CSE 332 Winter 2024
Lecture 4: Algorithm Analysis and Priority Queues

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Warm Up

Give the worst case running time for the following code

doSomething(List myList){
    n = myList.size();
    x = 0;
    for (i=0; i < n; i++)
    {
        for (j=0; j < i; j++)
        {
            x++;
        }
    }
    return x;
}

Questions to ask:
• What are the units of the input size?
• What are the operations we’re counting?
• For each line:
  • How many times will it run?
  • How long does it take to run?
  • Does this change with the input size?
\[1 + 2 + 3 + 4 + 5 + \cdots + n + \frac{n}{2}\]

\[n + 1 + n + 2 + n + 3 + \cdots + n + 1\]

\[\frac{n(n+1)}{2}\]
Goals for Algorithm Analysis

• Identify a function which maps the algorithm’s input size to a measure of resources used.
  • Domain of the function: sizes of the input
    • Number of characters in a string, number of items in a list, number of pixels in an image
  • Codomain of the function: counts of resources used
    • Number of times the algorithm adds two numbers together, number times the algorithm does a > or < comparison, maximum number of bytes of memory the algorithm uses at any time

• Important note: Make sure you know the “units” of your domain and codomain!
  • Domain = inputs to the function
  • Codomain = outputs to the function
Comparing
Comparing Running Times

• Suppose I have these algorithms, all of which have the same input/output behavior:
  • Algorithm A’s worst case running time is $10n + 900$
  • Algorithm B’s worst case running time is $100n – 50$
  • Algorithm C’s worst case running time is $\frac{n^2}{100}$

• Which algorithm is best?
What we need

• A way of comparing functions that:
  • Ignores constants and non-dominant terms
  • Looks at long term trends
    • Ignores “small” inputs
\[ f(n) = \Theta(g(n)) \]

\[ f(n) = O(g(n)) \]

\[ f(n) = \Omega(g(n)) \]
Asymptotic Notation

• $O(g(n))$
  - The set of functions with asymptotic behavior less than or equal to $g(n)$
  - Upper-bounded by a constant times $g$ for large enough values $n$
  - $f \in O(g(n)) \equiv \exists c > 0. \exists n_0 > 0. \forall n \geq n_0. f(n) \leq c \cdot g(n)$

• $\Omega(g(n))$
  - The set of functions with asymptotic behavior greater than or equal to $g(n)$
  - Lower-bounded by a constant times $g$ for large enough values $n$
  - $f \in \Omega(g(n)) \equiv \exists c > 0. \exists n_0 > 0. \forall n \geq n_0. f(n) \geq c \cdot g(n)$

• $\Theta(g(n))$
  - "Tightly" within constant of $g$ for large $n$
  - $\Omega(g(n)) \cap O(g(n))$
\[ \exists c > 0 \left( \forall n_0, n \in \mathbb{N} \right) \left( \forall a \right) \left( a > n \right) \]
Asymptotic Notation Example

• Show: $10n + 100 \in O(n^2)$
  • **Technique:** find values $c > 0$ and $n_0 > 0$ such that $\forall n > n_0. \ 10n + 100 \leq c \cdot n^2$
  • **Proof:**

\[
10n + 100 \leq 5n^2
\]

\[
-5n^2 + 10n + 100 \leq 0
\]

\[
-5(n^2 - 2n - 20) \leq 0
\]
Asymptotic Notation Example

• Show: $10n + 100 \in O(n^2)$
  
  • **Technique:** find values $c > 0$ and $n_0 > 0$ such that $\forall n \geq n_0. 10n + 100 \leq c \cdot n^2$
  
  • **Proof:** Let $c = 10$ and $n_0 = 6$. Show $\forall n \geq 6. 10n + 100 \leq 10n^2$

  
  $10n + 100 \leq 10n^2$

  $\equiv n + 10 \leq n^2$

  $\equiv 10 \leq n^2 - n$

  $\equiv 10 \leq n(n - 1)$

  This is True because $n(n - 1)$ is strictly increasing and $6(6 - 1) > 10$
\( \forall n \in \mathbb{O} (n) \)

\( c = 2 \)

\( n + 5 \geq n + n \)
Asymptotic Notation Example

• **Show:** $13n^2 - 50n \in \Omega(n^2)$
  
  • **Technique:** find values $c > 0$ and $n_0 > 0$ such that $\forall n \geq n_0. 13n^2 - 50n \geq c \cdot n^2$
  
  • **Proof:**
Asymptotic Notation Example

• Show: $13n^2 - 50n \in \Omega(n^2)$
  
  • **Technique:** find values $c > 0$ and $n_0 > 0$ such that $\forall n \geq n_0. 13n^2 - 50n \geq c \cdot n^2$
  
  • **Proof:** let $c = 12$ and $n_0 = 50$. Show $\forall n \geq 50. 13n^2 - 50n \geq 12n^2$

$$
13n^2 - 50n \geq 12n^2 \\
\equiv n^2 - 50n \geq 0 \\
\equiv n^2 \geq 50n \\
\equiv n \geq 50
$$

This is certainly true $\forall n \geq 50$. 

Asymptotic Notation Example

• Show: $n^2 \not\in O(n)$
Asymptotic Notation Example

• To Show: \( n^2 \notin O(n) \)
  
  • Technique: Contradiction
  
  • Proof: Assume \( n^2 \in O(n) \). Then \( \exists c, n_0 > 0 \text{ s.t. } \forall n > n_0, n^2 \leq cn \)
  
  Let us derive constant \( c \). For all \( n > n_0 > 0 \), we know:
  
  \[ cn \geq n^2, \quad c \geq n. \]
  
  Since \( c \) is lower bounded by \( n \), \( c \) cannot be a constant and make this True.
  
  Contradiction. Therefore \( n^2 \notin O(n) \).
Gaining Intuition

• When doing asymptotic analysis of functions:
  • If multiple expressions are added together, ignore all but the “biggest”
    • If $f(n)$ grows asymptotically faster than $g(n)$, then $f(n) + g(n) \in \Theta(f(n))$
  • Ignore all multiplicative constants
    • $f(n) + c \in \Theta(f(n))$ for any constant $c \in \mathbb{R}$
  • Ignore bases of logarithms
  • Do NOT ignore:
    • Non-multiplicative and non-additive constants (e.g. in exponents, bases of exponents)
    • Logarithms themselves

• Examples:
  • $4n + 5$
  • $0.5n \log n + 2n + 7$
  • $n^3 + 2^n + 3n$
  • $n \log(10n^2)$
More Examples

• Is each of the following True or False?
  • $4 + 3n \in O(n)$
  • $n + 2 \log n \in O(\log n)$
  • $\log n + 2 \in O(1)$
  • $n^{50} \in O(1.1^n)$
  • $3^n \in \Theta(2^n)$
Common Categories

- $O(1)$ “constant”
- $O(\log n)$ “logarithmic”
- $O(n)$ “linear”
- $O(n \log n)$ “log-linear”
- $O(n^2)$ “quadratic”
- $O(n^3)$ “cubic”
- $O(n^k)$ “polynomial”
- $O(k^n)$ “exponential”
Defining your running time function

• Worst-case complexity:
  • max number of steps algorithm takes on “most challenging” input
• Best-case complexity:
  • min number of steps algorithm takes on “easiest” input
• Average/expected complexity:
  • avg number of steps algorithm takes on random inputs (context-dependent)
• Amortized complexity:
  • max total number of steps algorithm takes on M “most challenging” consecutive inputs, divided by M (i.e., divide the max total sum by M).
ADT: Queue

• What is it?
  • A “First In First Out” (FIFO) collection of items

• What Operations do we need?
  • Enqueue
    • Add a new item to the queue
  • Dequeue
    • Remove the “oldest” item from the queue
  • Is_empty
    • Indicate whether or not there are items still on the queue
ADT: Priority Queue

• What is it?
  • A collection of items and their “priorities”
  • Allows quick access/removal to the “top priority” thing

• What Operations do we need?
  • insert(item, priority)
    • Add a new item to the PQ with indicated priority
    • Usually, smaller priority value means more important
  • deleteMin
    • Remove and return the “top priority” item from the queue
  • Is_empty
    • Indicate whether or not there are items still on the queue

• Note: the “priority” value can be any type/class so long as it’s comparable (i.e. you can use “<” or “compareTo” with it)
Priority Queue, example

PriorityQueue PQ = new PriorityQueue();
PQ.insert(5,5)
PQ.insert(6,6)
PQ.insert(1,1)
PQ.insert(3,3)
PQ.insert(8,8)
Print(PQ.deleteMin)
Print(PQ.deleteMin)
Print(PQ.deleteMin)
Print(PQ.deleteMin)
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Print(PQ.deleteMin)
Priority Queue, example

PriorityQueue PQ = new PriorityQueue();
PQ.insert(5,5)
PQ.insert(6,6)
PQ.insert(1,1)
Print(PQ.deleteMin)
PQ.insert(3,3)
Print(PQ.deleteMin)
Print(PQ.deleteMin)
PQ.insert(8,8)
Print(PQ.deleteMin)
Print(PQ.deleteMin)
Applications?
Thinking through implementations

<table>
<thead>
<tr>
<th>Data Structure</th>
<th>Worst case time to insert</th>
<th>Worst case time to deleteMin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unsorted Array</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unsorted Linked List</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sorted Circular Array</td>
<td></td>
<td></td>
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<tr>
<td>Sorted Linked List</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Binary Search Tree</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Assume we know the maximum size of the PQ in advance