CSE 332 Autumn 2023
Lecture 26: Topological Sort and Minimum Spanning Trees

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Bank Account

Public static final Object BANK = new Object();
class BankAccount {
    ...
    synchronized void withdraw(int amt) {...}
    synchronized void deposit(int amt) {...}
    synchronized void transferTo(int amt, BankAccount a) {
        this.withdraw(amt);
        a.deposit(amt);
    }
}

The Deadlock

Thread 1:

x.transferTo(1,y);  

acquire lock for account x b/c transferTo is synchronized  
acquire lock for account y b/c deposit is synchronized  
release lock for account y after deposit  
release lock for account x at end of transferTo

Thread 2:

y.transferTo(1,x);  

acquire lock for account y b/c transferTo is synchronized  
acquire lock for account x b/c deposit is synchronized  
release lock for account x after deposit  
release lock for account y at end of transferTo

Expected Behavior:
Thread 2 items from a stack are popped in LIFO order
The Deadlock

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Thread 2 items from a stack are popped in LIFO order

Thread 1:

x.transferTo(1,y);

Thread 2:

y.transferTo(1,x);

acquire lock for account x b/c transferTo is synchronized
acquire lock for account y b/c deposit is synchronized
release lock for account y after deposit
release lock for account x at end of transferTo

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release lock for account y at end of transferTo
Resolving Deadlocks

• Deadlocks occur when there are multiple locks necessary to complete a task and different threads may obtain them in a different order

• Option 1:
  • Have a coarser lock granularity
  • E.g. one lock for ALL bank accounts

• Option 2:
  • Have a finer critical section so that only one lock is needed at a time
  • E.g. instead of a synchronized transferTo, have the withdraw and deposit steps locked separately

• Option 3:
  • Force the threads to always acquire the locks in the same order
  • E.g. make transferTo acquire both locks before doing either the withdraw or deposit, make sure both threads agree on the order to acquire
Option 1: Coarser Locking

static final Object BANK = new Object();
class BankAccount {
    ...
    synchronized void withdraw(int amt) {...}
    synchronized void deposit(int amt) {...}
    void transferTo(int amt, BankAccount a) {
        synchronized(BANK){
            this.withdraw(amt);
            a.deposit(amt);
        }
    }
}
Option 2: Finer Critical Section

class BankAccount {

    synchronized void withdraw(int amt) {...}
    synchronized void deposit(int amt) {...}

    void transferTo(int amt, BankAccount a) {
        synchronized(this){
            this.withdraw(amt);
        }
        synchronized(a){
            a.deposit(amt);
        }
    }

}
Option 3: First Get All Locks In A Fixed Order

class BankAccount {

    synchronized void withdraw(int amt) {...}
    synchronized void deposit(int amt) {...}

    void transferTo(int amt, BankAccount a) {
        if (this.acctNum < a.acctNum) {
            synchronized(this) {
                synchronized(a) {
                    this.withdraw(amt);
                    a.deposit(amt);
                }
            }
        } else {
            synchronized(a) {
                synchronized(this) {
                    this.withdraw(amt);
                    a.deposit(amt);
                }
            }
        }
    }
}
Depth-First Search

- **Input:** a node $s$
- **Behavior:** Start with node $s$, visit one neighbor of $s$, then all nodes reachable from that neighbor of $s$, then another neighbor of $s$, ...
- **Output:**
  - Does the graph have a cycle?
  - A **topological sort** of the graph.
DFS (non-recursive)

```java
void dfs(graph, s){
    found = new Stack();
    found.pop(s);
    mark s as “visited”;
    While (!found.isEmpty()){
        current = found.pop();
        for (v : neighbors(current)){
            if (! v marked “visited”){
                mark v as “visited”;
                found.push(v);
            }
        }
    }
}
```

Running time: $\Theta(|V| + |E|)$
void dfs(graph, curr){
    mark curr as “visited”;  
    for (v : neighbors(current)){
        if (! v marked “visited”){
            dfs(graph, v);
        }
    }
    mark curr as “done”;  
}
boolean hasCycle(graph, curr) {
    mark curr as “visited”;
    cycleFound = false;
    for (v : neighbors(current)) {
        if (v marked “visited” && ! v marked “done”) {
            cycleFound = true;
        }
        if (! v marked “visited” && ! cycleFound) {
            cycleFound = hasCycle(graph, v);
        }
    }
    mark curr as “done”;
    return cycleFound;
}
Topological Sort

• A Topological Sort of a directed acyclic graph $G = (V, E)$ is a permutation of $V$ such that if $(u, v) \in E$ then $u$ is before $v$ in the permutation.
DFS Recursively

void dfs(graph, curr){
    mark curr as “visited”;
    for (v : neighbors(current)){
        if (! v marked “visited”){
            dfs(graph, v);
        }
    }
    mark curr as “done”;
}
DFS Recursively

```java
void dfs(graph, curr){
    mark curr as "visited";
    for (v : neighbors(current)){
        if (! v marked "visited"){
            dfs(graph, v);
        }
    }
    mark curr as "done";
}
```

Idea: List in reverse order by “done” time
DFS: Topological sort

List topSort(graph){
    List<Nodes> done = new List<>();
    for (Node v : graph.vertices){
        if (!v.visited){
            finishTime(graph, v, finished);
        }
    }
    done.reverse();
    return done;
}

void finishTime(graph, curr, finished){
    curr.visited = true;
    for (Node v : curr.neighbors){
        if (!v.visited){
            finishTime(graph, v, finished);
        }
    }
    done.add(curr)
}

Idea: List in reverse order by “done” time
Definition: Tree

A connected graph with no cycles

Note: A tree does not need a root, but they often do!
Definition: Tree

A connected graph with no cycles

Pick some arbitrary root node and rearrange tree
Definition: Spanning Tree

A Tree $T = (V_T, E_T)$ which connects (“spans”) all the nodes in a graph $G = (V, E)$.

How many edges does $T$ have? $V - 1$

Any set of V-1 edges in the graph that doesn’t have any cycles is guaranteed to be a spanning tree!

Any set of V-1 edges that connects all the nodes in the graph is guaranteed to be a spanning tree!
Definition: Minimum Spanning Tree

A Tree \( T = (V_T, E_T) \) which connects ("spans") all the nodes in a graph \( G = (V, E) \), that has minimal cost

\[
\text{Cost}(T) = \sum_{e \in E_T} w(e)
\]
Kruskal’s Algorithm

Start with an empty tree $A$
Add to $A$ the lowest-weight edge that does not create a cycle
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Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, $S$ and $V - S$.

Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. $(A, C)$.

A set of edges $R$ Respects a cut if no edges cross the cut, e.g. $R = \{(A, B), (E, G), (F, G)\}$. 
Cut Theorem

If a set of edges $A$ is a subset of a minimum spanning tree $T$, let $(S, V - S)$ be any cut which $A$ respects. Let $e$ be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.
Cut Theorem

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Cut Theorem

If a set of edges $A$ is a subset of a minimum spanning tree $T$, let $(S, V − S)$ be any cut which $A$ respects. Let $e$ be the least-weight edge which crosses $(S, V − S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.
Proof of Kruskal’s Algorithm

Start with an empty tree $A$
Repeat $V - 1$ times:
   Add the min-weight edge that doesn’t cause a cycle

**Proof:** Suppose we have some arbitrary set of edges $A$ that Kruskal’s has already selected to include in the MST. $e = (F,G)$ is the edge Kruskal’s selects to add next

We know that there cannot exist a path from $F$ to $G$ using only edges in $A$ because $e$ does not cause a cycle

We can cut the graph therefore into 2 disjoint sets:
- nodes reachable from $G$ using edges in $A$
- All other nodes

$e$ is the minimum cost edge that crosses this cut, so by the Cut Theorem, Kruskal’s is optimal!
Kruskal’s Algorithm Runtime

Start with an empty tree $A$
Repeat $V - 1$ times:
Add the min-weight edge that doesn’t cause a cycle

Keep edges in a Disjoint-set data structure (very fancy)
$O(E \log V)$
General MST Algorithm

Start with an empty tree $A$

Repeat $V - 1$ times:

Pick a cut $(S, V - S)$ which $A$ respects (typically implicitly)

Add the min-weight edge which crosses $(S, V - S)$
Prim’s Algorithm
Start with an empty tree $A$
Repeat $V - 1$ times:
   Pick a cut $(S, V - S)$ which $A$ respects
   Add the min-weight edge which crosses $(S, V - S)$

$S$ is all endpoint of edges in $A$
$e$ is the min-weight edge that grows the tree
Prim’s Algorithm
Start with an empty tree $A$
Pick a start node
Repeat $V - 1$ times:
Add the min-weight edge which connects to node in $A$ with a node not in $A$
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Start with an empty tree $A$
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Prim’s Algorithm
Start with an empty tree $A$
Pick a start node
Repeat $V - 1$ times:
   - Add the min-weight edge which connects to node in $A$ with a node not in $A$

Keep edges in a Heap $O(E \log V)$
Dijkstra’s Algorithm

```java
int dijkstras(graph, start, end){
    PQ = new minheap();
    PQ.insert(0, start); // priority=0, value=start
    start.distance = 0;
    while (!PQ.isEmpty){
        current = PQ.extractmin();
        if (current.known){ continue;}
        current.known = true;
        for (neighbor : current.neighbors){
            if (!neighbor.known){
                new_dist = current.distance + weight(current,neighbor);
                if(neighbor.dist != ∞){ PQ.insert(new_dist, neighbor);}
                else if (new_dist < neighbor.distance){
                    neighbor.distance = new_dist;
                    PQ.decreaseKey(new_dist,neighbor); }
            }
        }
    }
    return end.distance;
}
```
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                else if (new_dist < neighbor.distance){
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            }
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    }
    return end.distance;
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