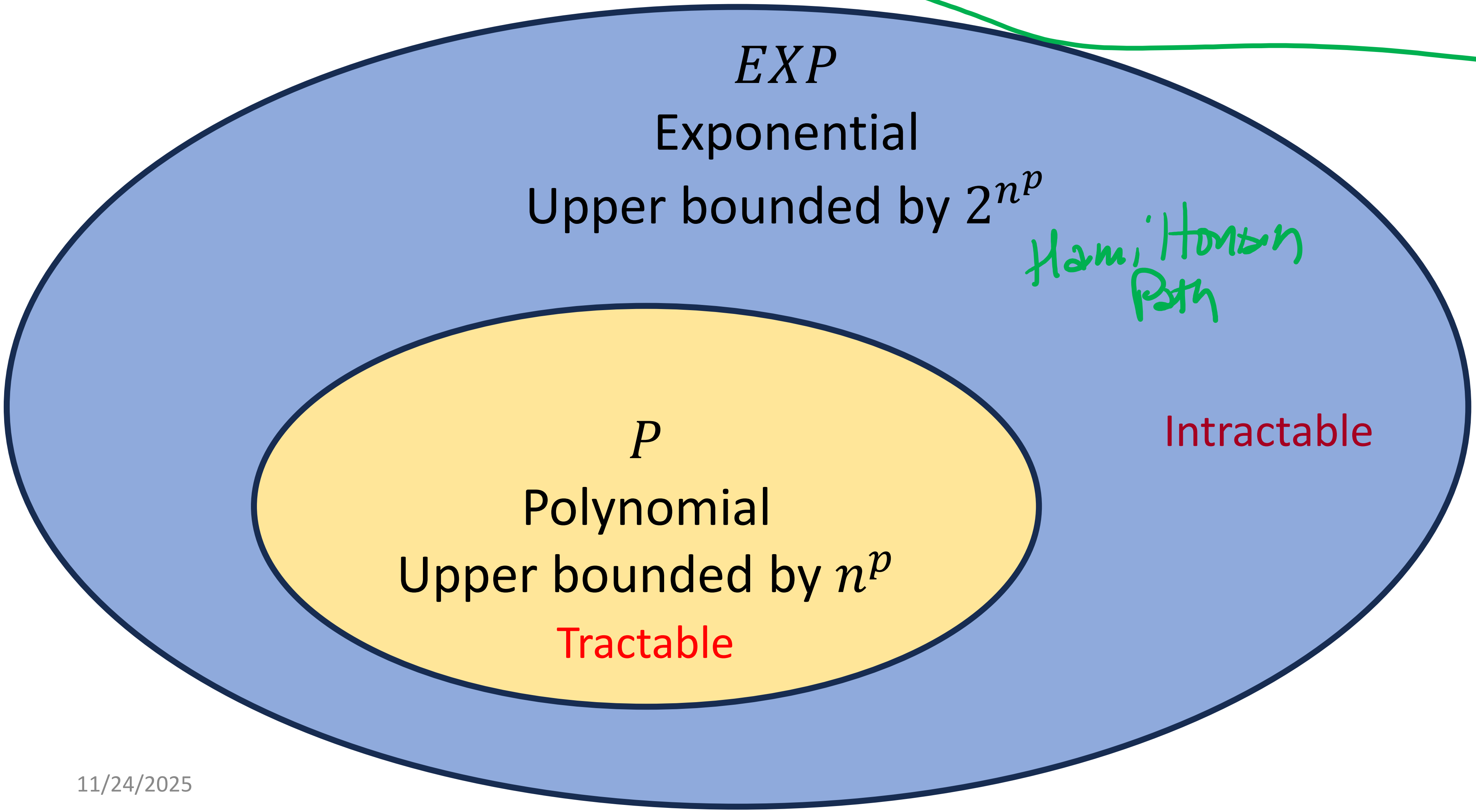
EXP and P

Important!

$P \subset EXP$

Every problem within P is also within EXP

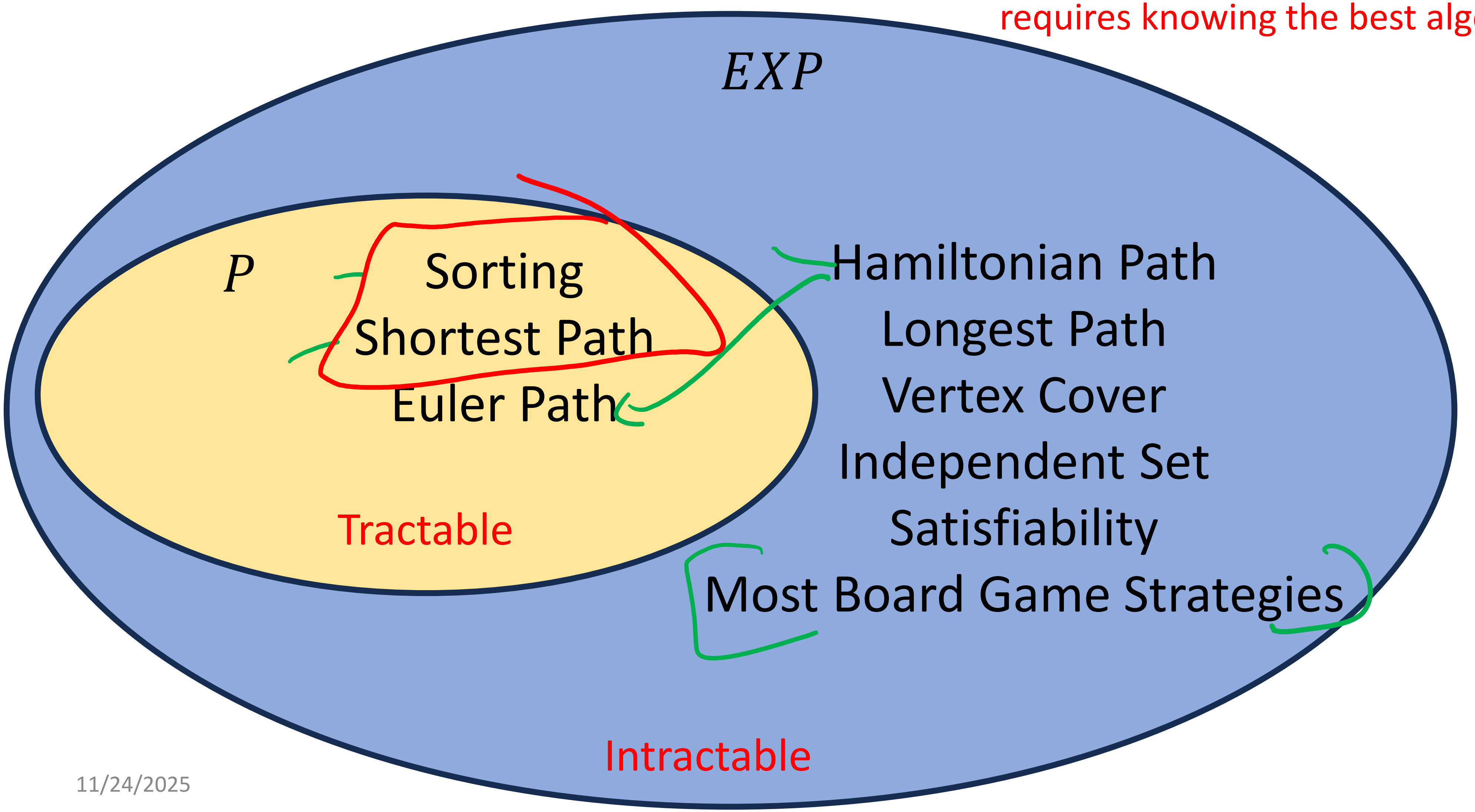
The intractable ones are the problems within EXP but NOT P



Members

Important!

Some of the problems we've listed in EXP could also be members of P. Since membership is determined by a problem's *most* efficient algorithm, knowing if a problem belongs to P requires knowing the best algorithm possible!



Studying Complexity and Tractability

- Organizing problems into complexity classes helps us to reason more carefully and flexibly about tractability
- The goal for each problem is to either
  - Find an efficient algorithm if it exists
    - i.e. show it belongs to P
  - Prove that no efficient algorithm exists
    - i.e. show it does not belong to P
- Complexity classes allow us to reason about sets of problems at a time, rather than each problem individually
  - If we can find more precise classes to organize problems into, we might be able to draw conclusions about the entire class
  - It may be easier to show a problem belongs to class C than to P, so it may help to show that $C \subseteq P$

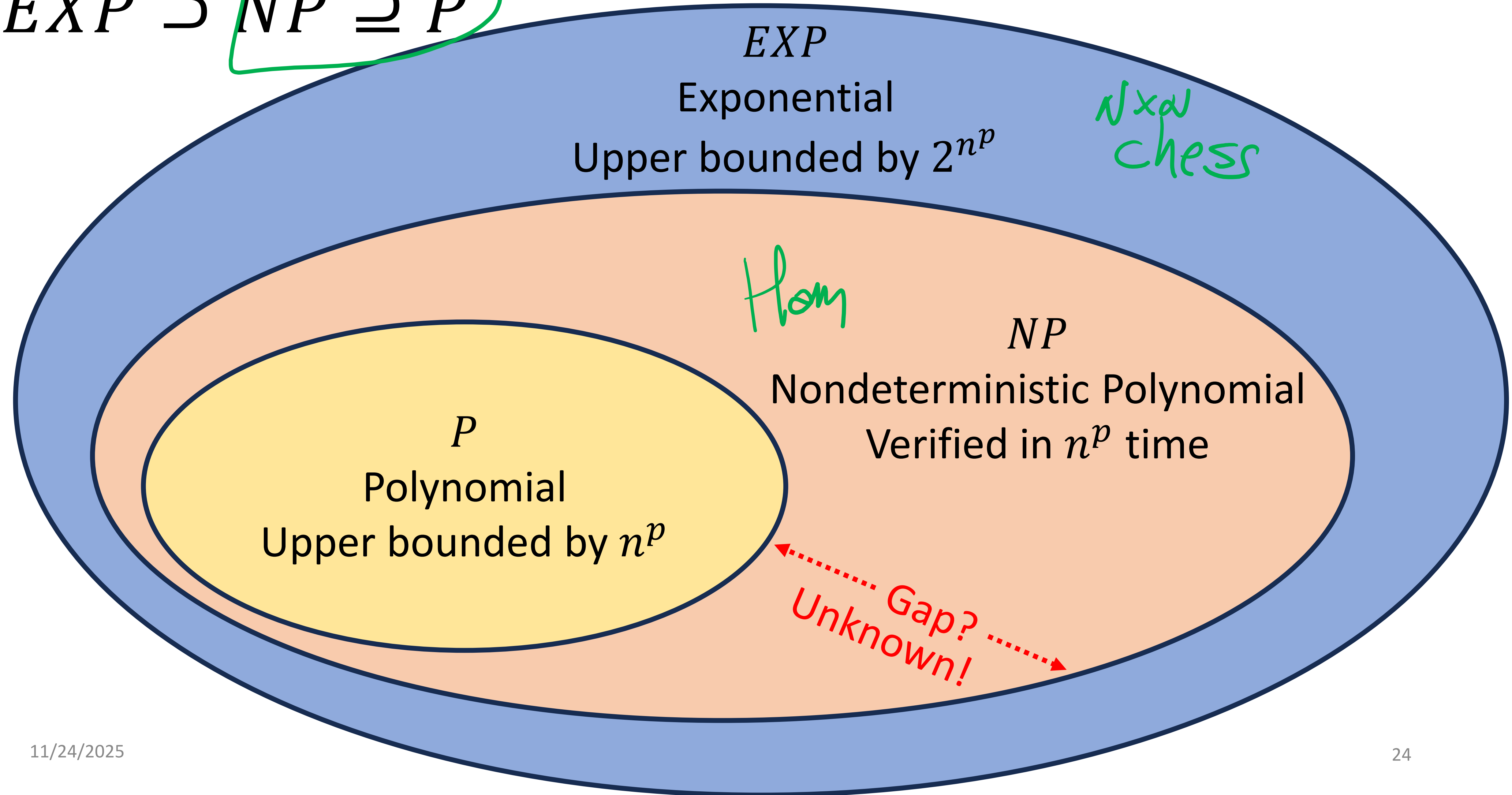
Some problems in EXP seem "easier"

- There are some problems that we do not have polynomial time algorithms to solve, but provided answers are easy to check
- Hamiltonian Path:
  - It's "hard" to look at a graph and determine whether it has a Hamiltonian Path
  - It's "easy" to look at a graph and a candidate path together and determine whether THAT path is a Hamiltonian Path
    - It's easy to **verify** whether a given path is a Hamiltonian path

Class NP

- NP
  - The set of problems for which a candidate solution can be **verified** in polynomial time
  - Stands for "Non-deterministic Polynomial"
    - Corresponds to algorithms that can guess a solution (if it exists), that solution is then verified to be correct in polynomial time
    - Can also think of as allowing a special operation that allows the algorithm to magically guess the right choice at each step of an exhaustive search
- $P \subseteq NP$
  - Why?

$$EXP \supset NP \supseteq P$$



Solving and Verifying Hamiltonian Path

- Give an algorithm to solve Hamiltonian Path
  - Input: $G = (V, E)$
  - Output: True if G has a Hamiltonian Path
  - Algorithm: Check whether each permutation of V is a path.
    - Running time: $|V|!$, so does not show whether it belongs to P
- Give an algorithm to verify Hamiltonian Path
  - Input: $G = (V, E)$ and a sequence of nodes
  - Output: True if that sequence of nodes is a Hamiltonian Path
  - Algorithm:
    - Check that each node appears in the sequence exactly once
    - Check that the sequence is a path
    - Running time: $O(V \cdot E)$, so it belongs to NP

$EXP \supset NP \supseteq P$

$P = NP$ or $P \subset NP$