

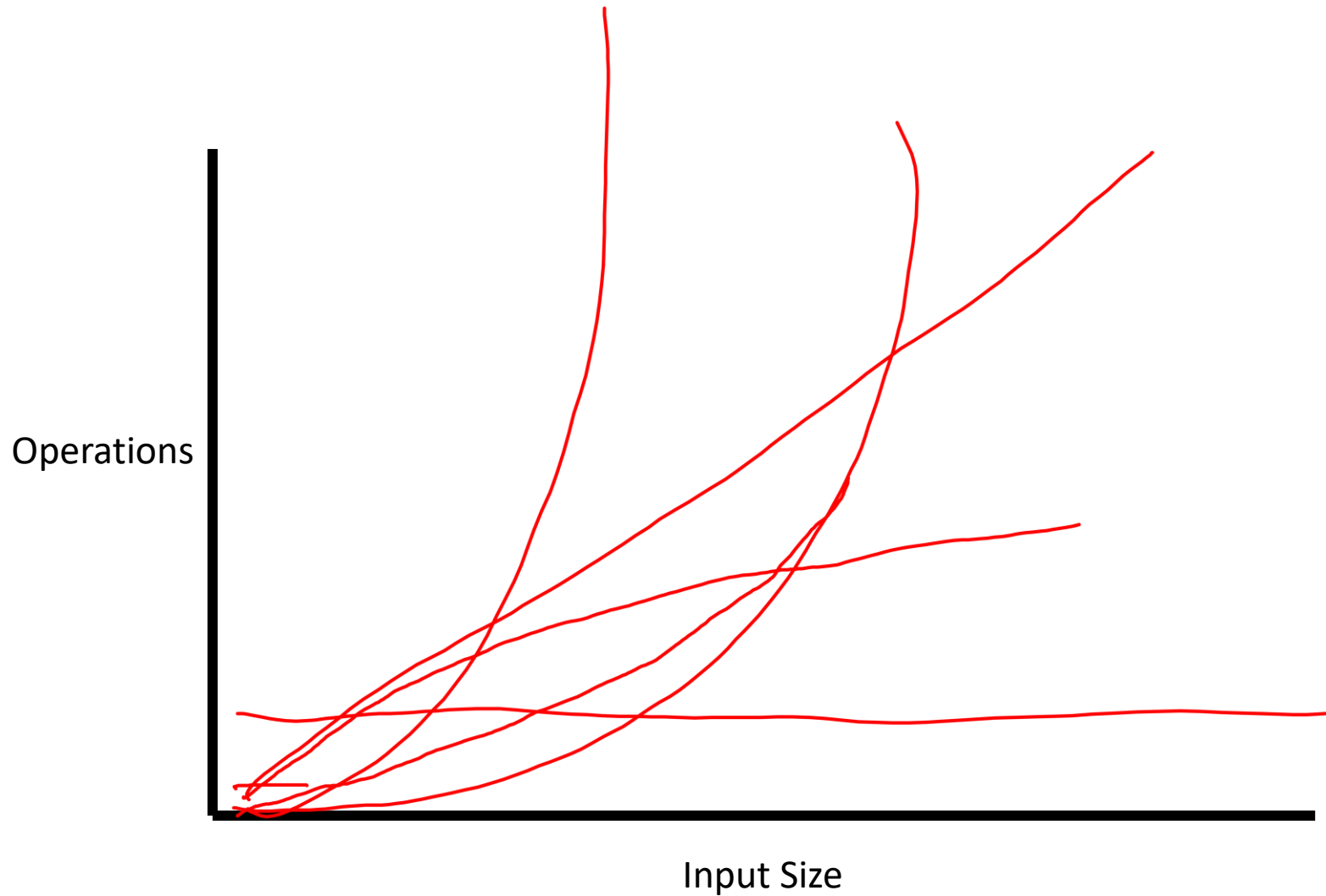
CSE 332 Summer 2024

Lecture 23: P & NP

Nathan Brunelle

<http://www.cs.uw.edu/332>

Running Times



Running times we've seen:

- $\Theta(1)$
- $\Theta(\log n)$
- $\Theta(n)$
- $\Theta(n \log n)$
- $\Theta(n^2)$
- $\Theta(2^n)$

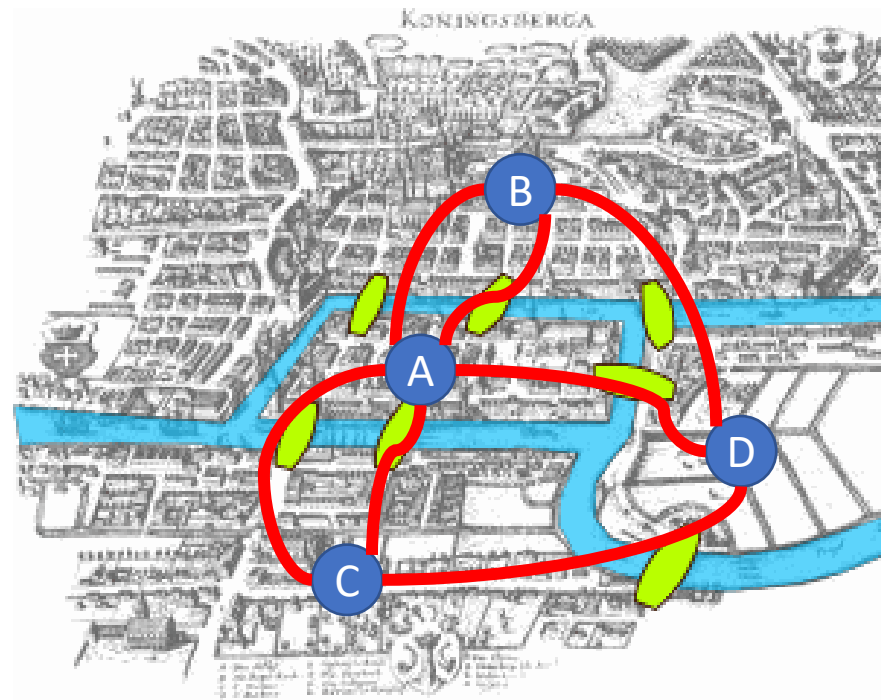


Running Times

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

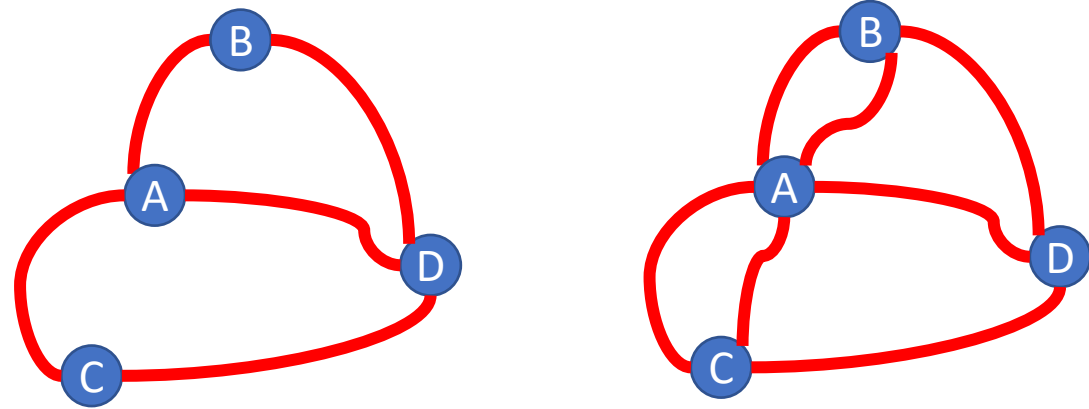
	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1,000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

7 Bridges of Königsberg



The Pregel River runs through the city of Königsberg, creating 2 islands. Among these 2 islands and the 2 sides of the river, there are 7 bridges. Is there any path starting at one landmass which crosses each bridge exactly once?

Euler Path Problem



- Path:

- A sequence of nodes v_1, v_2, \dots such that for every consecutive pair are connected by an edge (i.e. (v_i, v_{i+1}) is an edge for each i in the path)

- Euler Path:

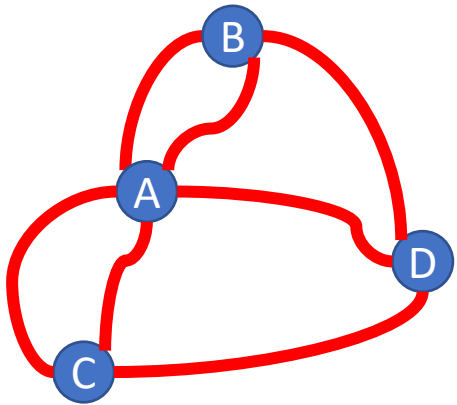
- A path such that every edge in the graph appears exactly once
 - If the graph is not simple then some pairs need to appear multiple times!

- Euler path problem:

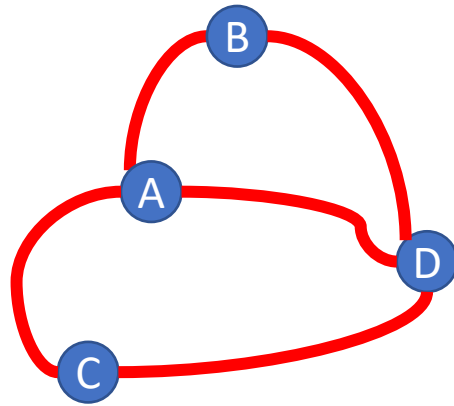
- Given an undirected graph $G = (V, E)$, does there exist an Euler path for G ?

Examples

- Which of the graphs below have an Euler path?

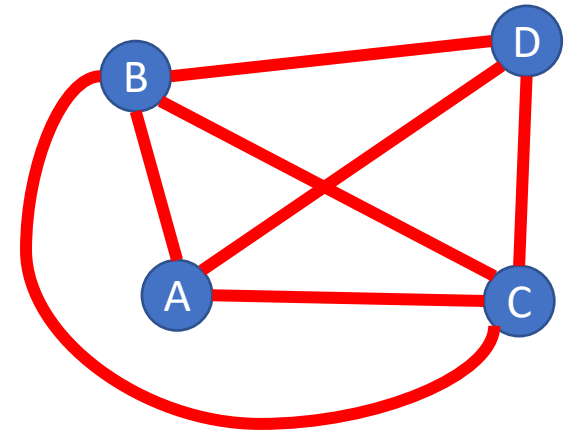


No Euler path exists!



Euler path exists!

A, B, D, A, C, D

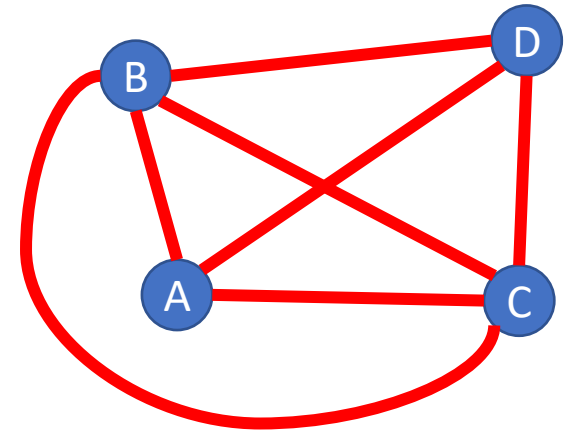
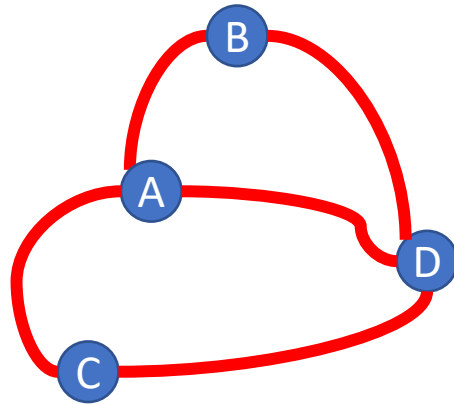
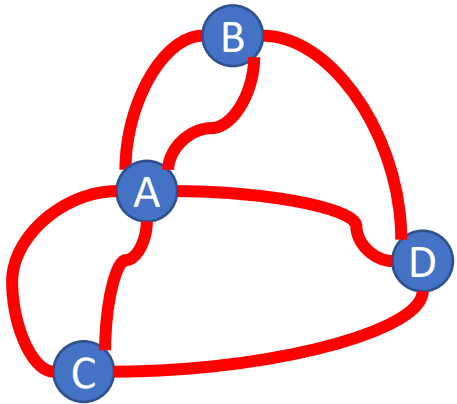


Euler path exists!

A, B, C, D, A, C, B, D

Euler's Theorem

- A graph has an Euler Path if and only if it is connected and has exactly 0 or 2 nodes with odd degree.

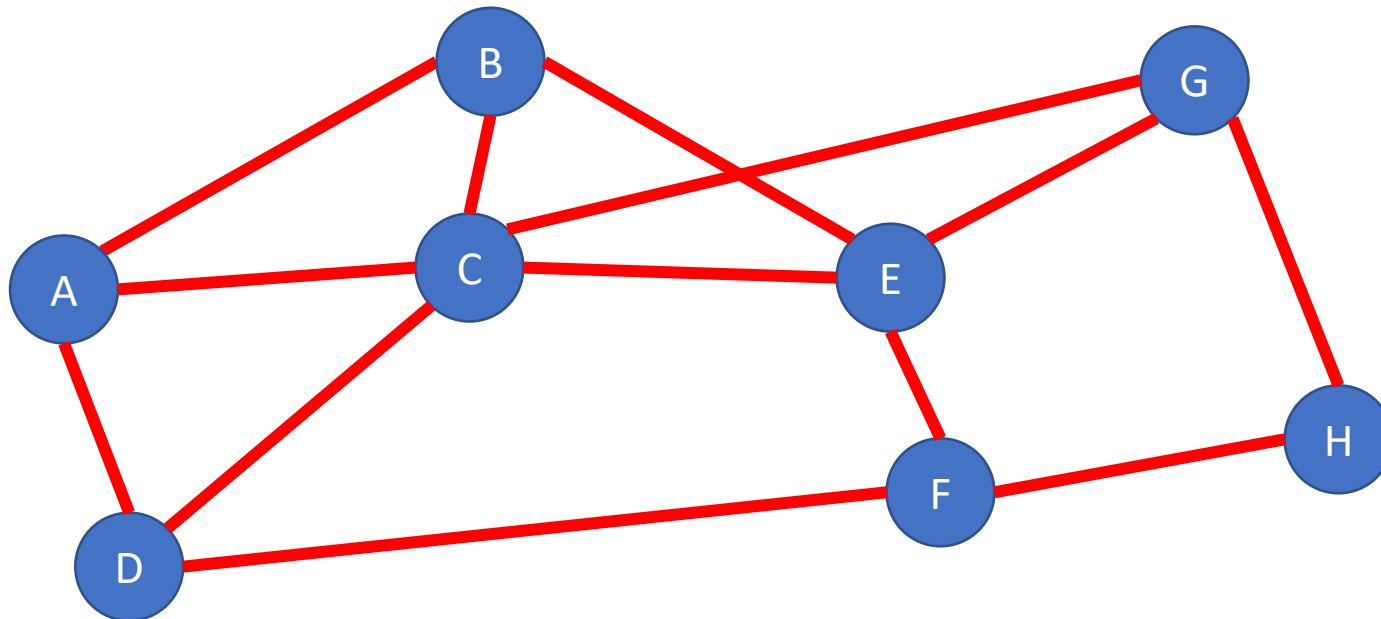


Algorithm for the Euler Path Problem

- Given an undirected graph $G = (V, E)$, does there exist an Euler path for G ?
- Algorithm:
 - Check if the graph is connected
 - Check the degree of each node
 - If the number of nodes with odd degree is 0 or 2, return true
 - Otherwise return false
- Running time?

A Seemingly Similar Problem

- Hamiltonian Path:
 - A path that includes every node in the graph exactly once
- Hamiltonian Path Problem:
 - Given a graph $G = (V, E)$, does that graph have a Hamiltonian Path?



True!
A, B, C, E, G, H, F, D

Algorithms for the Hamiltonian Path Problem

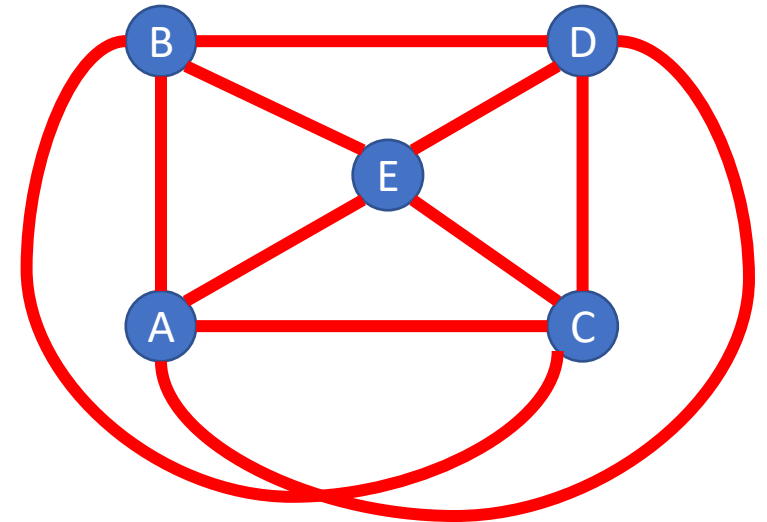
- Option 1:
 - Explore all possible simple paths through the graph
 - Check to see if any of those are length V
- Option 2:
 - Write down every sequence of nodes
 - Check to see if any of those are a path
- Both options are examples of an **Exhaustive Search (“Brute Force”) algorithm**

Option 2: List all sequences, look for a path

- Running time:
 - $G = (V, E)$
 - Number of permutations of V is $|V|!$
 - $n! = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 2 \cdot 1$
 - How does $n!$ compare with 2^n ?
 - $n! \in \Omega(2^n)$
 - Exponential running time!

Option 1: Explore all simple paths, check for one of length V

- Running time:
 - $G = (V, E)$
 - Number of paths
 - Pick a first node ($|V|$ choices)
 - Pick a neighbor (up to $|V| - 1$ choices)
 - Pick a neighbor (up to $|V| - 2$ choices)
 - Repeat $|V| - 1$ total times
 - Overall: $|V|!$ paths
 - Exponential running time



Tractability

- Tractable:
 - Feasible to solve in the “real world”
- Intractable:
 - Infeasible to solve in the “real world”
- Whether a problem is considered “tractable” or “intractable” depends on the use case
 - For machine learning, big data, etc. tractable might mean $O(n)$ or even $O(\log n)$
 - For most applications it’s more like $O(n^3)$ or $O(n^2)$
- A strange pattern:
 - Most “natural” problems are either done in small-degree polynomial (e.g. n^2) or else exponential time (e.g. 2^n)
 - It’s rare to have problems which require a running time of n^5 , for example

Complexity Classes

- A Complexity Class is a set of problems (e.g. sorting, Euler path, Hamiltonian path)
 - The problems included in a complexity class are those whose most efficient algorithm has a specific upper bound on its running time (or memory use, or...)
- Examples:
 - The set of all problems that can be solved by an algorithm with running time $O(n)$
 - Contains: Finding the minimum of a list, finding the maximum of a list, buildheap, summing a list, etc.
 - The set of all problems that can be solved by an algorithm with running time $O(n^2)$
 - Contains: everything above as well as sorting, Euler path
 - The set of all problems that can be solved by an algorithm with running time $O(n!)$
 - Contains: everything we've seen in this class so far

Complexity Classes and Tractability

- To explore what problems are and are not tractable, we give some complexity classes special names:
- Complexity Class P :
 - Stands for “Polynomial”
 - The set of problems which have an algorithm whose running time is $O(n^p)$ for some choice of $p \in \mathbb{R}$.
 - We say all problems belonging to P are “Tractable”
- Complexity Class EXP :
 - Stands for “Exponential”
 - The set of problems which have an algorithm whose running time is $O(2^{n^p})$ for some choice of $p \in \mathbb{R}$
 - We say all problems belonging to $EXP - P$ are “Intractable”
 - Disclaimer: Really it’s all problems outside of P , and there are problems which do not belong to EXP , but we’re not going to worry about those in this class

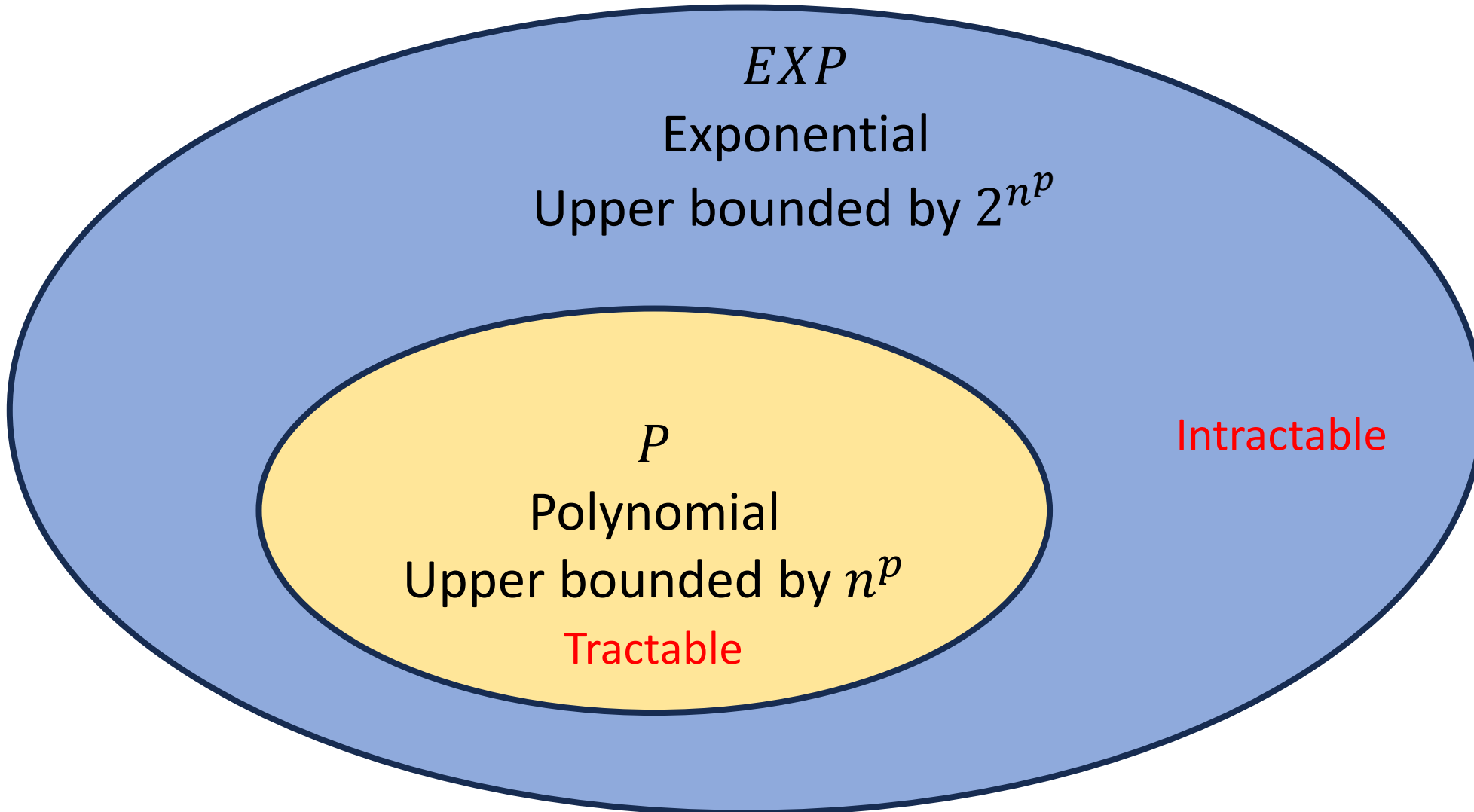
EXP and P

Important!

$$P \subset EXP$$

Every problem within P is also within EXP

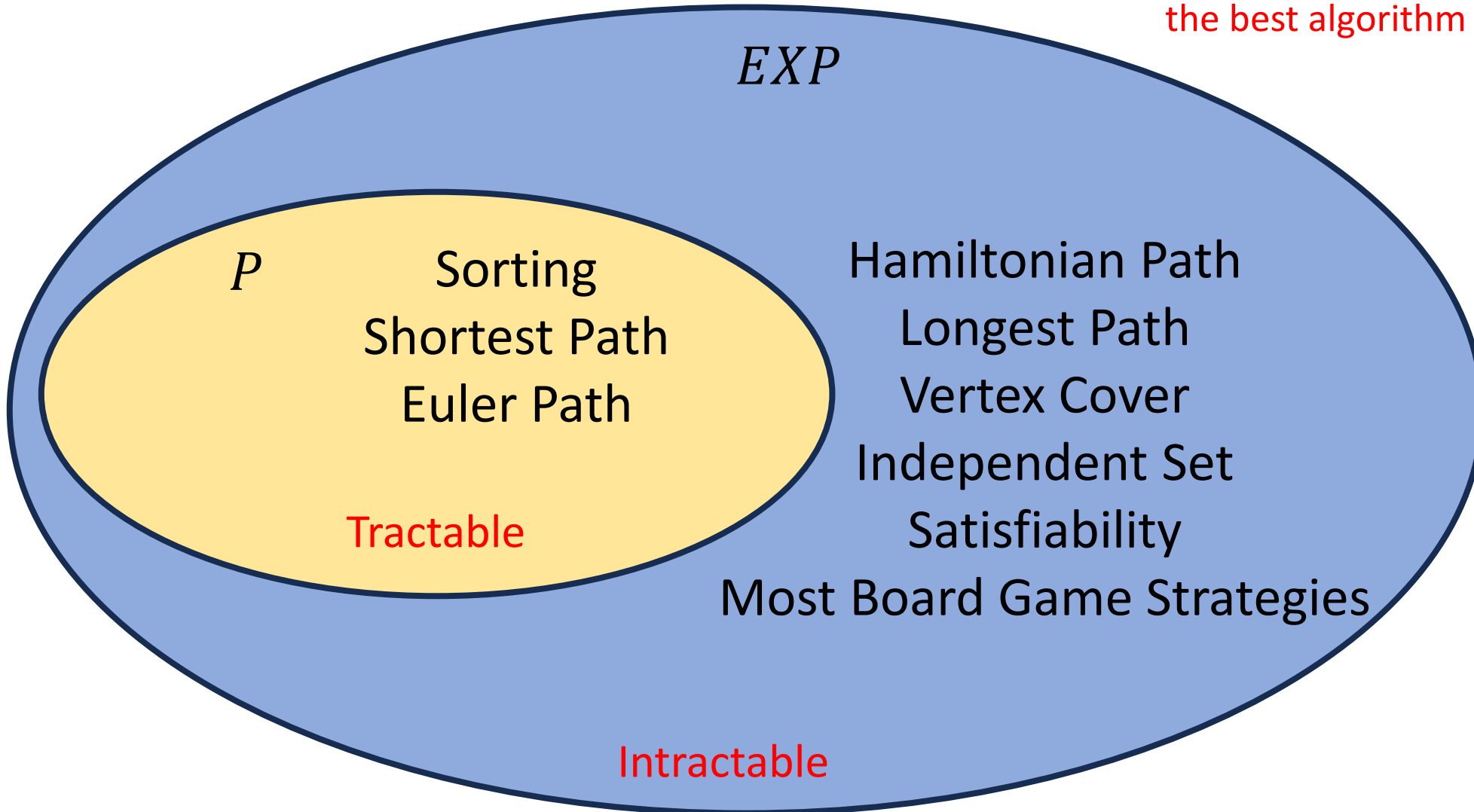
The intractable ones are the problems within EXP but NOT P



Members

Important!

Some of the problems listed in *EXP* could also be members of *P*
Since membership is determined by a problem's *most* efficient algorithm, knowing if a problem belongs to *P* requires knowing the best algorithm possible!



Studying Complexity and Tractability

- Organizing problems into complexity classes helps us to reason more carefully and flexibly about tractability
- The goal for each problem is to either
 - Find an efficient algorithm if it exists
 - i.e. show it belongs to P
 - Prove that no efficient algorithm exists
 - i.e. show it does not belong to P
- Complexity classes allow us to reason about sets of problems at a time, rather than each problem individually
 - If we can find more precise classes to organize problems into, we might be able to draw conclusions about the entire class
 - It may be easier to show a problem belongs to class C than to P , so it may help to show that $C \subseteq P$

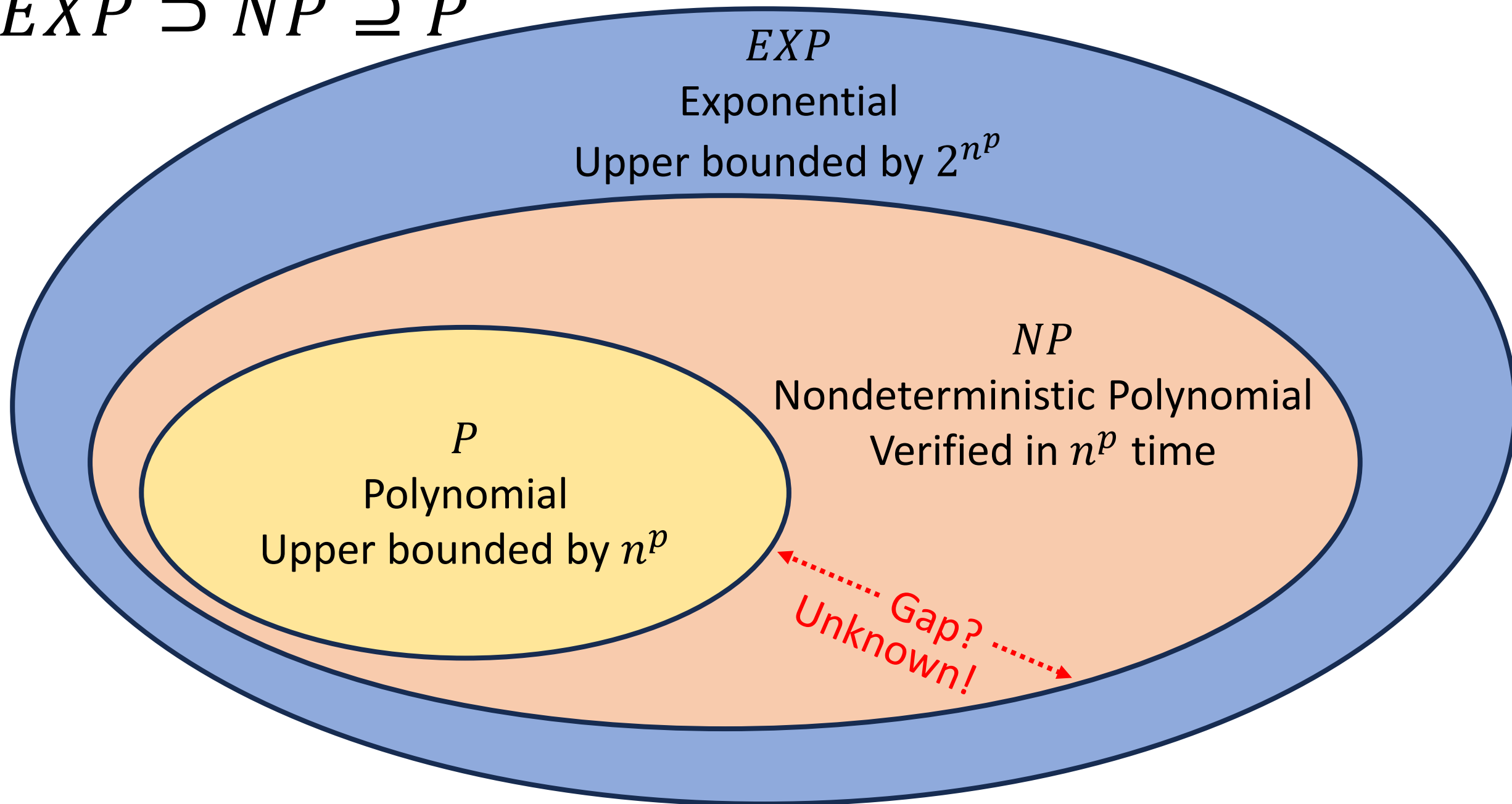
Some problems in *EXP* seem “easier”

- There are some problems that we do not have polynomial time algorithms to solve, but provided answers are easy to check
- Hamiltonian Path:
 - It’s “hard” to look at a graph and determine whether it has a Hamiltonian Path
 - It’s “easy” to look at a graph and a candidate path together and determine whether THAT path is a Hamiltonian Path
 - It’s easy to **verify** whether a given path is a Hamiltonian path

Class NP

- NP
 - The set of problems for which a candidate solution can be verified in polynomial time
 - Stands for “Non-deterministic Polynomial”
 - Corresponds to algorithms that can guess a solution (if it exists), that solution is then verified to be correct in polynomial time
 - Can also think of as allowing a special operation that allows the algorithm to magically guess the right choice at each step of an exhaustive search
- $P \subseteq NP$
 - Why?

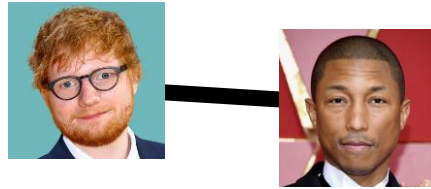
$$EXP \supset NP \supseteq P$$



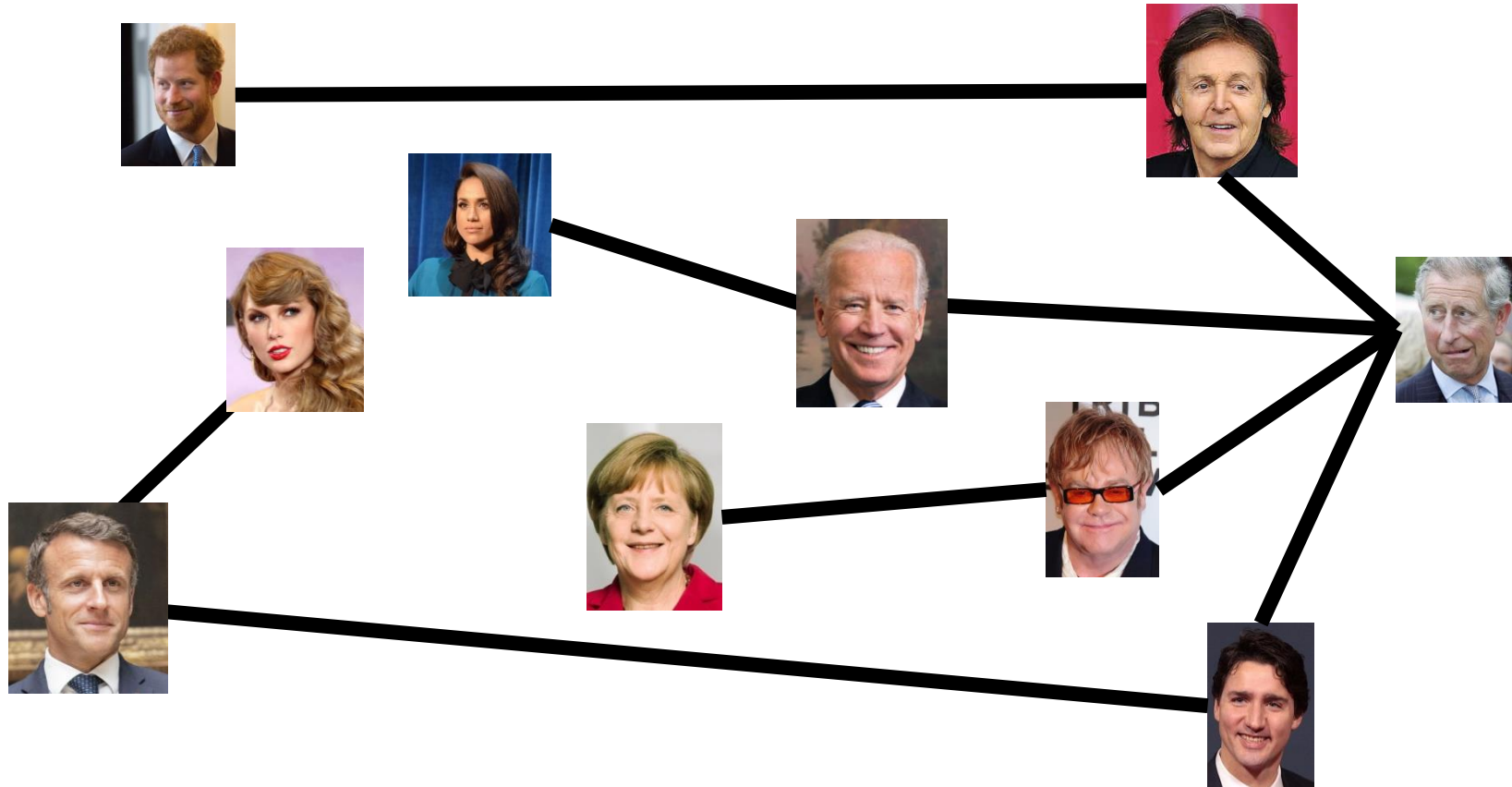
Solving and Verifying Hamiltonian Path

- Give an algorithm to solve Hamiltonian Path
 - Input: $G = (V, E)$
 - Output: True if G has a Hamiltonian Path
 - Algorithm: Check whether each permutation of V is a path.
 - Running time: $|V|!$, so does not show whether it belongs to P
- Give an algorithm to verify Hamiltonian Path
 - Input: $G = (V, E)$ and a sequence of nodes
 - Output: True if that sequence of nodes is a Hamiltonian Path
 - Algorithm:
 - Check that each node appears in the sequence exactly once
 - Check that the sequence is a path
 - Running time: $O(V \cdot E)$, so it belongs to NP

Party Problem



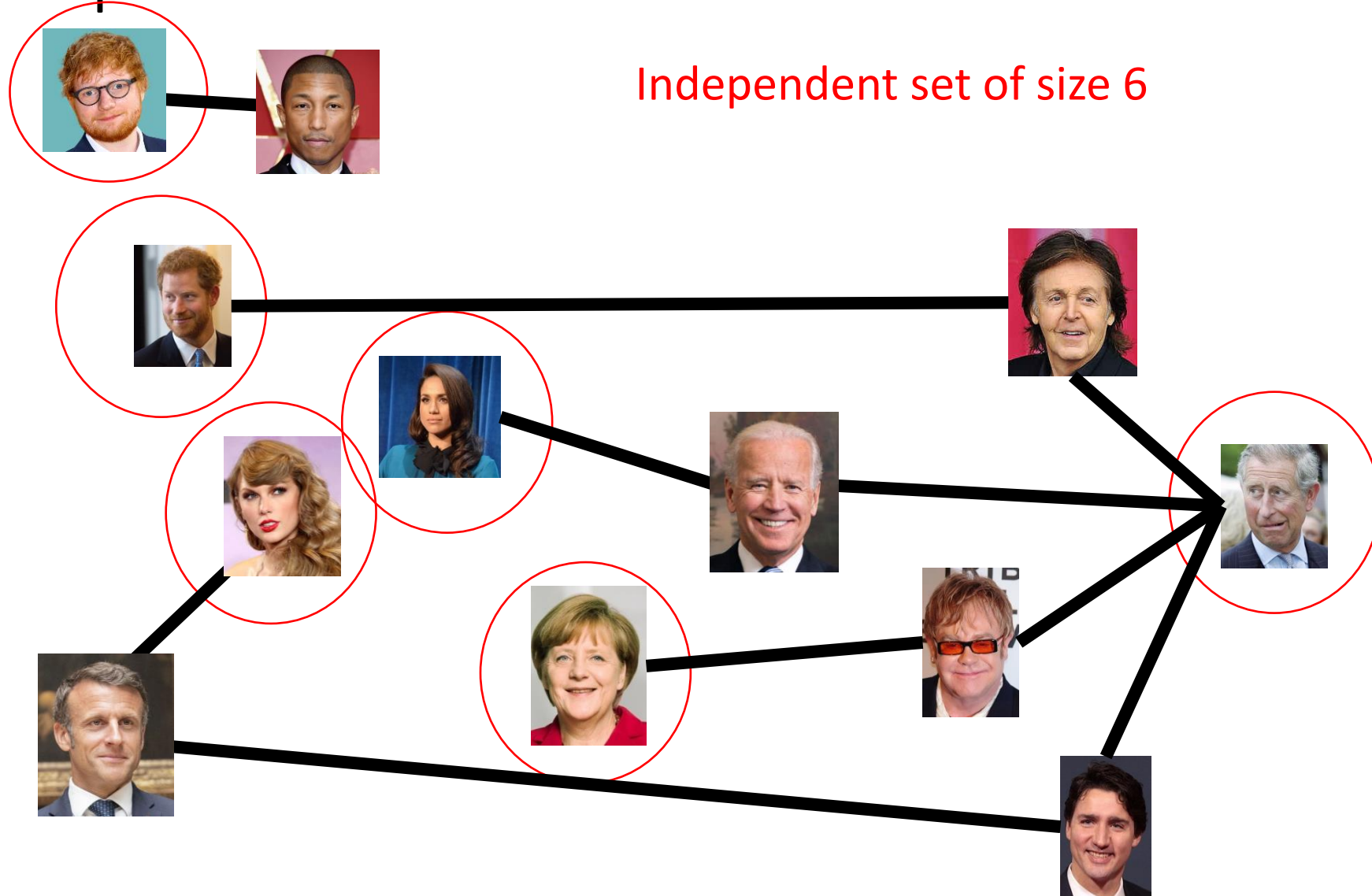
Draw Edges between people who don't get along
How many people can I invite to a party if everyone must get along?



Independent Set

- Independent set:
 - $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Independent Set Problem:
 - Given a graph $G = (V, E)$ and a number k , determine whether there is an independent set S of size k

Example

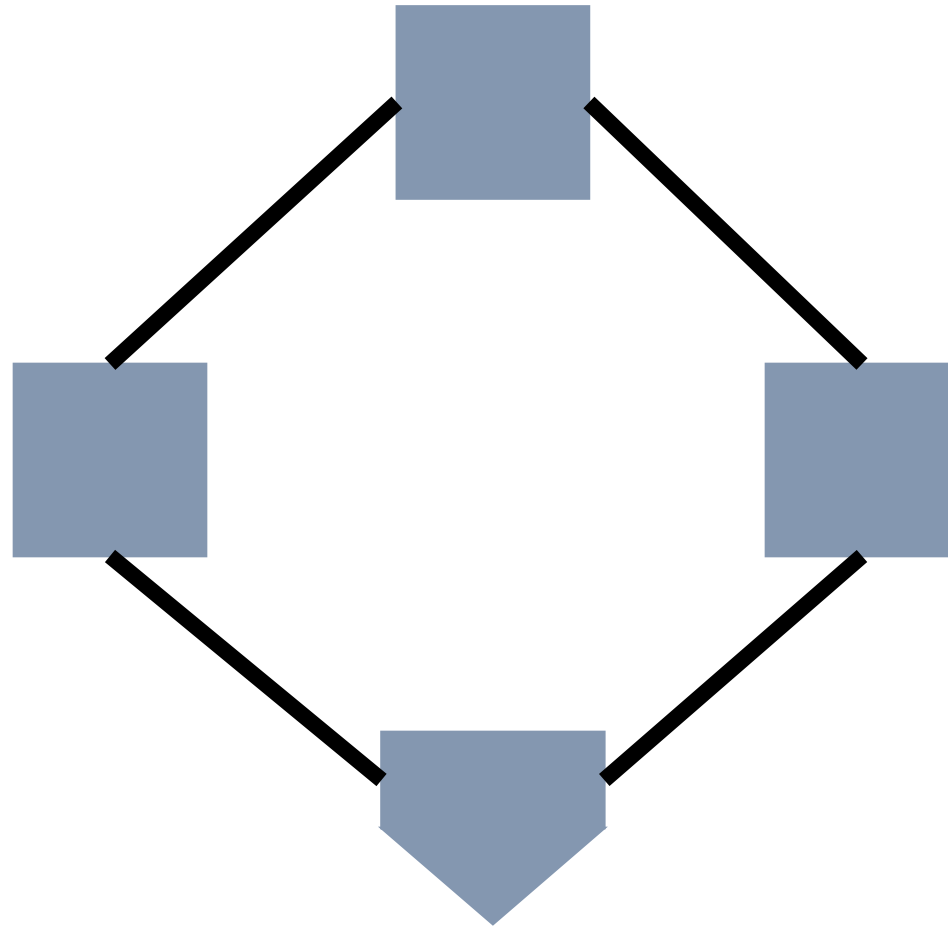


Independent set of size 6

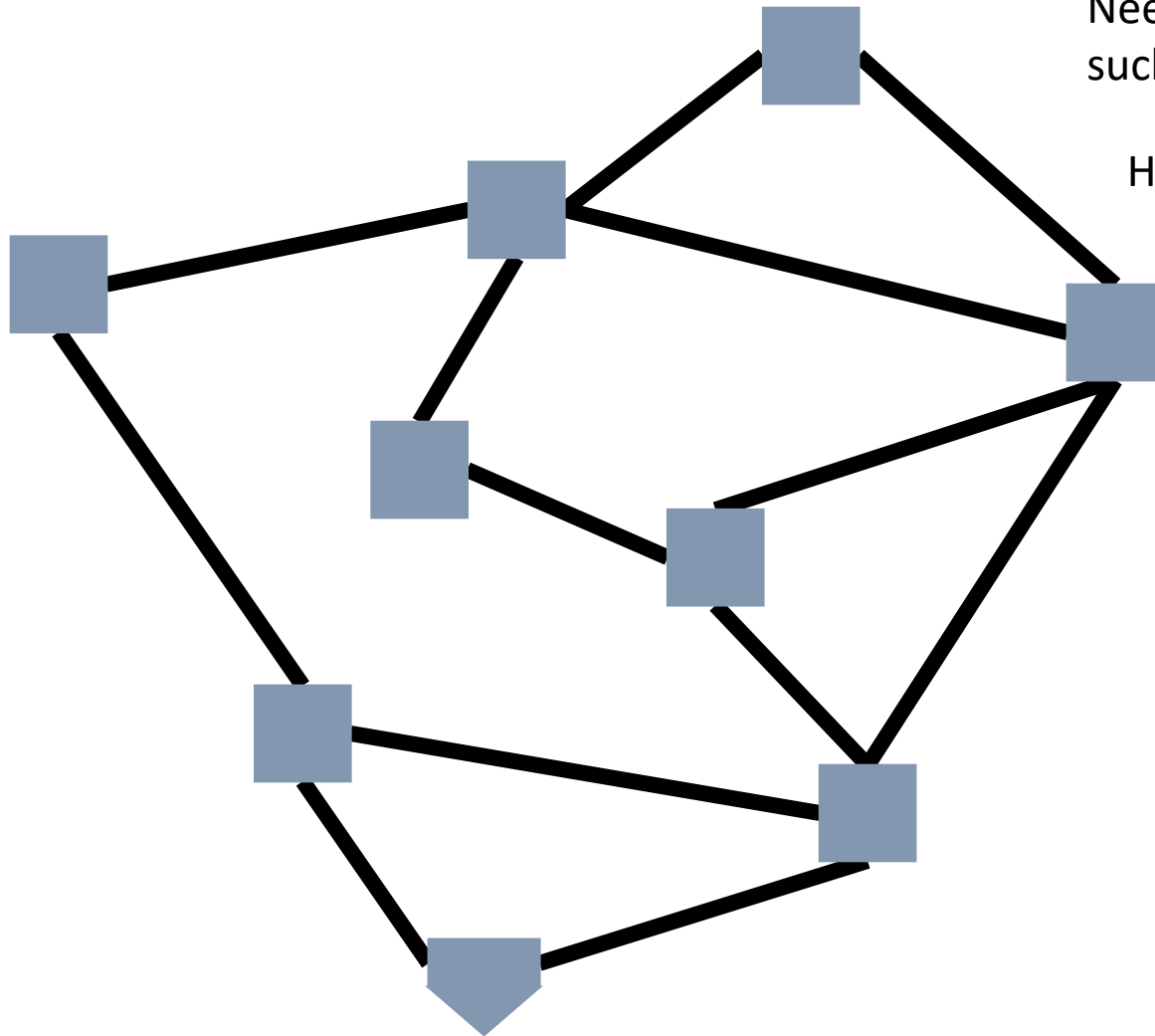
Solving and Verifying Independent Set

- Give an algorithm to solve independent set
 - Input: $G = (V, E)$ and a number k
 - Output: True if G has an independent set of size k
- Give an algorithm to verify independent set
 - Input: $G = (V, E)$, a number k , and a set $S \subseteq V$
 - Output: True if S is an independent set of size k

Generalized Baseball



Generalized Baseball



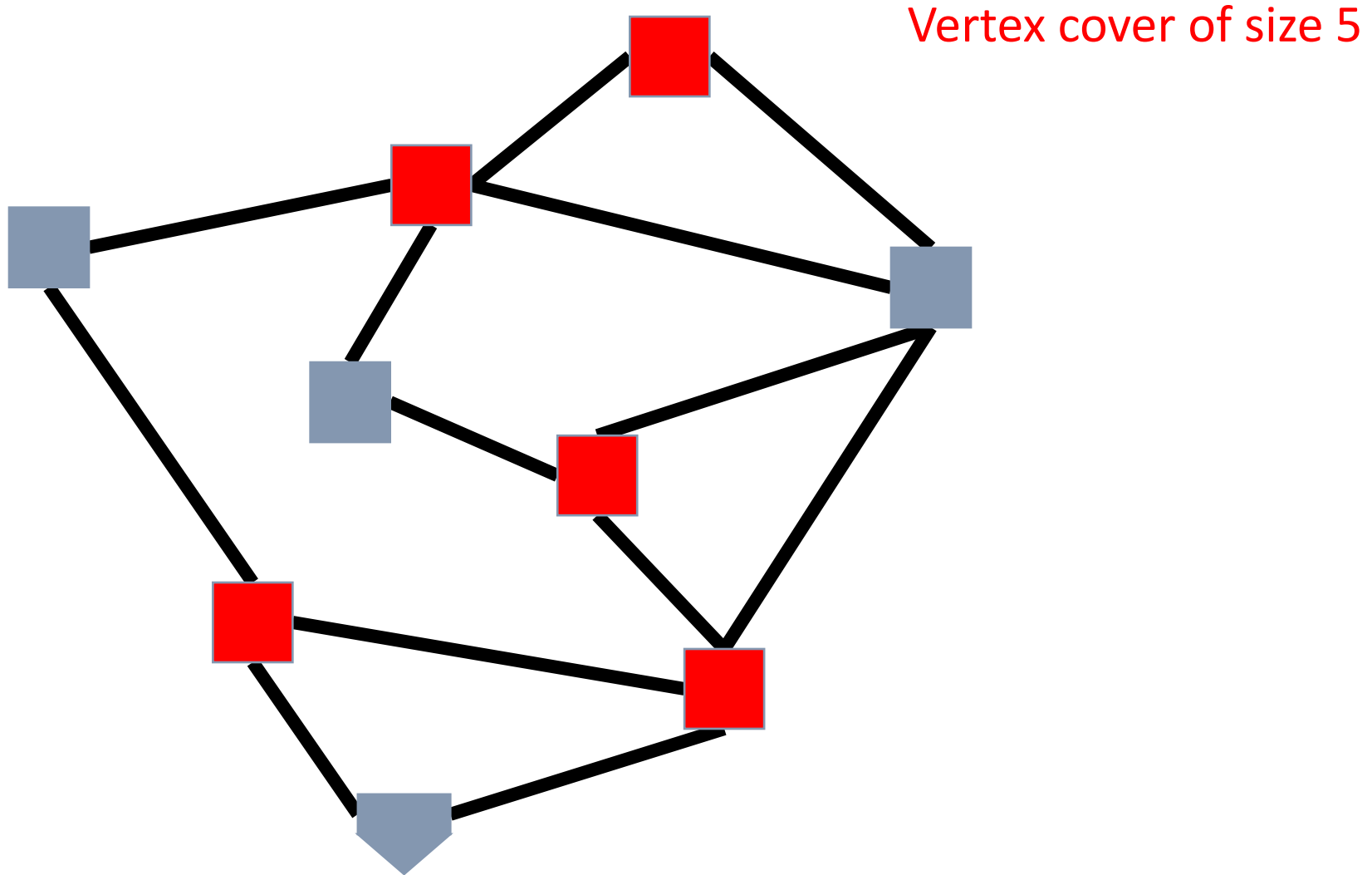
Need to place defenders on bases such that every edge is defended

How many defenders would suffice?

Vertex Cover

- Vertex Cover:
 - $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Vertex Cover Problem:
 - Given a graph $G = (V, E)$ and a number k , determine if there is a vertex cover C of size k

Example

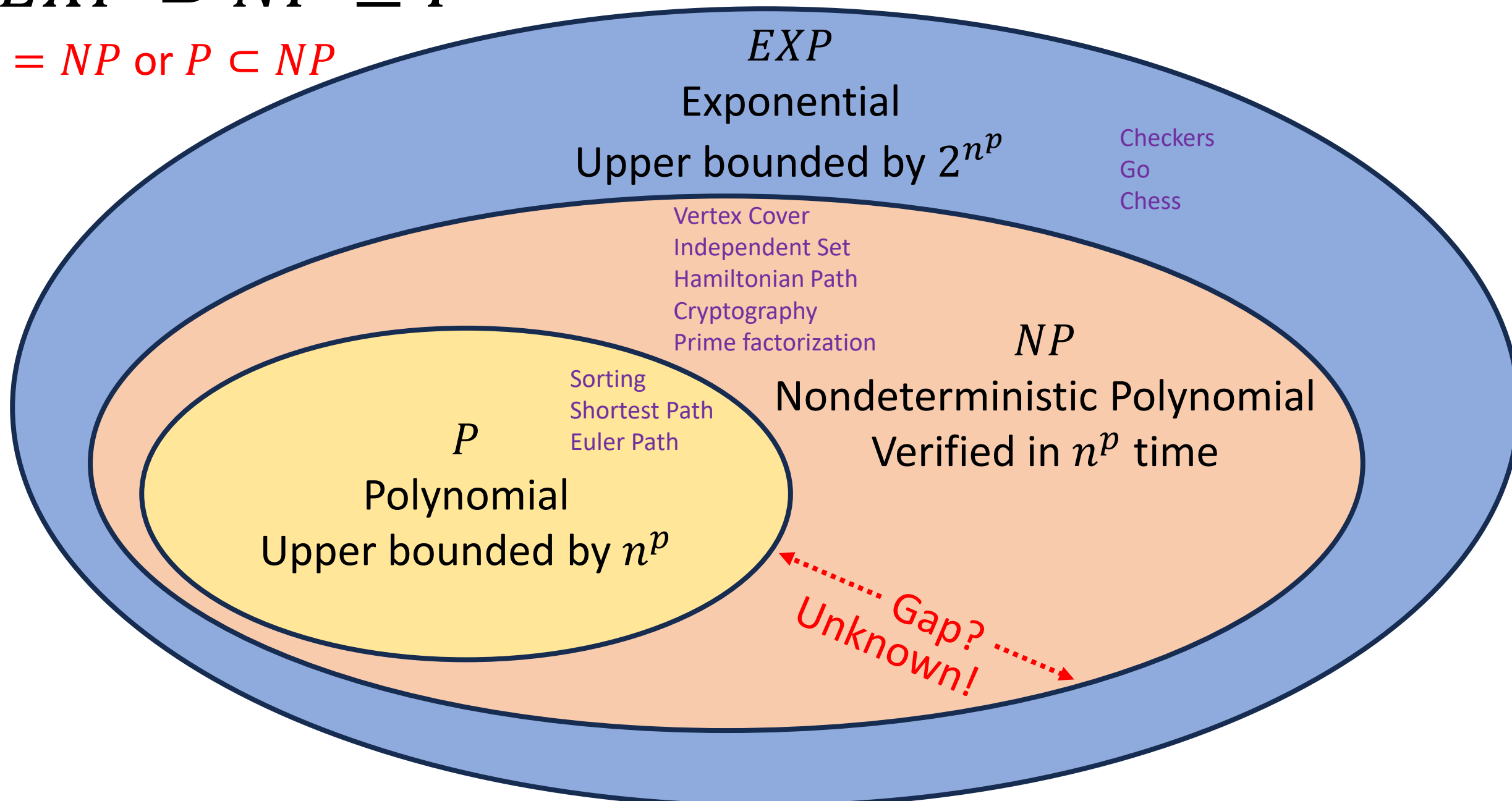


Solving and Verifying Vertex Cover

- Give an algorithm to solve vertex cover
 - Input: $G = (V, E)$ and a number k
 - Output: True if G has a vertex cover of size k
- Give an algorithm to verify vertex cover
 - Input: $G = (V, E)$, a number k , and a set $S \subseteq E$
 - Output: True if S is a vertex cover of size k

$$EXP \supset NP \supseteq P$$

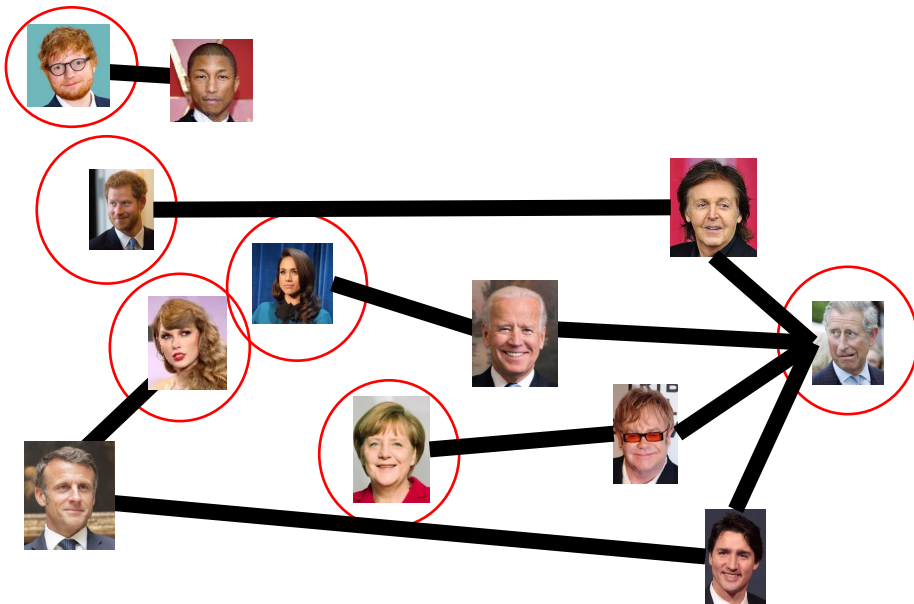
$P = NP$ or $P \subset NP$



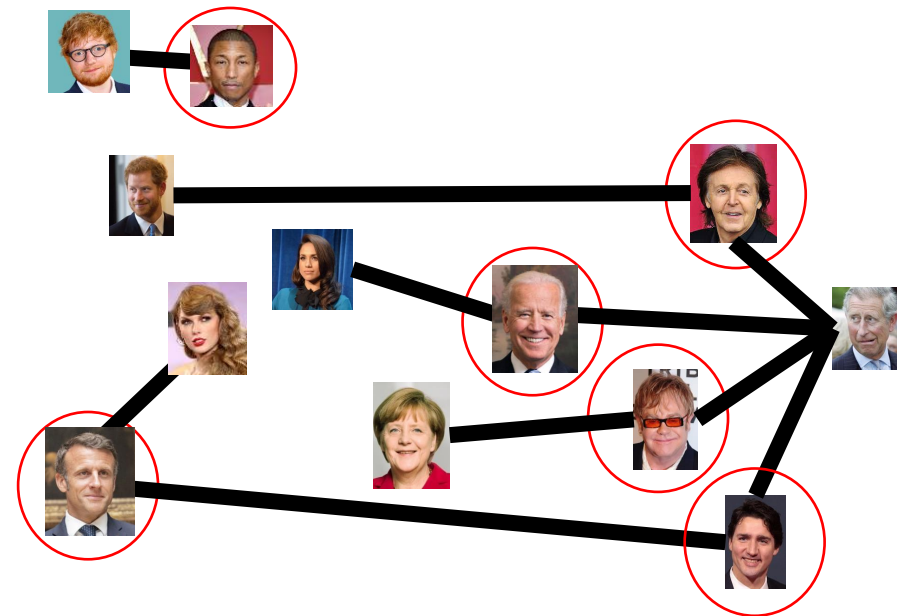
Way Cool!

S is an independent set of G iff $V - S$ is a vertex cover of G

Independent Set



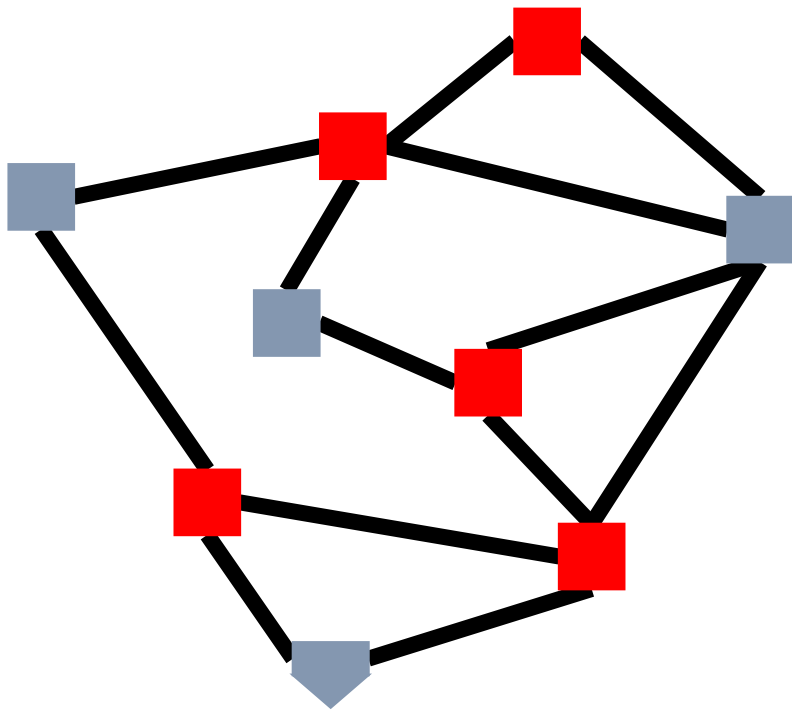
Vertex Cover



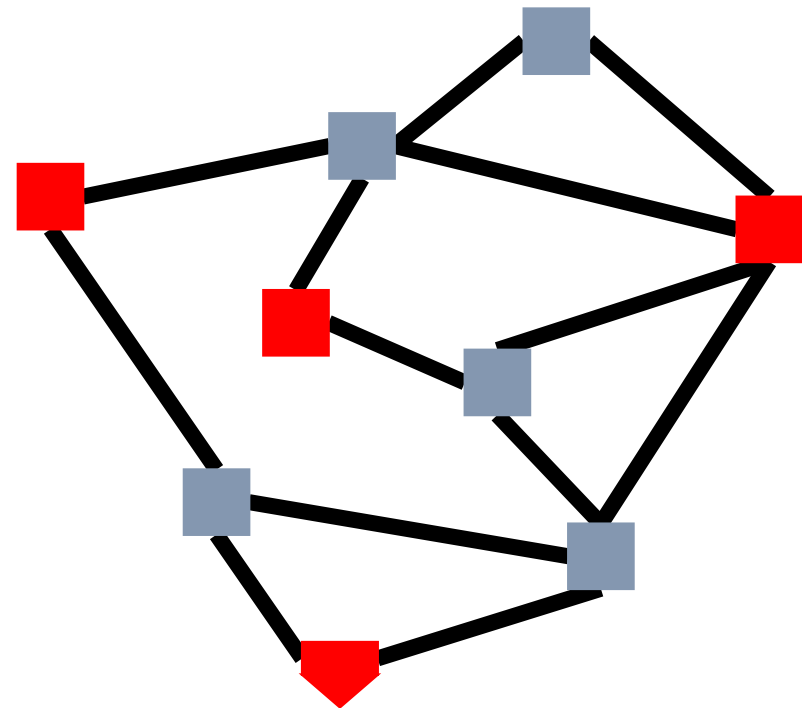
Way Cool!

S is an independent set of G iff $V - S$ is a vertex cover of G

Vertex Cover



Independent Set



Solving Vertex Cover and Independent Set

- Algorithm to solve vertex cover
 - Input: $G = (V, E)$ and a number k
 - Output: True if G has a vertex cover of size k
 - Check if there is an Independent Set of G of size $|V| - k$
- Algorithm to solve independent set
 - Input: $G = (V, E)$ and a number k
 - Output: True if G has an independent set of size k
 - Check if there is a Vertex Cover of G of size $|V| - k$

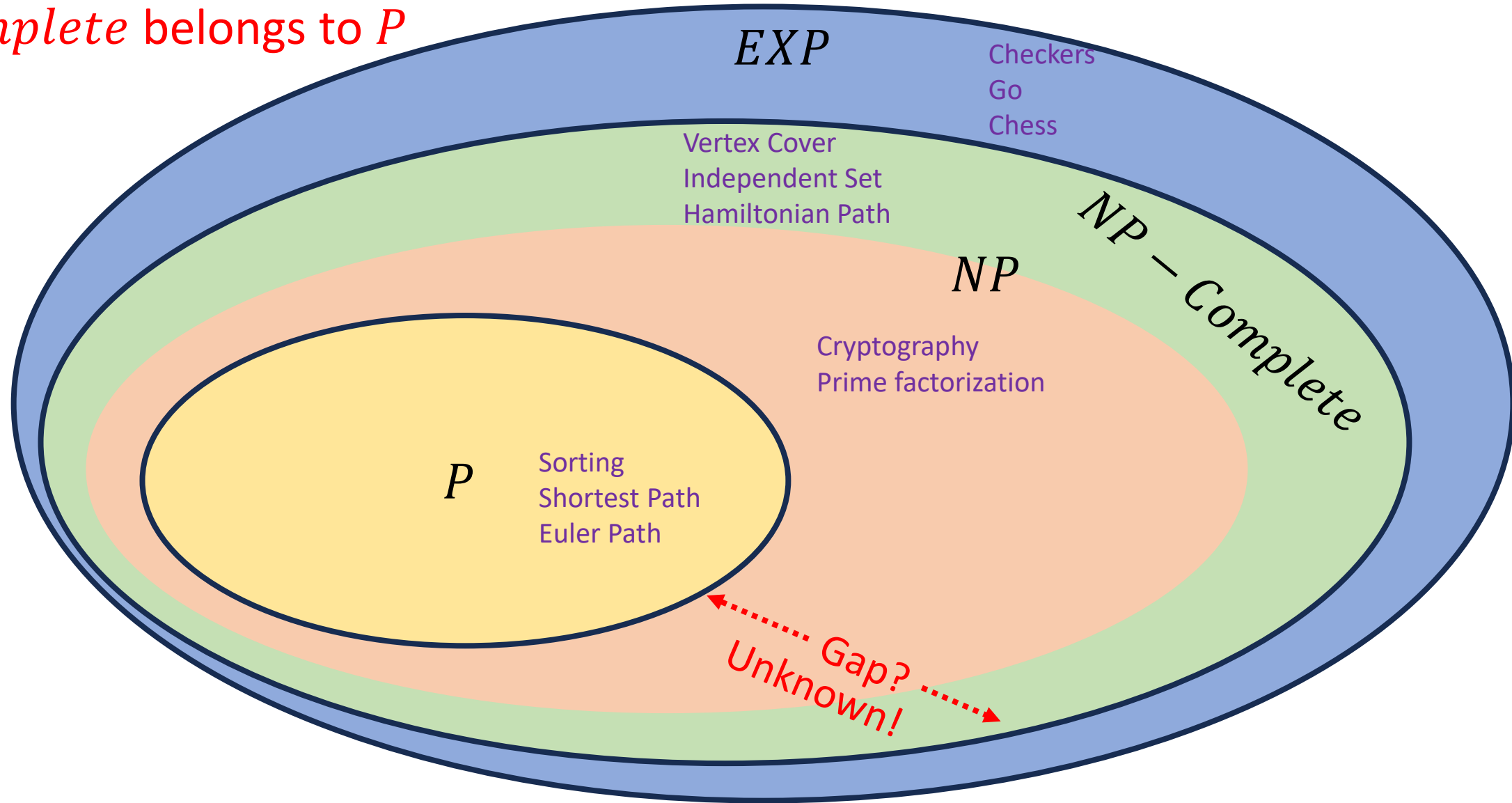
Either both problems belong to P , or else neither does!

NP-Complete

- A set of “together they stand, together they fall” problems
- The problems in this set either all belong to P , or none of them do
- Intuitively, the “hardest” problems in NP
- Collection of problems from NP that can all be “transformed” into each other in polynomial time
 - Like we could transform independent set to vertex cover, and vice-versa
 - We can also transform vertex cover into Hamiltonian path, and Hamiltonian path into independent set, and ...

$EXP \supset NP - Complete \supseteq NP \supseteq P$

$P = NP$ iff some problem from
 $NP - Complete$ belongs to P



Overview

- Problems not belonging to P are considered intractable
- The problems within NP have some properties that make them seem like they might be tractable, but we've been unsuccessful with finding polynomial time algorithms for many
- The class $NP - Complete$ contains problems with the properties:
 - All members are also members of NP
 - All members of NP can be transformed into every member of $NP - Complete$
 - Therefore if any one member of $NP - Complete$ belongs to P , then $P = NP$

Why should YOU care?

- If you can find a polynomial time algorithm for any *NP – Complete* problem then:
 - You will win \$1million
 - You will win a Turing Award
 - You will be world famous
 - You will have done something that no one else on Earth has been able to do in spite of the above!
- If you are told to write an algorithm a problem that is *NP – Complete*
 - You can tell that person everything above to set expectations
 - Change the requirements!
 - **Approximate the solution:** Instead of finding a path that visits every node, find a path that visits at least 75% of the nodes
 - **Add Assumptions:** problem might be tractable if we can assume the graph is acyclic, a tree
 - **Use Heuristics:** Write an algorithm that’s “good enough” for small inputs, ignore edge cases