Dictionary (Map) ADT

• Contents:
  • Sets of key+value pairs
  • Keys must be comparable

• Operations:
  • insert(key, value)
    • Adds the (key,value) pair into the dictionary
    • If the key already has a value, overwrite the old value
      • Consequence: Keys cannot be repeated
  • find(key)
    • Returns the value associated with the given key
  • delete(key)
    • Remove the key (and its associated value)
Less Naïve attempts

• Binary Search Trees (Friday)
• Tries (Project)
• AVL Trees (Today)
• B-Trees (this week)
• HashTables (next week)
• Red-Black Trees (not included in this course)
• Splay Trees (not included in this course)
## Dictionary Data Structures

<table>
<thead>
<tr>
<th>Data Structure</th>
<th>Time to insert</th>
<th>Time to find</th>
<th>Time to delete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unsorted Array</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Unsorted Linked List</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Sorted Array</td>
<td>$\Theta(n)$</td>
<td>$\Theta(\log n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Sorted Linked List</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Binary Search Tree</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>AVL Tree</td>
<td>$\Theta(\log n)$</td>
<td>$\Theta(\log n)$</td>
<td>$\Theta(\log n)$</td>
</tr>
</tbody>
</table>
Binary Search Tree

- **Binary Tree**
  - Definition:
    - Every node has at most 2 children

- **Order Property**
  - All keys in the left subtree are smaller than the root
  - All keys in the right subtree are larger than the root
  - Apply recursively

- **Why?**
  - Makes searching quicker
    - Worst case: tree’s height
Find Operation (recursive)

find(key, root)
    if (root == Null)
        return Null;
    if (key == root.key)
        return root.value;
    if (key < root.key)
        return find(key, root.left);
    if (key > root.key)
        return find(key, root.right);
    return Null;
Find Operation (iterative)

find(key, root) {
    while (root != Null && key != root.key) {
        if (key < root.key) {
            root = root.left;
        } else if (key > root.key) {
            root = root.right;
        }
    }
    if (root == Null) {
        return Null;
    }
    return root.value;
}
Insert Operation (iterative)

```java
insert(key, value, root){
    if (root == Null){ this.root = new Node(key, value); }
    parent = Null;
    while (root != Null && key != root.key){
        parent = root;
        if (key < root.key){ root = root.left; }
        else if (key > root.key){ root = root.right; }
    }
    if (root != Null){ root.value = value; }
    else if (key < parent.key){ parent.left = new Node(key, value); }
    else{ parent.right = new Node (key, value); }
}
```

Note: Insert happens only at the leaves!
Delete Operation (iterative)

```plaintext
delete(key, root){
    while (root != Null && key != root.key){
        if (key < root.key){ root = root.left; }
        else if (key > root.key){ root = root.right; }
    }
    if (root == Null){ return; }
    // Now root is the node to delete, what happens next?
}
```
Delete – 3 Cases

• 0 Children (i.e. it’s a leaf)
  - Point to null

• 1 Child
  - Replace node with the max on L or min on R
Finding the Max and Min

• Max of a BST:
  • Right-most Thing

• Min of a BST:
  • Left-most Thing

maxNode(root){
    if (root == Null){ return Null; }
    while (root.right != Null){
        root = root.right;
    }
    return root;
}

minNode(root){
    if (root == Null){ return Null; }
    while (root.left != Null){
        root = root.left;
    }
    return root;
}
Delete Operation (iterative)

def delete(key, root):
    while (root != Null && key != root.key):
        if (key < root.key){ root = root.left; }
        else if (key > root.key){ root = root.right; }
    
    if (root == Null){ return; }
    if (root has no children){
        make parent point to Null instead;
    }
    if (root has one child){
        make parent point to that child instead;
    }
    if (root has two children){
        make parent point to either the max from the left or min from the right
    }
Improving the worst case

• How can we get a better worst case running time?
“Balanced” Binary Search Trees

• We get better running times by having “shorter” trees
• Trees get tall due to them being “sparse” (many one-child nodes)
• Idea: modify how we insert/delete to keep the tree more “full”
Idea 1: Both Subtrees of Root have same # Nodes
Idea 2: Both Subtrees of Root have same height
Idea 3: Both Subtrees of every Node have same # Nodes
Idea 4: Both Subtrees of every Node have same height
AVL Tree

• A Binary Search tree that maintains that the left and right subtrees of every node have heights that differ by at most one.
  • height of left subtree and height of right subtree off by at most 1
  • Not too weak (ensures trees are short)
  • Not too strong (works for any number of nodes)

• Idea of AVL Tree:
  • When you insert/delete nodes, if tree is “out of balance” then modify the tree
  • Modification = “rotation”
Is it an AVL Tree?

A: No
B: Yes
Using AVL Trees

• Each node has:
  • Key
  • Value
  • Height
  • Left child
  • Right child
Inserting into an AVL Tree

• Starts out the same way as BST:
  • “Find” where the new node should go
  • Put it in the right place (it will be a leaf)

• Next check the balance
  • If the tree is still balanced, you’re done!
  • Otherwise we need to do rotations
Insert Example

```
10
  9
  3
  1
  2
  0
  6
  11
  7
  16
```
Insert Example

-1

3

9

11

16

1

2

6

7

0

Rotation
Not Balanced!

Solution: rotate the whole tree to the right

Height = 3
Height = 1
Balanced!
Right Rotation

- Make the left child the new root
- Make the old root the right child of the new
- Make the new root’s right subtree the old root’s left subtree
Insert Example

20

Diagram of a tree with nodes labeled 0 to 20.
Not Balanced!

Solution: rotate the deepest imbalance to the left
Balanced!
Left Rotation

- Make the right child the new root
- Make the old root the left child of the new
- Make the new root’s left subtree the old root’s right subtree
Insertion Story So Far

• After insertion, update the heights of the node’s ancestors
• Check for imbalance
• If there’s imbalance then at the deepest root of imbalance:
  • If the left subtree was deeper then rotate right
  • If the right subtree was deeper then rotate left

This is incomplete! There are some cases where this doesn’t work!
Insertion Story So Far

- After insertion, update the heights of the node’s ancestors
- Check for imbalance
- If there’s imbalance then at the deepest root of imbalance:
  - Case LL: If we inserted in the left subtree of the left child then rotate right
  - Case RR: If we inserted in the right subtree of the right child then rotate left
  - Case LR: If we inserted into the right subtree of the left child then ???
  - Case RL: If we inserted into the left subtree of the right child then ???

Cases LR and RL require 2 rotations!
Case LR

- From “root” of the deepest imbalance:
  - Rotate left at the left child
  - Rotate right at the root
Case LR in General

- Imbalance caused by inserting in the left child’s right subtree
- Rotate left at the left child
- Rotate right at the imbalanced node
Case RL in General

- Imbalance caused by inserting in the right child’s left subtree
- Rotate right at the right child
- Rotate left at the imbalanced node
Insert Summary

• After a BST insertion, update the heights of the node’s ancestors
• Check for imbalance

• If there’s imbalance then at the deepest root of imbalance:
  • Case LL: If we inserted in the left subtree of the left child then: rotate right
  • Case RR: If we inserted in the right subtree of the right child then: rotate left
  • Case LR: If we inserted into the right subtree of the left child then: rotate left at the left child and then rotate right at the root
  • Case RL: If we inserted into the left subtree of the right child then: rotate right at the right child and then rotate left at the root