CSE 332 Summer 2024 Lecture 4: Recurrences

Nathan Brunelle

http://www.cs.uw.edu/332

Warm Up: Practice with O

Show $n^2 + 3n$ belongs to $O(4n^3)$

More Examples

- Is each of the following True or False?
  - $4 + 3n \in O(n)$
  - $n + 2\log n \in O(\log n)$
  - $\log n + 2 \in O(1)$
  - $n^{50} \in O(1.1^n)$
  - $3^n \in \Theta(2^n)$

Gaining Intuition

- When doing asymptotic analysis of functions:
  - If multiple expressions are added together, ignore all but the "biggest"
    - If $f(n)$ grows asymptotically faster than $g(n)$, then $f(n) + g(n) \in \Theta\big(f(n)\big)$
  - Ignore all multiplicative constants
    - $f(n) + c \in \Theta\big(f(n)\big)$ for any constant $c \in \mathbb{R}$
  - Ignore bases of logarithms
  - Do NOT ignore:
    - Non-multiplicative and non-additive constants (e.g. in exponents, bases of exponents)
    - Logarithms themselves
- Examples:
  - $4n + 5$
  - $0.5n\log n + 2n + 7$
  - $n^3 + 2^n + 3n$
  - $n\log(10n^2)$

Common Categories

- $O(1)$ "constant"
- $O(\log n)$ "logarithmic"
- $O(n)$ "linear"
- $O(n \log n)$ "log-linear"
- $O(n^2)$ "quadratic"
- $O(n^3)$ "cubic"
- $O(n^k)$ "polynomial"
- $O(k^n)$ "exponential"

Defining your running time function

- Worst-case complexity:
  - max number of steps algorithm takes on “most challenging” input
- Best-case complexity:
  - min number of steps algorithm takes on “easiest” input
- Average/expected complexity:
  - avg number of steps algorithm takes on random inputs (context-dependent)
- Amortized complexity:
  - max total number of steps algorithm takes on M “most challenging” consecutive inputs, divided by M (i.e., divide the max total sum by M).

Beware!

- Worst case, Best case, amortized are ways to select a function
- O, Ω, Θ are ways to compare functions
- You can mix and match!
- The following statements totally make sense!
  - The worst case running time of my algorithm is $\Omega(n^3)$
  - The best case running time of my algorithm is $O(n)$
  - The best case running time of my algorithm is $\Theta(2^n)$

Recursive Binary Search

5	8	13	42	75	79	88	90	95	99
0	1	2	3	4	5	6	7	8	9

```java
public static boolean binarySearch(List<Integer> lst, int k){
        return binarySearch(lst, k, 0, lst.size());
}
private static boolean binarySearch(List<Integer> lst, int k, int start, int end){
    if(start == end)
        return false;
    int mid = start + (end-start)/2;
    if(lst.get(mid) == k){
        return true;
    } else if(lst.get(mid) > k){
        return binarySearch(lst, k, start, mid);
    } else{
        return binarySearch(lst, k, mid+1, end);
    }
}
```

Analysis of Recursive Algorithms

- Overall structure of recursion:
  - Do some non-recursive "work"
  - Do one or more recursive calls on some portion of your input
  - Do some more non-recursive "work"
  - Repeat until you reach a base case
- Running time: $T(n) = T(p_1) + T(p_2) + \cdots + T(p_x) + f(n)$
  - The time it takes to run the algorithm on an input of size n is:
  - The sum of how long it takes to run the same algorithm on each smaller input
  - Plus the total amount of non-recursive work done at that step
- Usually:
  - $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$
    - Called "divide and conquer"
  - $T(n) = T(n - c) + f(n)$
    - Called "chip and conquer"

How Efficient Is It?

- $T(n) = 1 + T\left(\left\lceil \frac{n}{2} \right\rceil\right)$
- Base case: $T(1) = 1$

$T(n) =$ "cost" of running the entire algorithm on an array of length n

Let’s Solve the Recurrence!

$T(1) = 1$

$T(n) = 1 + \cancel{T\left(\frac{n}{2}\right)}$

$1 + \cancel{T\left(\frac{n}{4}\right)}$

$1 + \cancel{T\left(\frac{n}{8}\right)}$

$\ddots$

1

Substitute until $T(1)$
So $\log_2 n$ steps

$$T(n) = \sum_{i=1}^{\log_2 n} 1 = \log_2 n$$

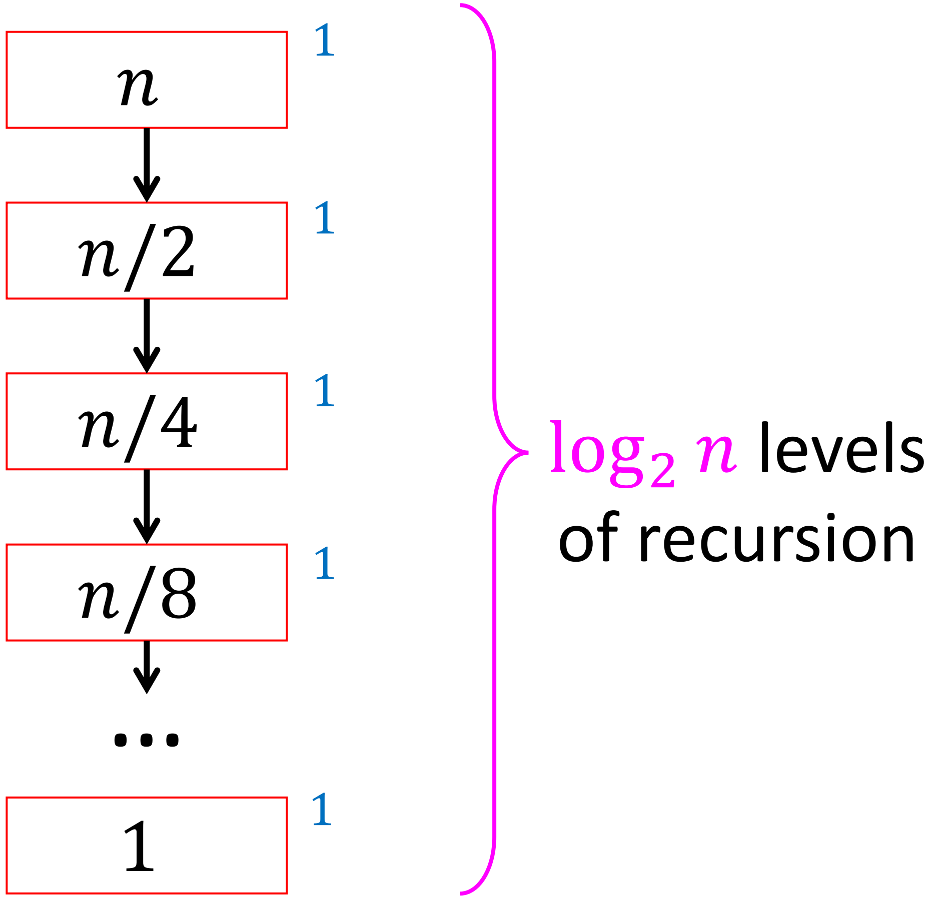
$$T(n) \in \Theta(\log n)$$

Make our method "prettier"

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

- Draw a picture of the recursion
- Identify the work done per stack frame
- Add up all the work!
    - Sum is the answer!
    - In this case $\Theta(\log_2 n)$

The "Tree Method"



Recursive Linear Search

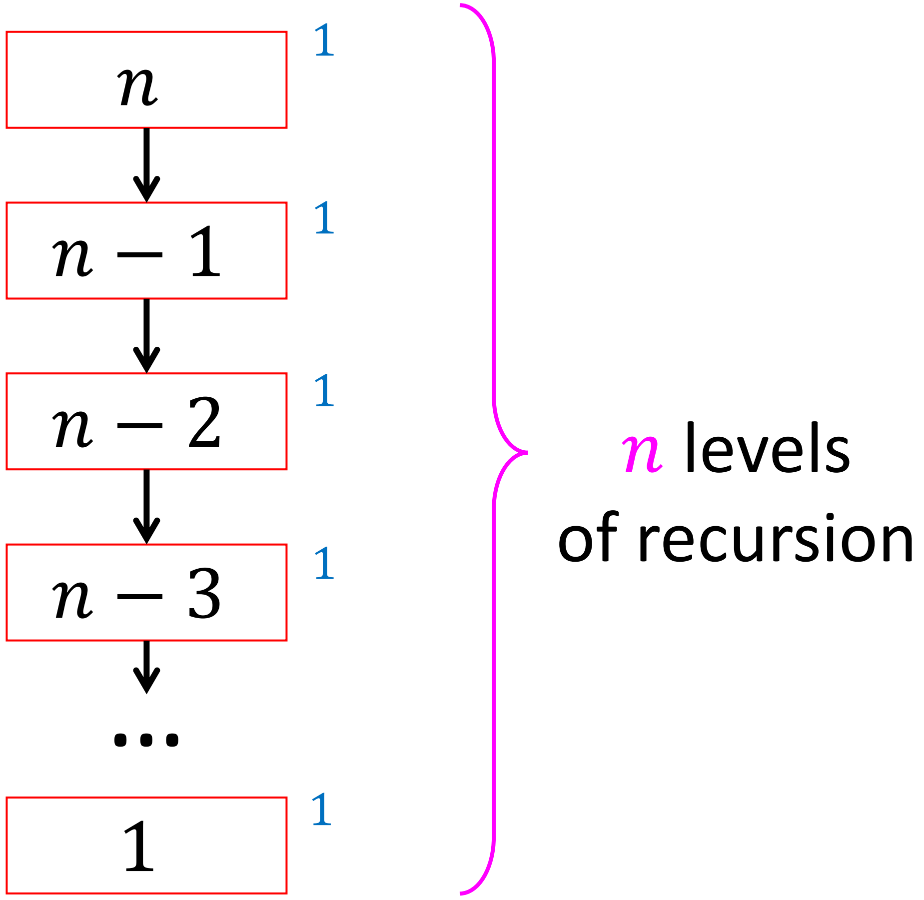
5	8	13	42	75	79	88	90	95	99
0	1	2	3	4	5	6	7	8	9

```java
public static boolean linearSearch(List<Integer> lst, int k){
        return linearSearch(lst, k, 0, lst.size());
    }
private static boolean linearSearch(List<Integer> lst, int k, int start, int end){
    if(start == end){
        return false;
    } else if(lst.get(start) == k){
        return true;
    } else{
        return linearSearch(lst, k, start+1, end);
    }
}
```

Make our method “prettier”

$$T(n) = T(n-1) + 1$$

- Draw a picture of the recursion
- Identify the work done per stack frame
- Add up all the work!



Running time: $\Theta(n)$

Recursive List Summation

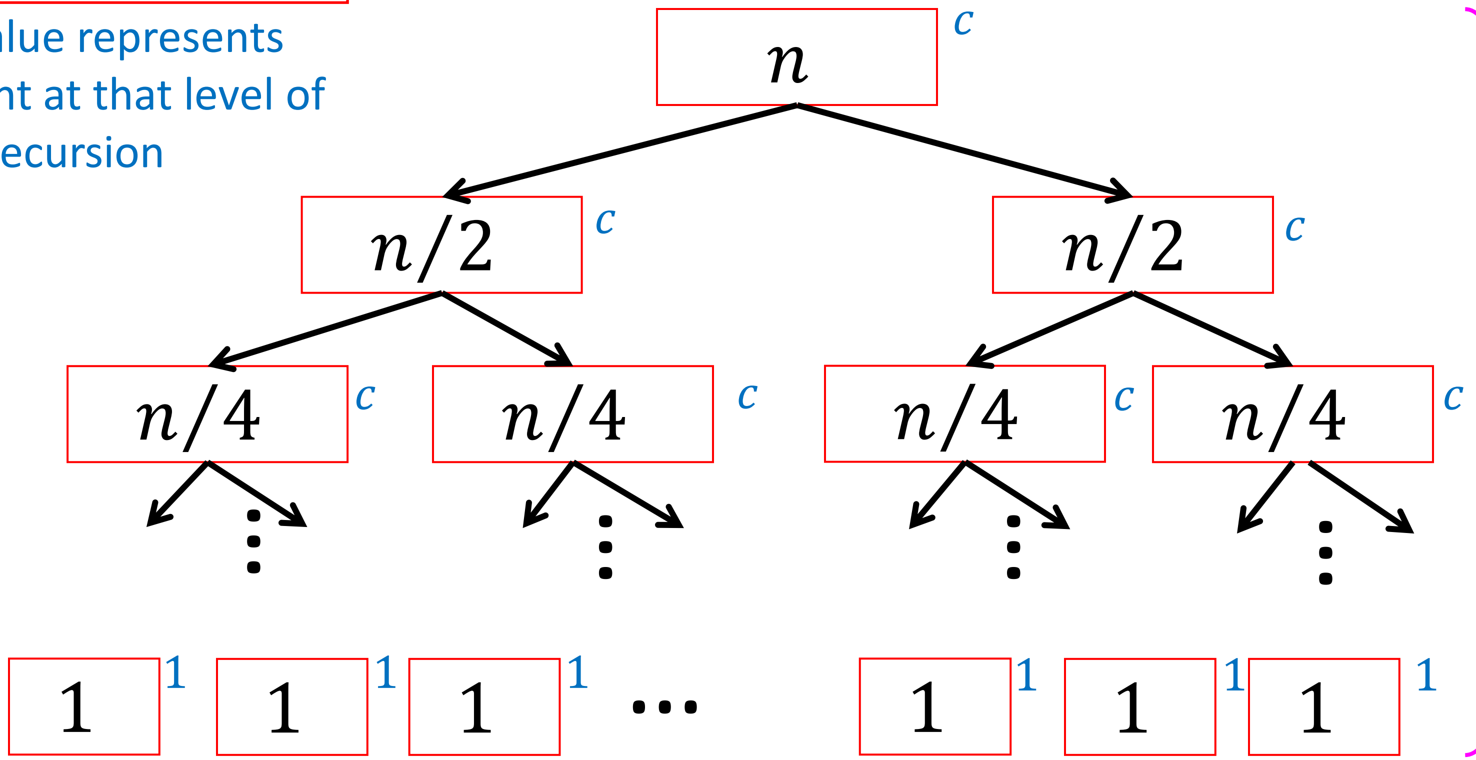
```
sum(list){
        return sum_helper(list, 0, list.size);
}
sum_helper(list, low, high){
        if (low == high){ return 0; }
        if (low == high-1){ return list[low]; }
        middle = (high+low)/2;
        return sum_helper(list, low, middle) + sum_helper(list, middle, high);
}
```

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{2}\right) + c$$



$\Rightarrow 2^i \cdot c$ work per level

$\log_2 n$ levels of recursion

$$T(n) = \sum_{i=1}^{\log_2 n} 2^i \cdot c$$

Recursive List Summation

$$T(n) = \sum_{i=1}^{\log_2 n} 2^i \cdot c$$

$$= c \cdot \sum_{i=1}^{\log_2 n} 2^i$$

$$= c\left(\frac{1 - 2^{\log_2 n}}{1 - 2}\right)$$

Let’s do some more!

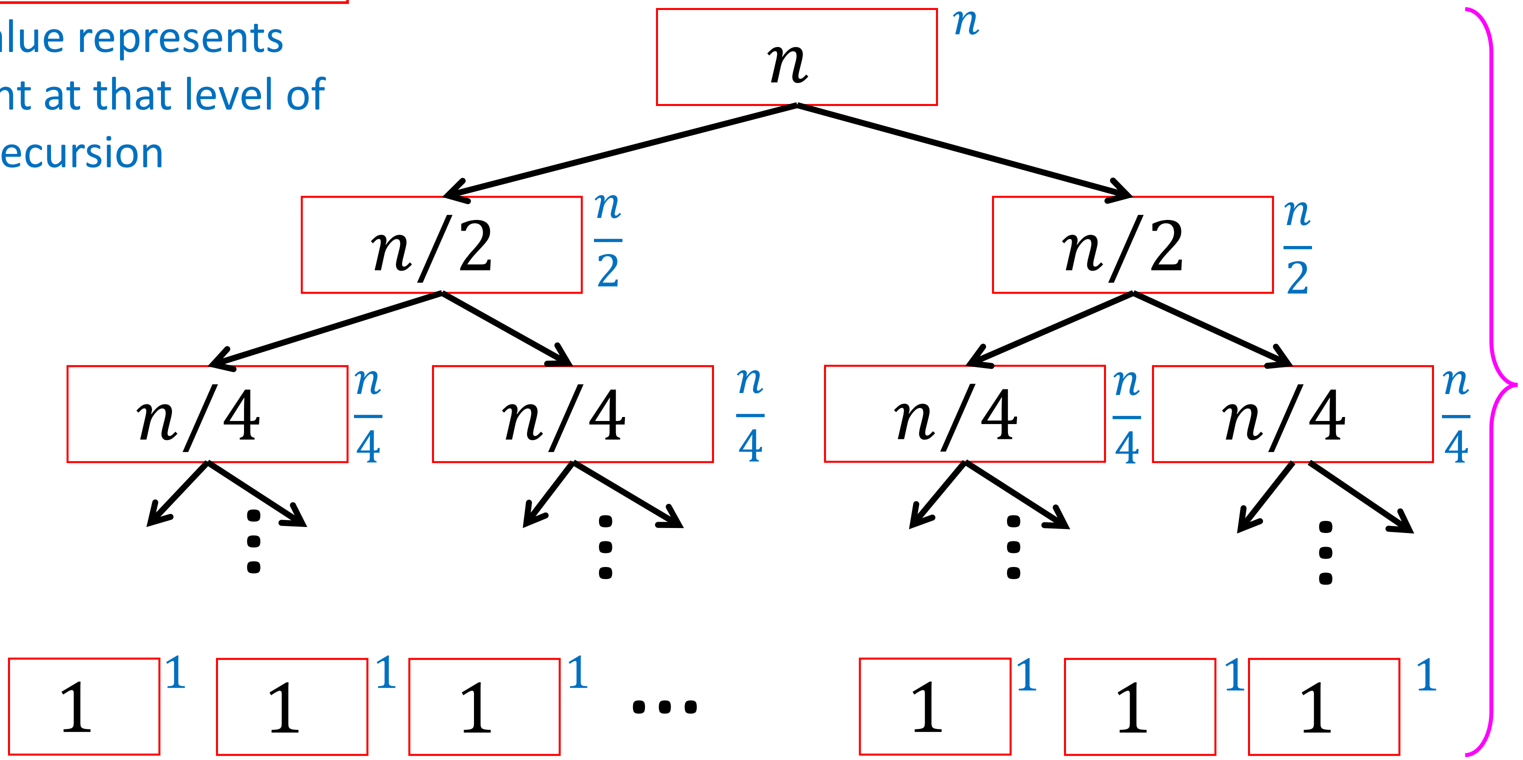
- For each, assume the base case is $n = 1$ and $T(1) = 1$
- $T(n) = 2T\left(\frac{n}{2}\right) + n$
- $T(n) = 2T\left(\frac{n}{2}\right) + n^2$
- $T(n) = 2T\left(\frac{n}{8}\right) + 1$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



$\Rightarrow n$ work per level

$\log_2 n$ levels of recursion

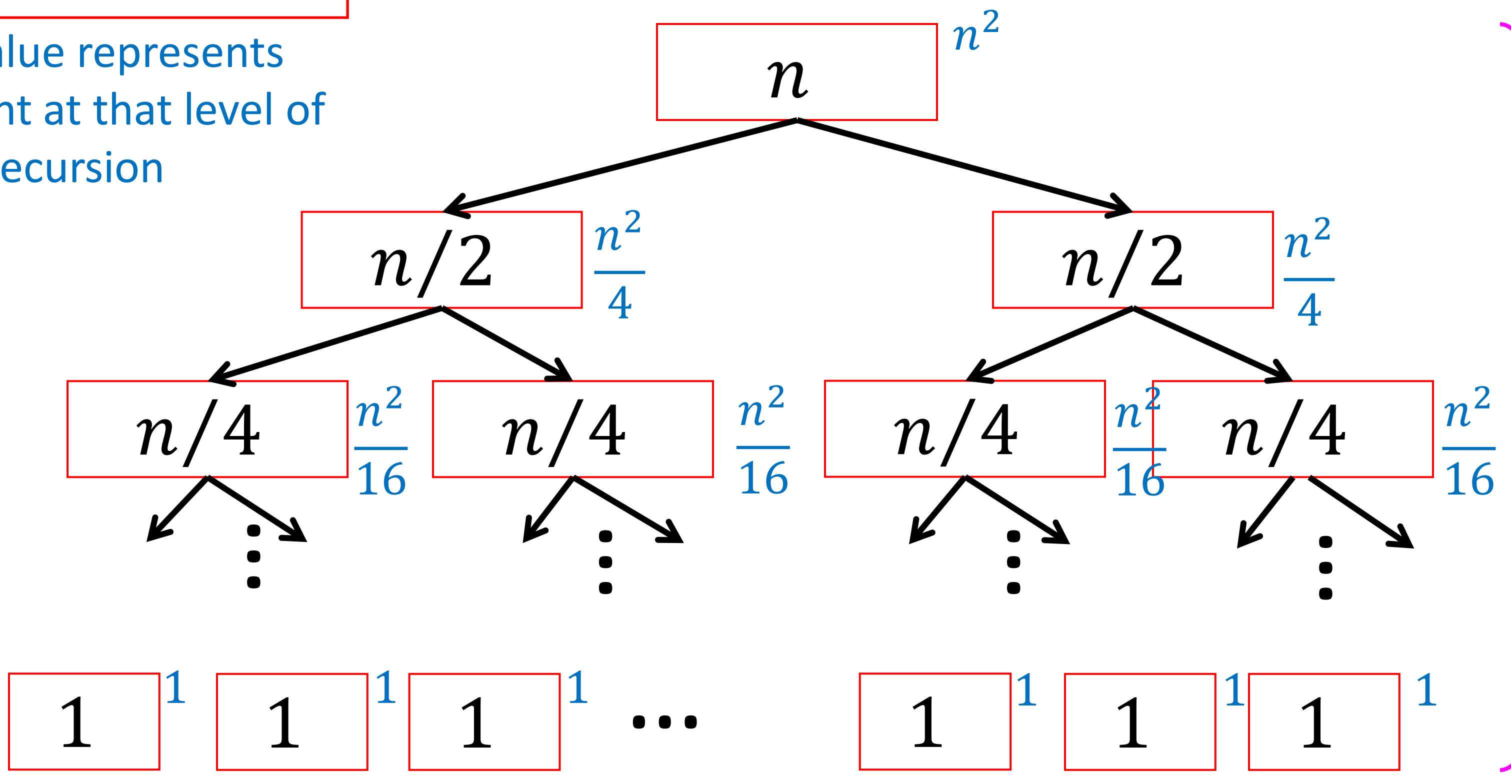
$$T(n) = \sum_{i=1}^{\log_2 n} n$$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2$$



$\Rightarrow$?? work per level

$\log_2 n$ levels of recursion

$$T(n) = \sum_{i=1}^{\log_2 n} ??$$

Recursive List Summation

$$T(n) = \sum_{i=1}^{\log_2 n} \frac{n^2}{2^i}$$

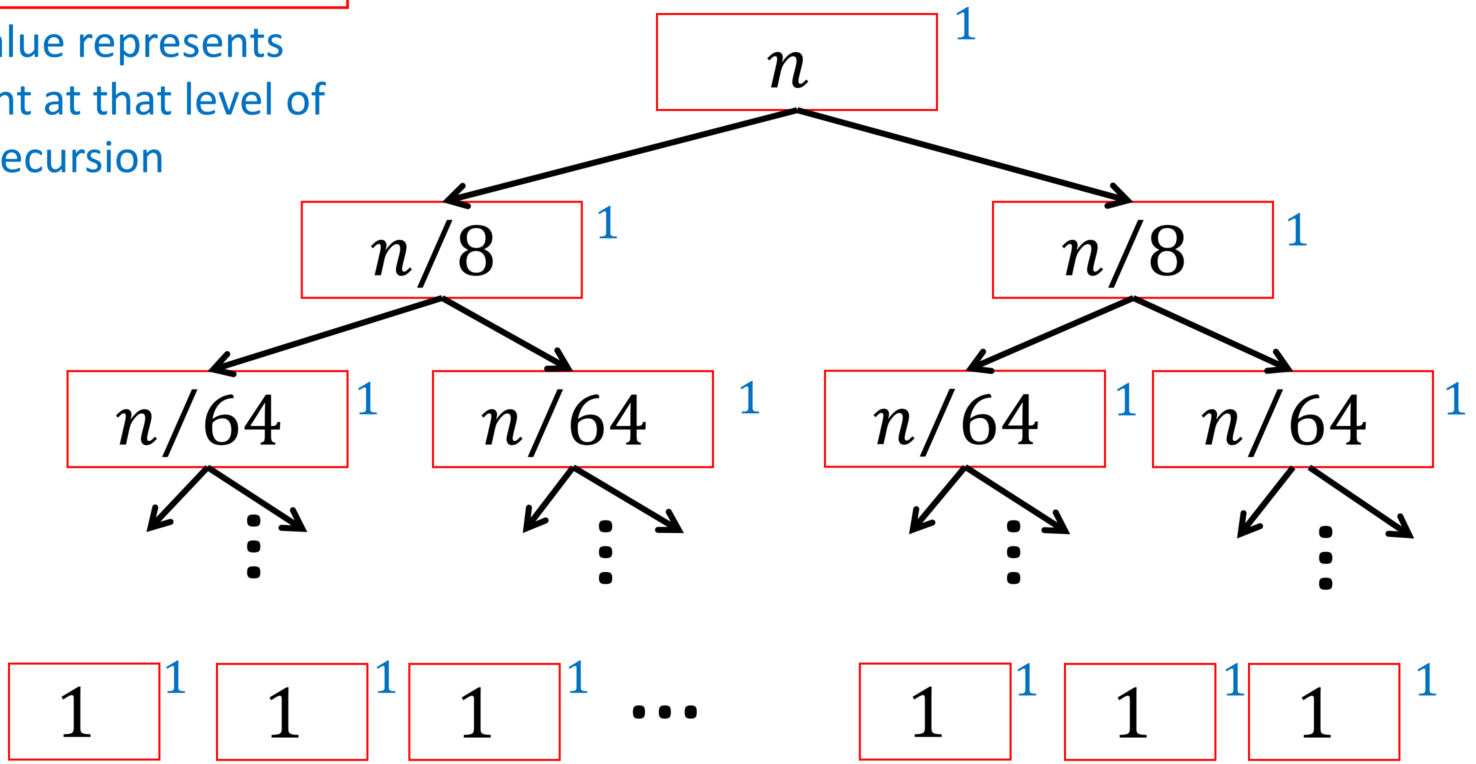
$$= n^2 \cdot \sum_{i=1}^{\log_2 n} \left(\frac{1}{2}\right)^i$$

Tree Method

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T\left(\frac{n}{8}\right) + 1$$



$\Rightarrow 2^i$ work per level

$\log_8 n$ levels of recursion

$$T(n) = \sum_{i=1}^{\log_8 n} 2^i$$

Recursive List Summation

$$T(n) = \sum_{i=1}^{\log_8 n} 2^i$$

$$= \left(\frac{1 - 2^{\log_8 n}}{1 - 2} \right)$$

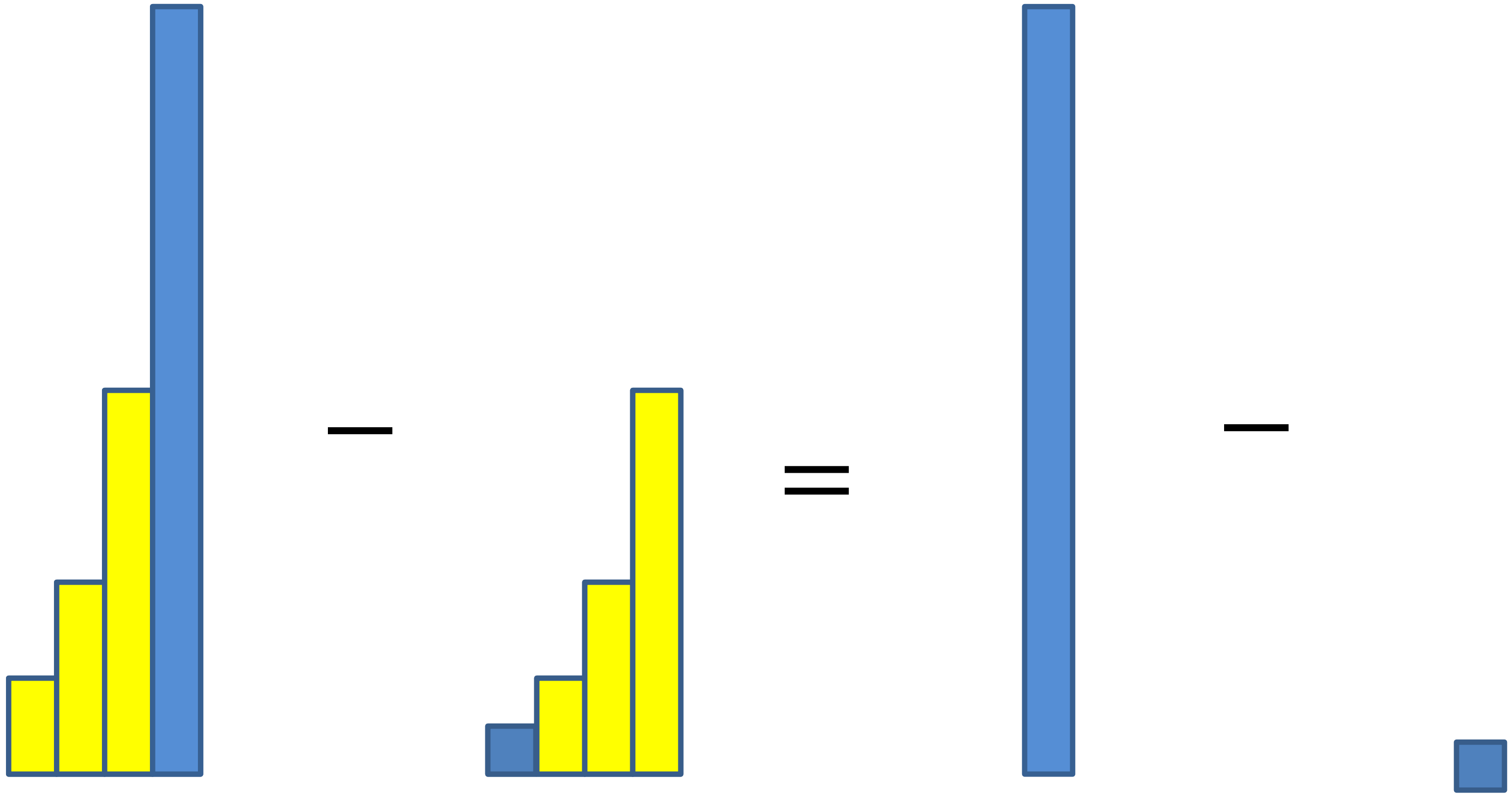
$$= 2^{\log_8 n} - 1$$

$$= n^{\log_8 2} = n^{\frac{1}{3}}$$

$$\sum_{i=0}^{L} a^i$$

Finite Geometric Series

If $a > 1$



The series multiplied by a

$(1 + a + a^2 + \cdots + a^L)a$

The series

$(1 + a + a^2 + \cdots + a^L)1$

The next term in the series

a^{L+1}

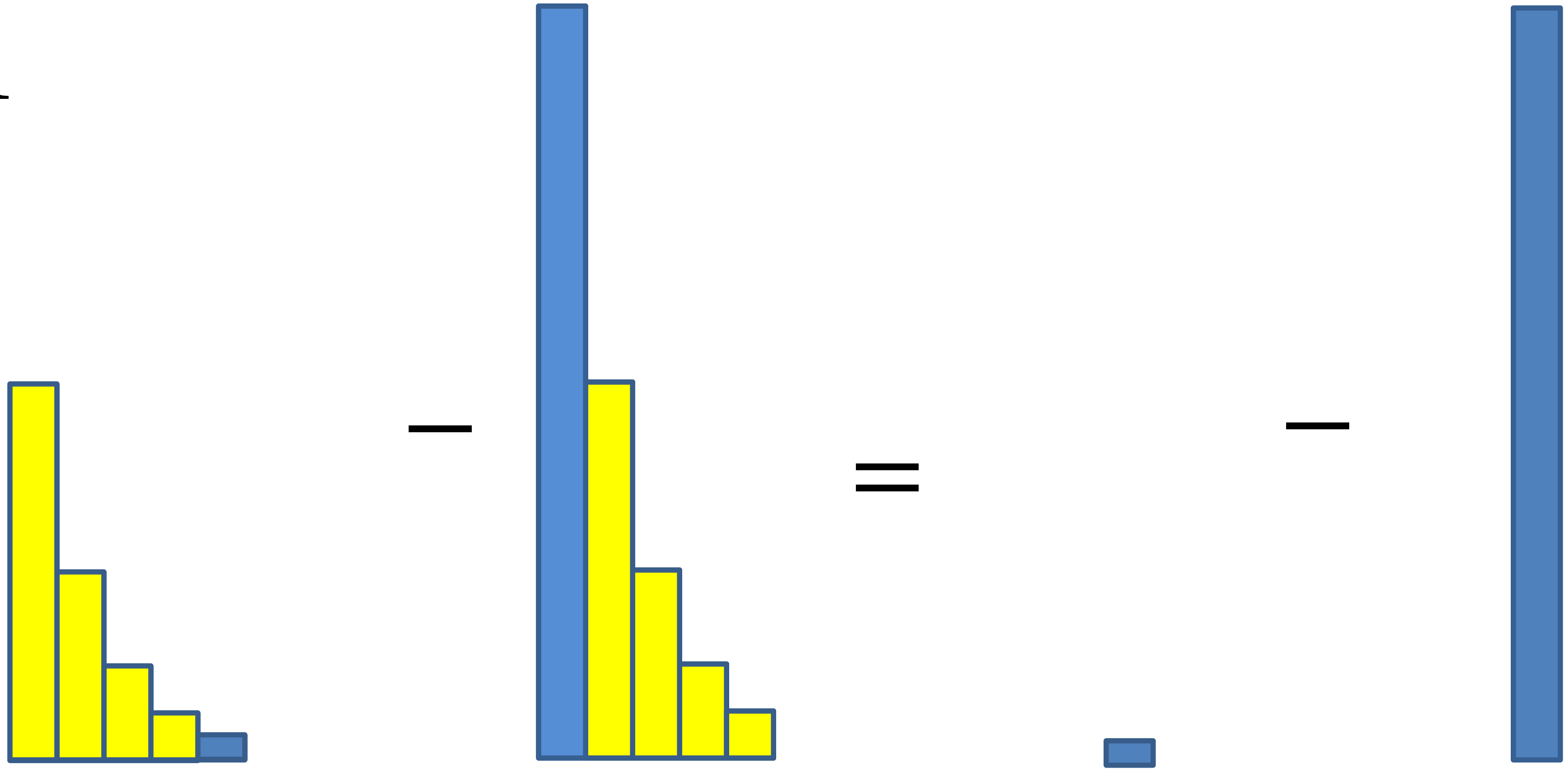
The first term

1

$$\sum_{i=0}^{L} a^i$$

Finite Geometric Series

If $a < 1$



The series multiplied by a − The series = The next term in the series − The first term

$$\underbrace{(1 + a + a^2 + \cdots + a^L)a \quad - \quad (1 + a + a^2 + \cdots + a^L)1 \quad = \quad a^{L+1} \quad - \quad 1}_{\text{Solve for the series}}$$