CSE 332 Summer 2024
Lecture 16: Graphs

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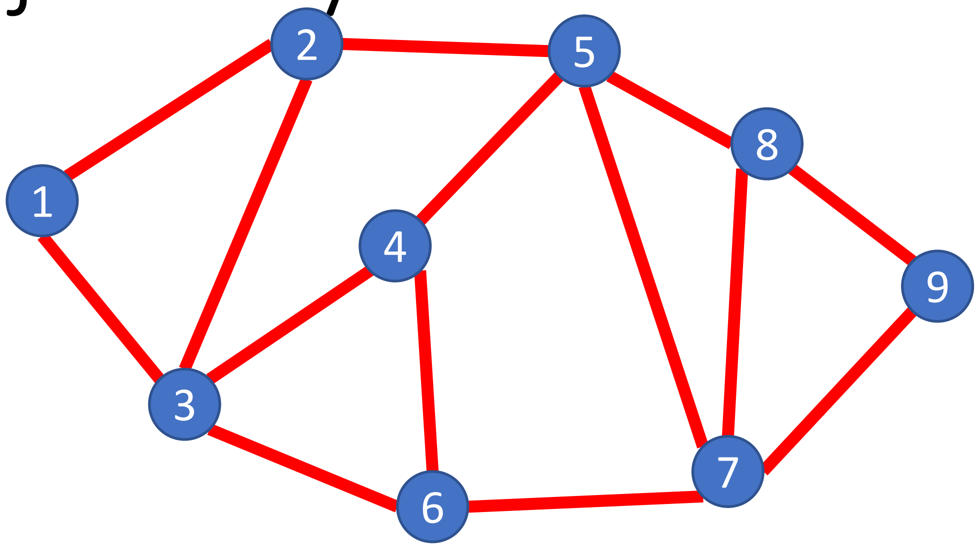
http://www.cs.uw.edu/332

Adjacency List



<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n+m)$

Add Edge (v,w): $\Theta(\deg(v))$

Remove Edge (v,w): $\Theta(\deg(v))$

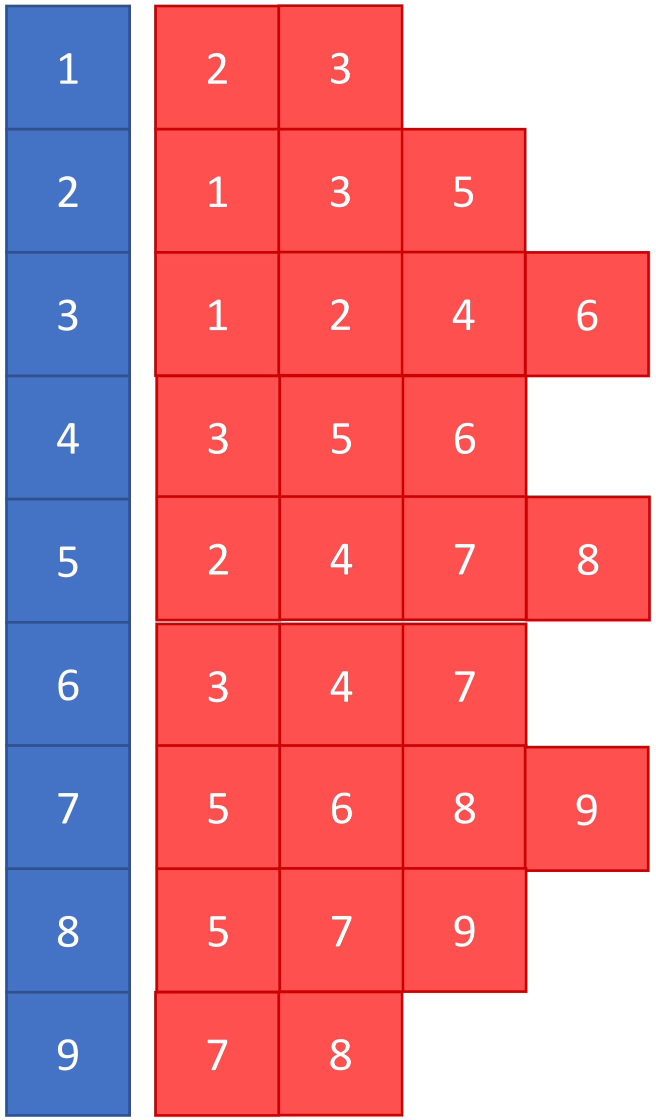
Check if Edge (v,w) Exists: $\Theta(\deg(v))$

Get Neighbors (incoming): $\Theta(n+m)$

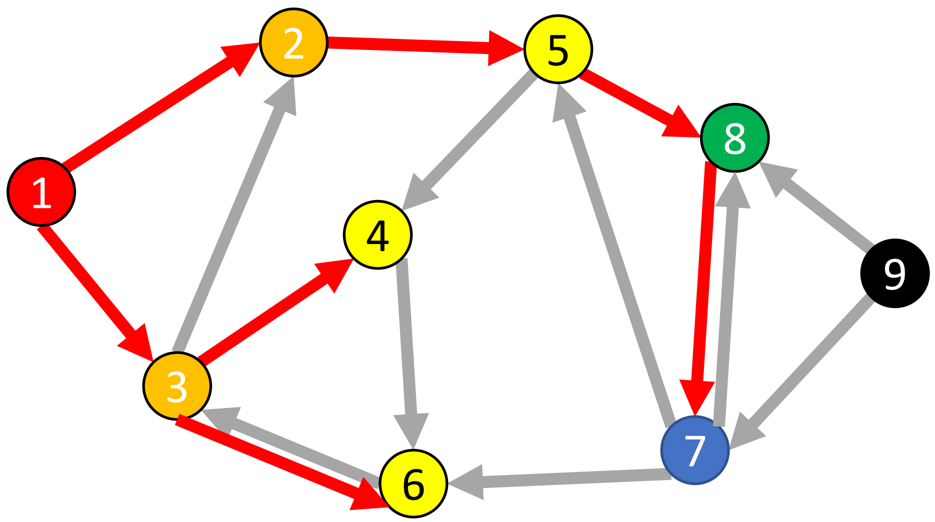
Get Neighbors (outgoing): $\Theta(\deg(v))$

$|V| = n$

$|E| = m$



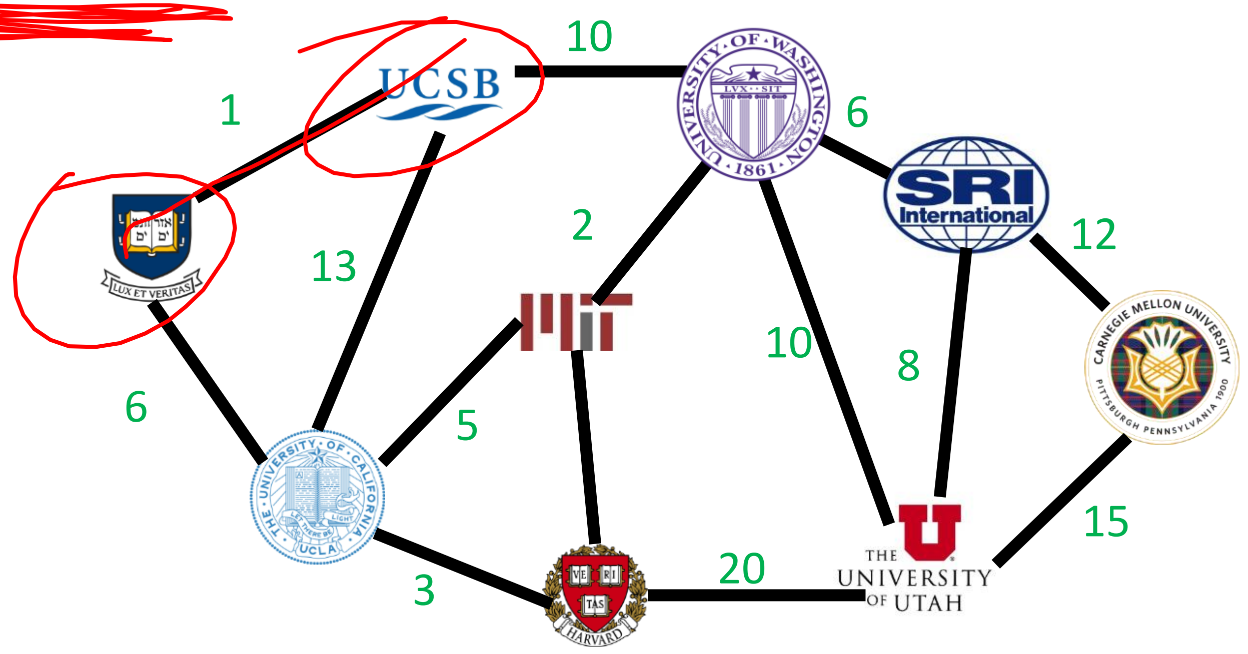
Shortest Path (unweighted)



Idea: when it's seen, remember its "layer" depth!

```
int shortestPath(graph, s, t){
    found = new Queue();
    layer = 0;
    found.enqueue(s);
    mark s as "visited";
    While (!found.isEmpty()){
        current = found.dequeue();
        layer = depth of current;
        for (v : neighbors(current)){
            if (! v marked "visited"){
                mark v as "visited";
                depth of v = layer + 1;
                found.enqueue(v);
            }
        }
    }
    return depth of t;
}
```

Single-Source Shortest Path



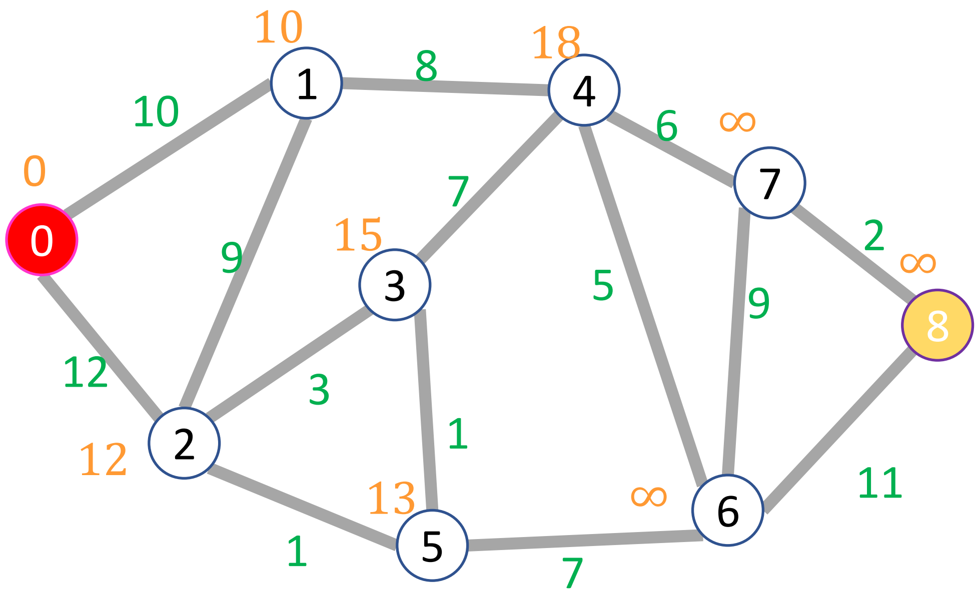
Find the quickest way to get from UW to each of these other places

Given a graph $G = (V, E)$ and a start node $s \in V$, for each $v \in V$ find the least-weight path from $s \to v$ (call this weight $\delta(s, v)$)

(assumption: all edge weights are positive)

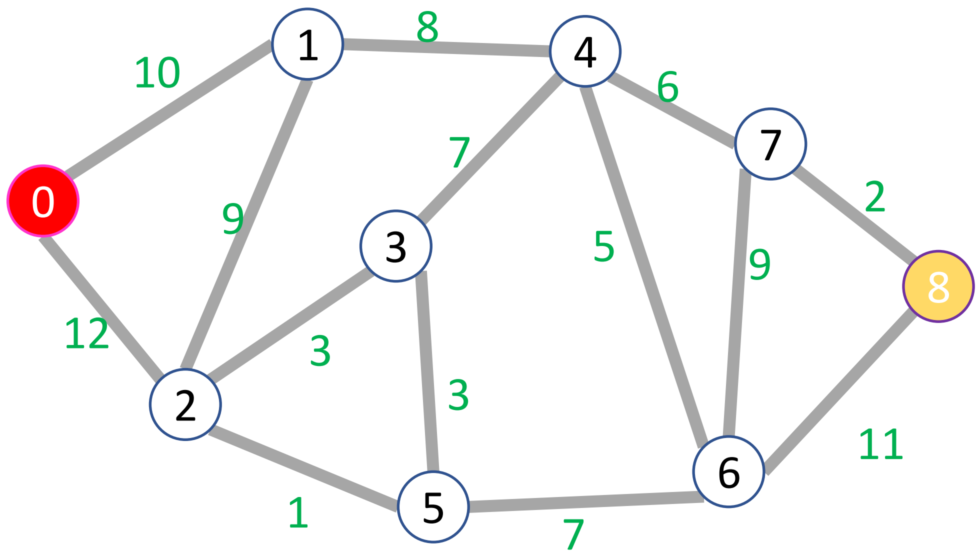
Dijkstra's Algorithm

- Input: graph with **no negative edge weights**, start node s, end node t
- Behavior: Start with node s, repeatedly go to the incomplete node "nearest" to s, stop when
- Output:
  - Distance from start to end
  - Distance from start to every node



Dijkstra's Algorithm

```
int dijkstras(graph, start, end){
        distances = [∞, ∞, ∞,…];  // one index per node
        done = [False,False,False,…];  // one index per node
        PQ = new minheap();
        PQ.insert(0, start);  // priority=0, value=start
        distances[start] = 0;
        while (!PQ.isEmpty){
                current = PQ.extract();
                done[current] = true;
                for (neighbor : current.neighbors){
                        if (!done[neighbor]){
                                new_dist = distances[current]+weight(current,neighbor);
                                if(distances[neighbor] == ∞){
                                        distances[neighbor] = new_dist;
                                        PQ.insert(new_dist, neighbor);
                                }
                                if (new_dist < distances[neighbor]){
                                        distances[neighbor] = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return distances[end]
}
```

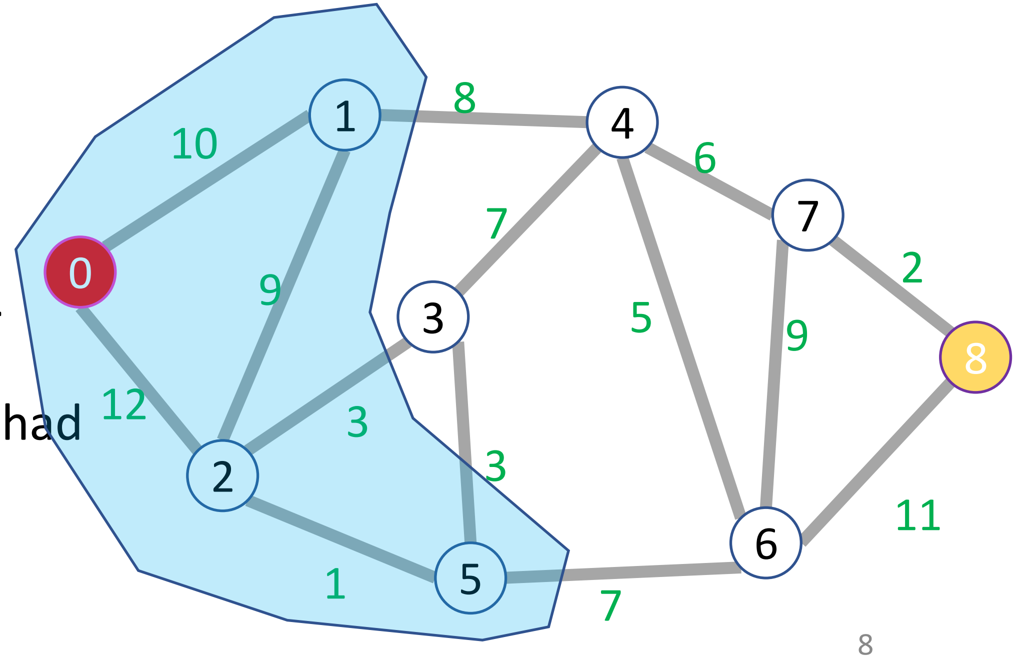


Dijkstra's Algorithm: Running Time

- How many total priority queue operations are necessary?
  - How many times is each node added to the priority queue?
    - At most once
  - How many times might a node's priority be changed?
    - Indegree of that node
- What's the running time of each priority queue operation?
  - $\log |V|$
- Overall running time:
  - $|\mathrm{V}| \log |V| + |E| \log |V|$
  - $\Theta(|E| \log |V|)$

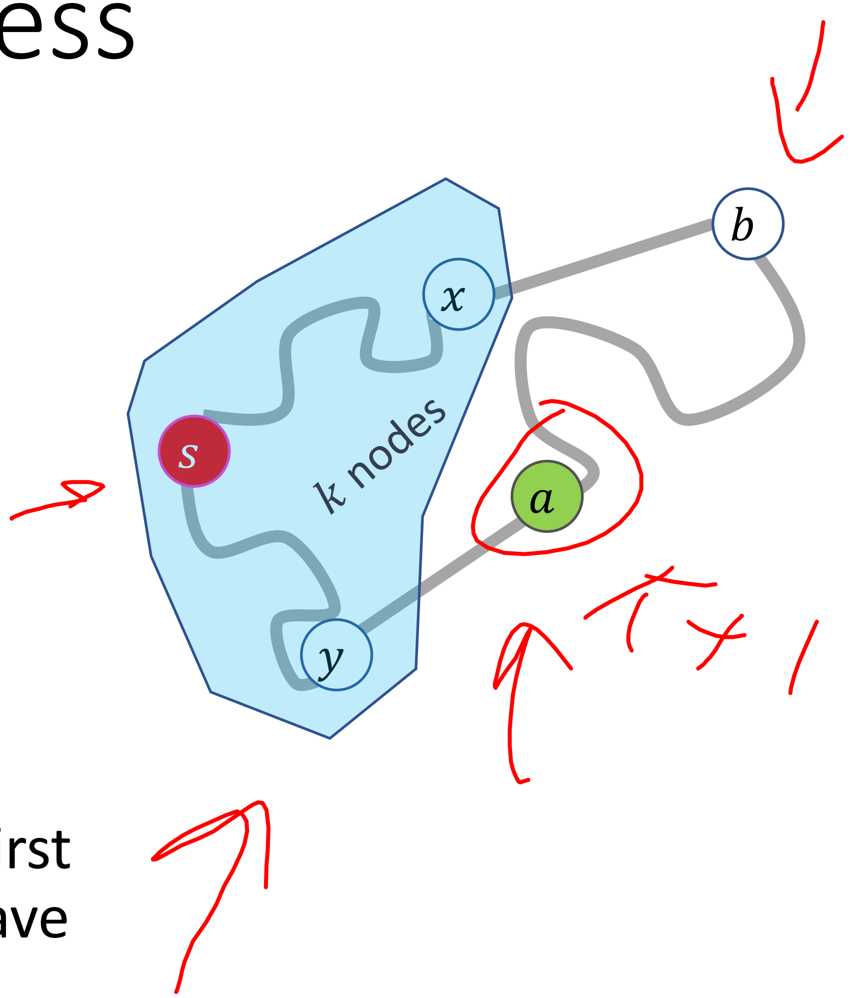
Dijkstra's Algorithm: Correctness

- Claim: when a node is removed from the priority queue, we have found its shortest path
- Induction over number of completed nodes
- Base Case:
  - When 1 node has been removed, its priority matches the weight of its shortest path
  - That one node is the start, it had priority 0
  - 0 is the cost of the shortest path from start to start
- Inductive Step:
  - Assume that k nodes have been removed so far
  - Assume that for all them, their priority matched their shortest path cost
  - Show that the next node that's removed $(k+1)$ also had its priority match its shortest path cost.



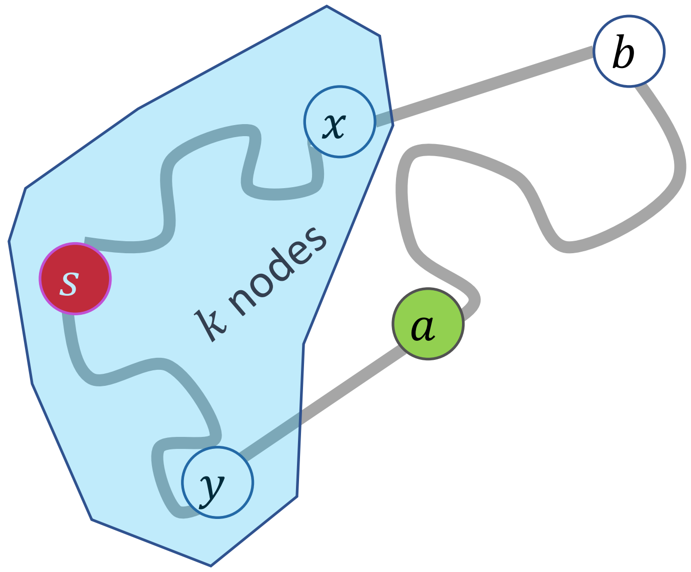
Dijkstra's Algorithm: Correctness

- Claim: when a node is removed from the priority queue, its distance is that of the shortest path
- Induction over number of completed nodes
- Base Case: Only the start node removed
  - It is indeed 0 away from itself
- Inductive Step:
  - If we have correctly found shortest paths for the first k nodes, then when we remove node $k+1$ we have found its shortest path



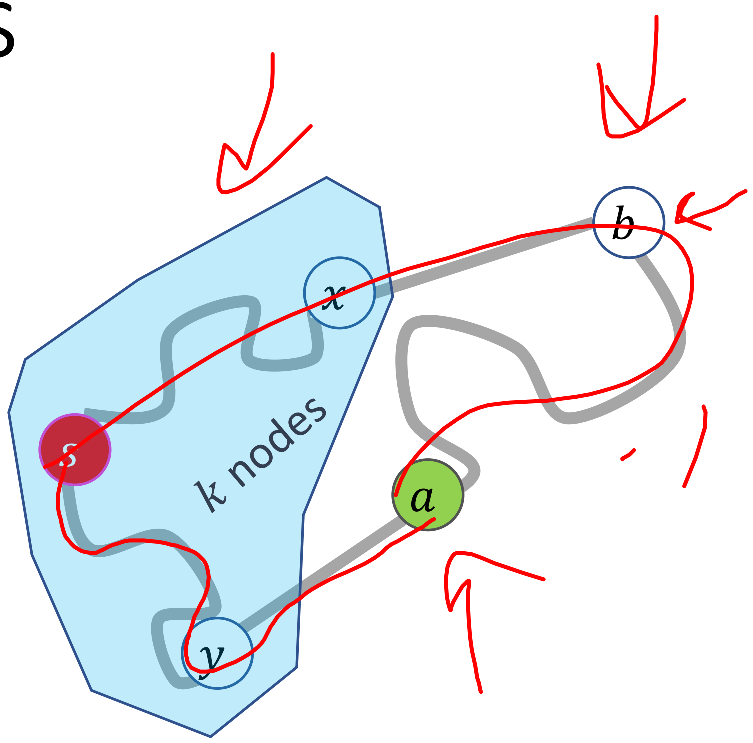
Dijkstra’s Algorithm: Correctness

- Suppose a is the next node removed from the priority queue. What do we know bout a?
  - We have a path from s to a
  - a has the lowest priority of all “incomplete” nodes, e.g. b
  - a has an edge with a already completed node



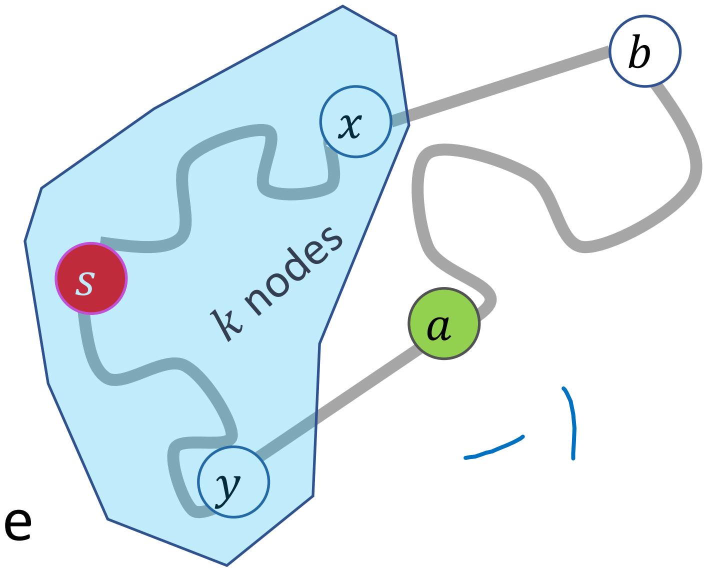
Dijkstra's Algorithm: Correctness

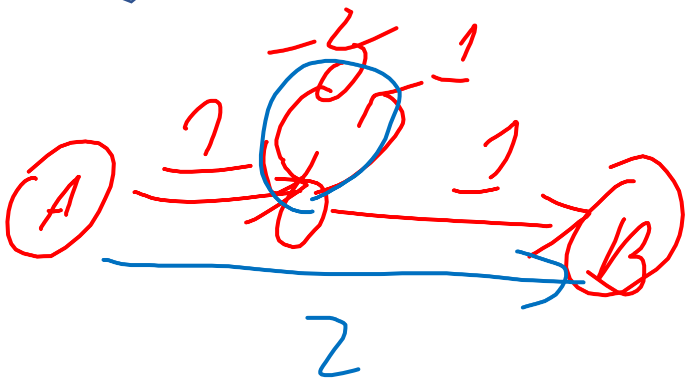
- Suppose a is the next node removed from the priority queue.
  - No other incomplete node has a shorter path discovered so far (e.g. b)
- Claim: no undiscovered path to a could be shorter
  - Consider any other incomplete node b that is 1 edge away from a complete node
  - a is the closest node that is one away from a complete node
  - Thus no path that includes b can be a shorter path to a
  - Therefore the shortest path to a must use only complete nodes, and therefore we have found it already!



Dijkstra’s Algorithm: Correctness

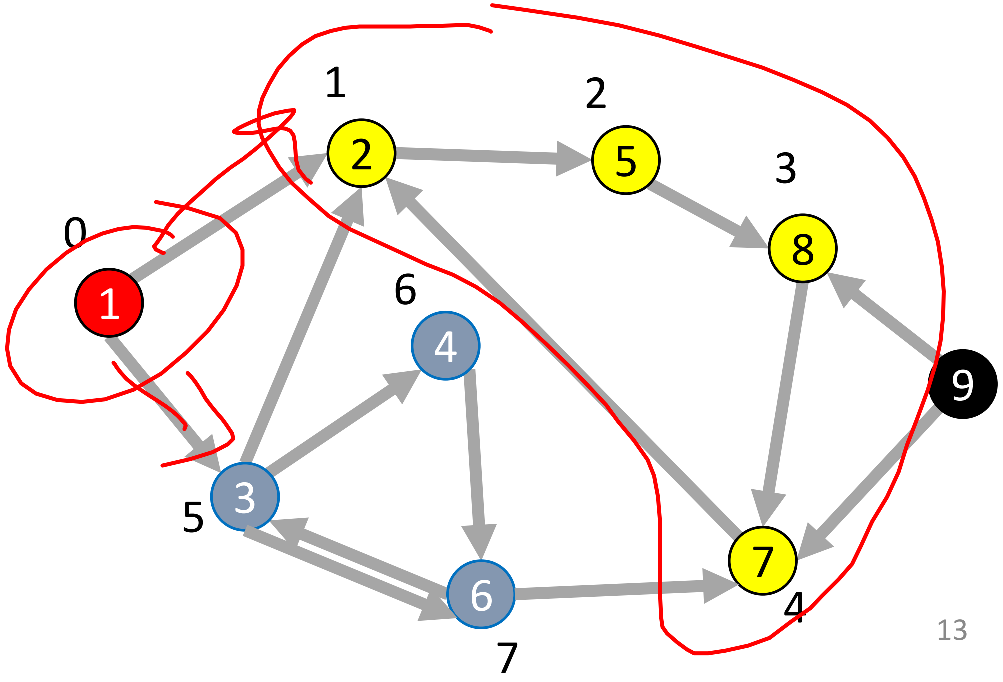
- Suppose a is the next node removed from the queue.
  - No other node incomplete node has a shorter path discovered so far
- Claim: no undiscovered path to a could be shorter
  - Consider any other incomplete node b that is 1 edge away from a complete node
  - a is the closest node that is one away from a complete node
  - Thus no path that includes b can be a shorter path to a
    - Only because no path from b to a can have negative weight!
  - Therefore the shortest path to a must use only complete nodes, and therefore we have found it already!





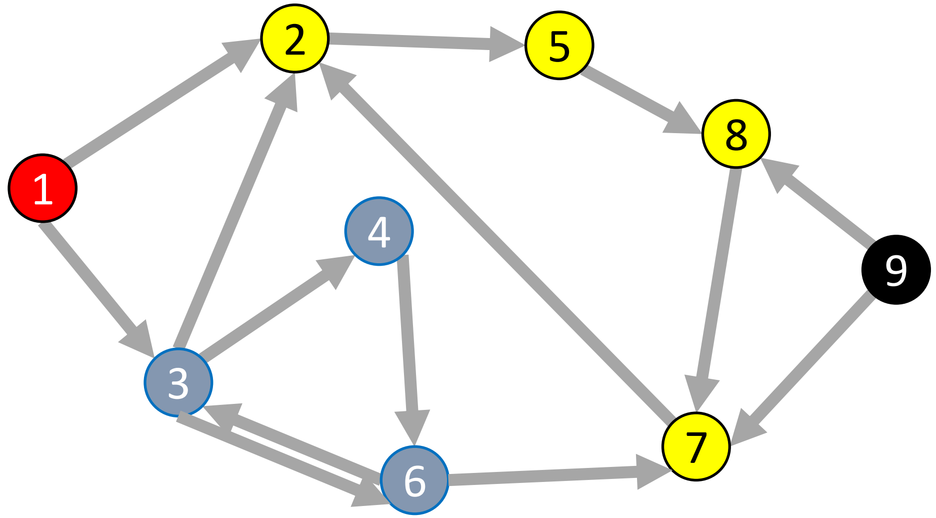
Depth-First Search

- Input: a node s
- Behavior: Start with node s, visit one neighbor of s, then all nodes reachable from that neighbor of s, then another neighbor of s,…
  - Before moving on to the second neighbor of s, visit everything reachable from the first neighbor of s
- Output:
  - Does the graph have a cycle?
  - A **topological sort** of the graph.



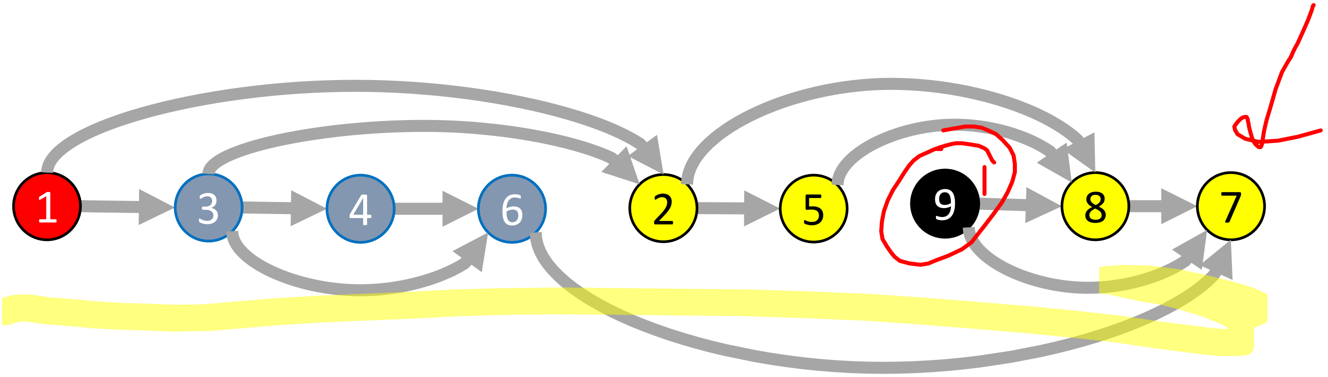
DFS Recursively (more common)

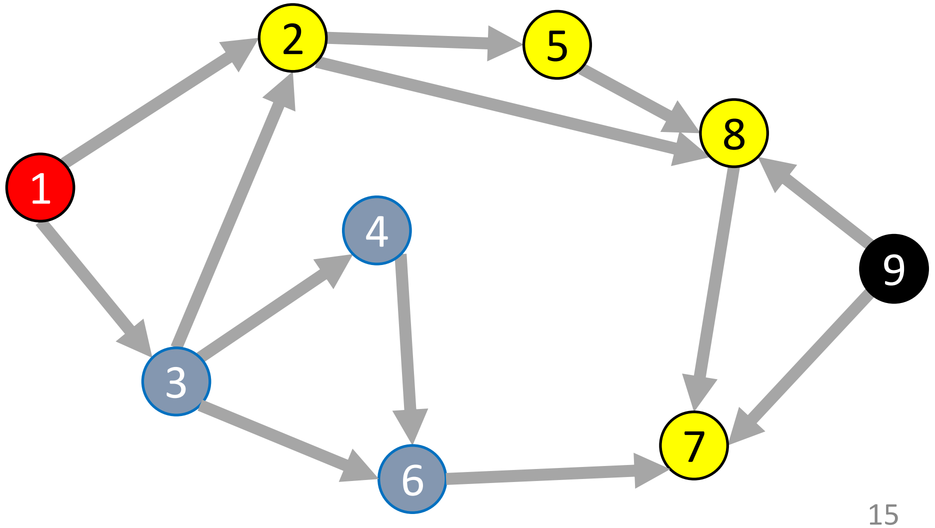
```
void dfs(graph, curr){
        mark curr as "visited";
        for (v : neighbors(current)){
                if (! v marked "visited"){
                        dfs(graph, v);
                }
        }
        mark curr as "done";
}
```



Topological Sort

- A Topological Sort of a **directed acyclic graph** $\boldsymbol{G = (V, E)}$ is a permutation of V such that if $(u, v) \in E$ then u is before v in the permutation

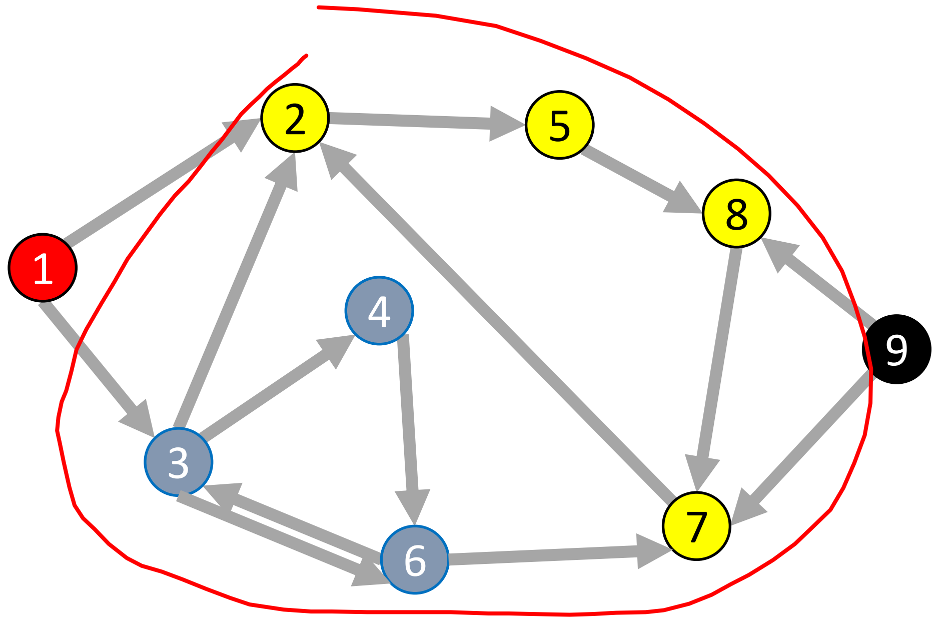




DFS Recursively

```
void dfs(graph, curr){
        mark curr as "visited";
        for (v : neighbors(current)){
                if (! v marked "visited"){
                        dfs(graph, v);
                }
        }
        mark curr as "done";
}
```

Idea: List in reverse order by "done" time



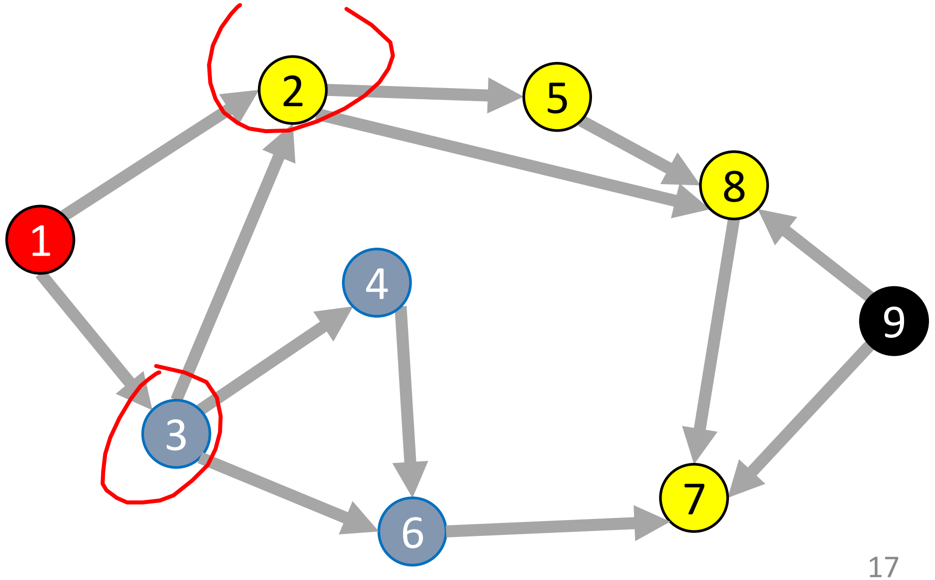
DFS: Topological sort

```
List topSort(graph){
        List<Node> finished = new List<>();
        for (Node v : graph.vertices){
                if (!v.visited){
                        finishTime(graph, v, finished);
                }
        }
        finished.reverse();
        return finished;
}

void finishTime(graph, curr, finished){
        curr.visited = true;
        for (Node v : curr.neighbors){
                if (!v.visited){
                        finishTime(graph, v, finished);
                }
        }
        finished.add(curr)
}
```

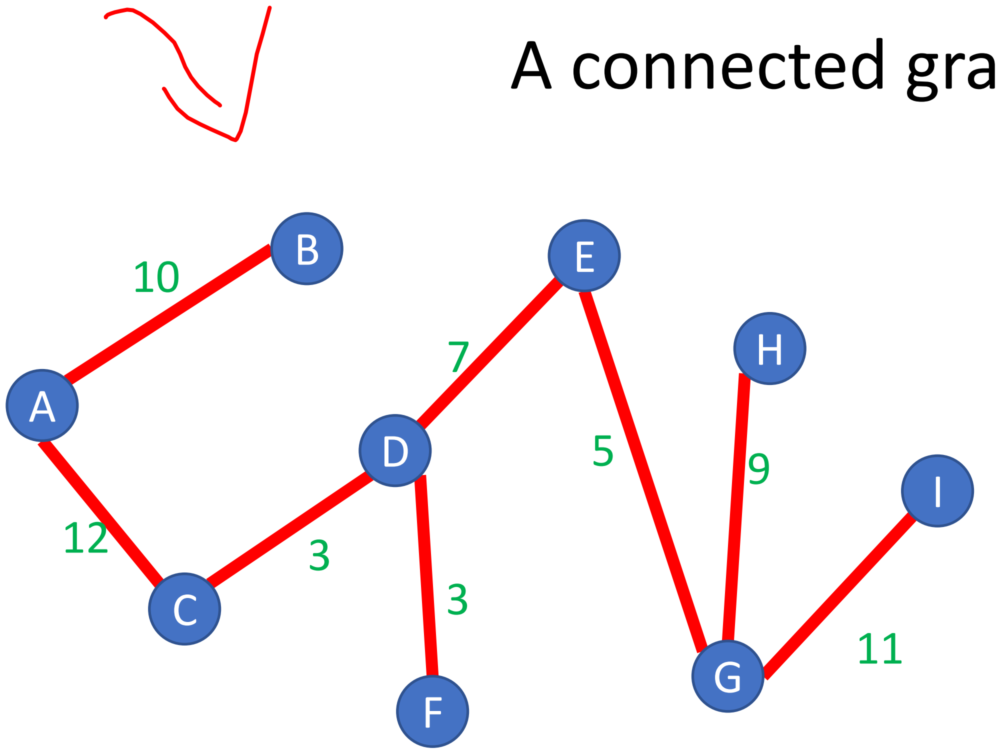
Idea: List in reverse order by “done” time

finished:									
	9	1	2	5	8	3	4	6	7



Definition: Tree

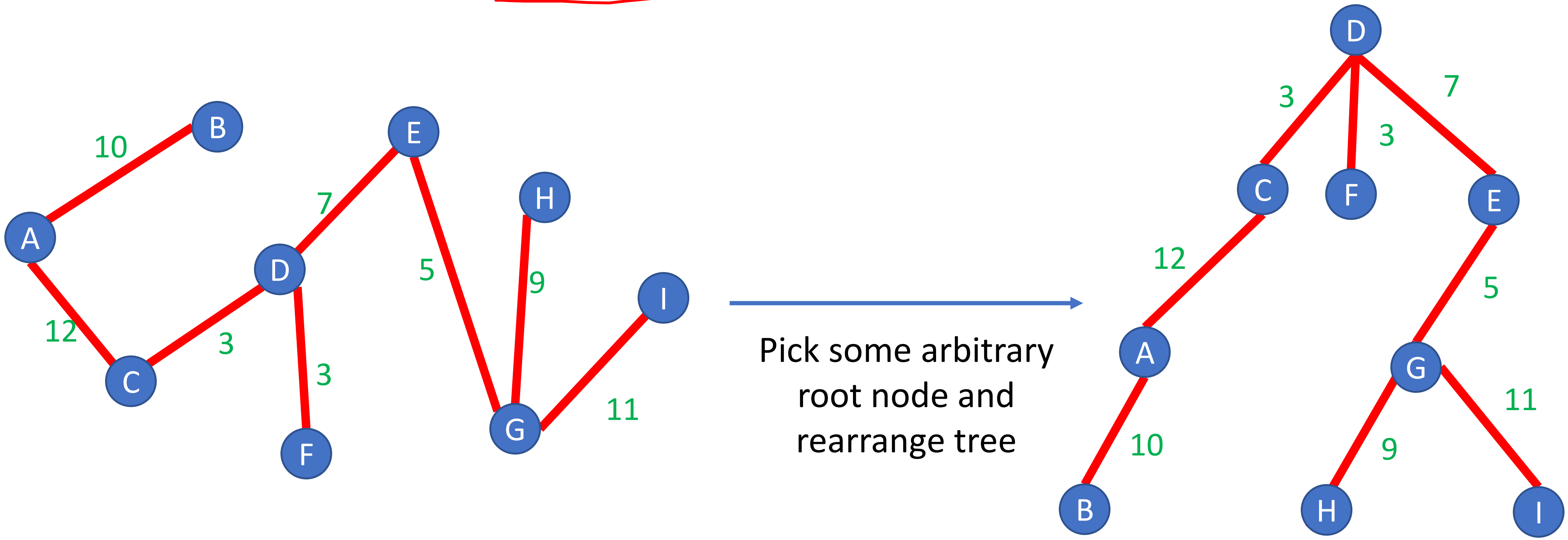
A connected graph with no cycles



Note: A tree does not need a root, but they often do!

Definition: Tree

A connected graph with no cycles



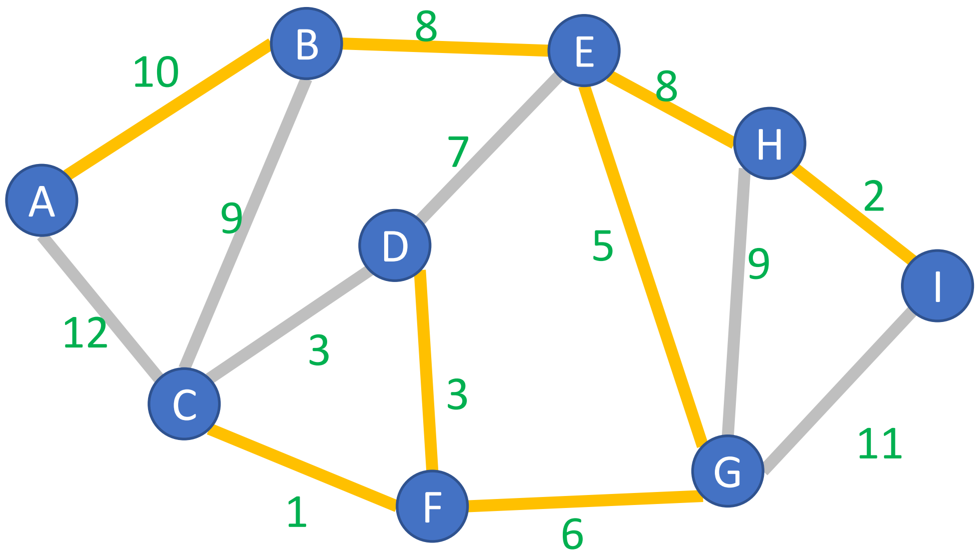
Definition: Spanning Tree

1) connected
2) acyclic
3) V-1

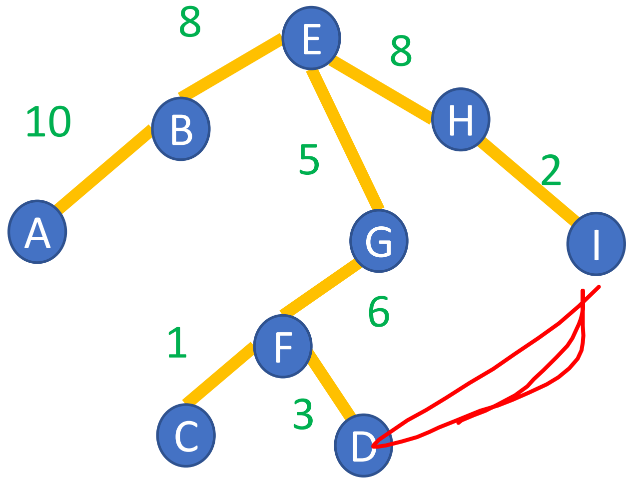
A Tree $T = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$

How many edges does T have?

$V - 1$



Pick some arbitrary root node and rearrange tree

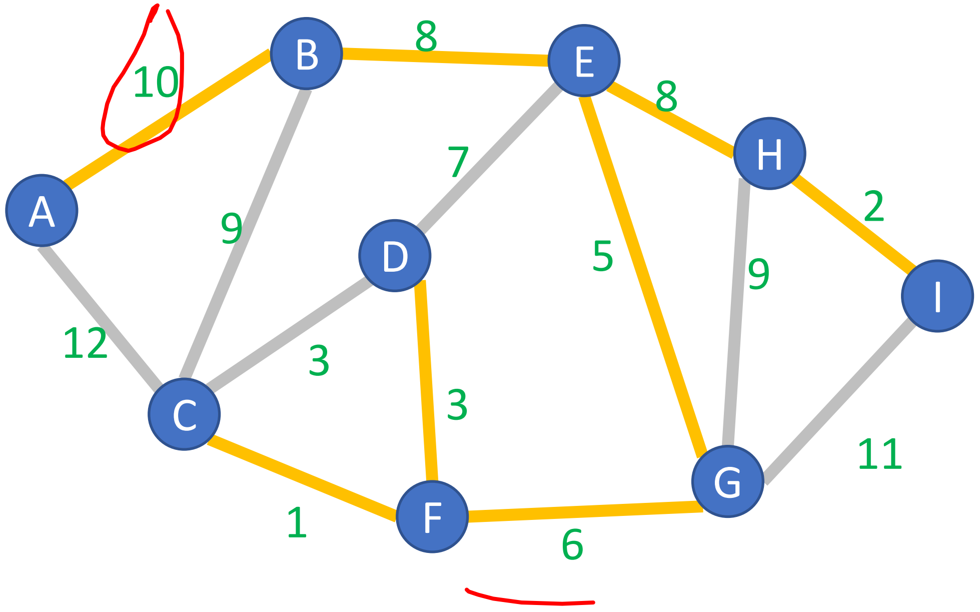


Any set of V-1 edges in the graph that doesn't have any cycles is guaranteed to be a spanning tree!

~~Any set of V-1 edges that~~ connects all the nodes in the graph is guaranteed to be a spanning tree!

Definition: Minimum Spanning Tree

A Tree $\boldsymbol{T} = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$, that has minimal cost

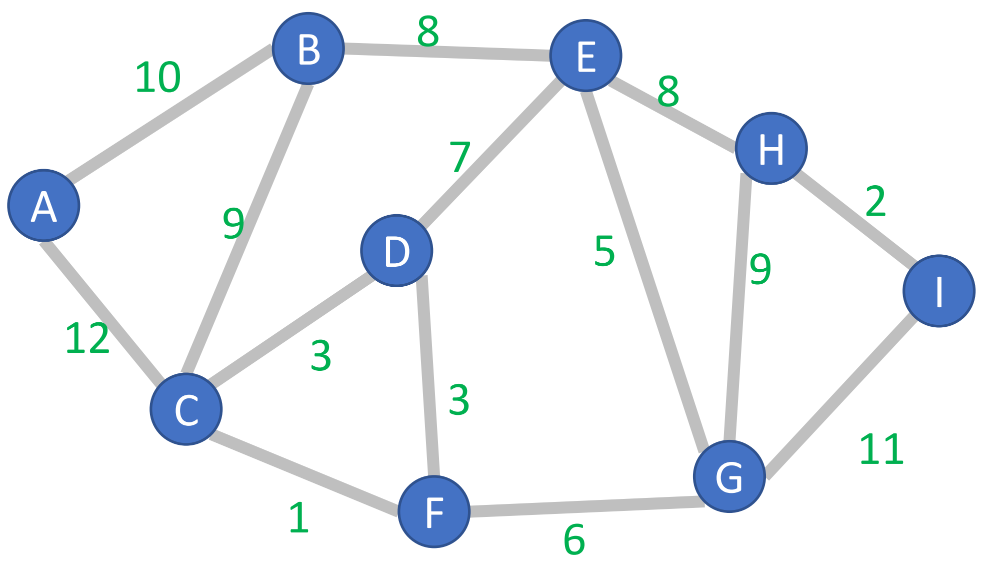


$$Cost(T) = \sum_{e \in E_T} w(e)$$

Kruskal's Algorithm

Start with an empty tree A

Add to A the lowest-weight edge that does not create a cycle



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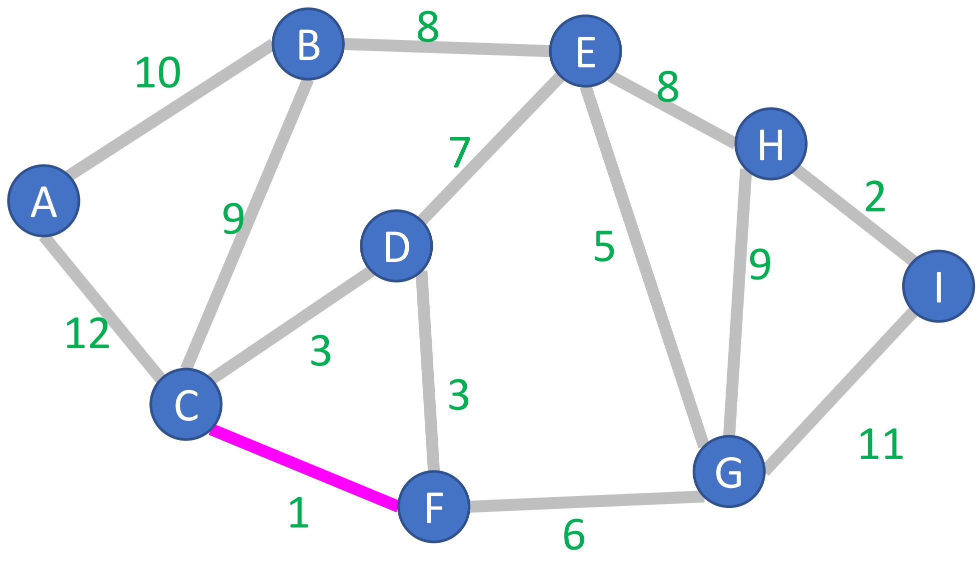

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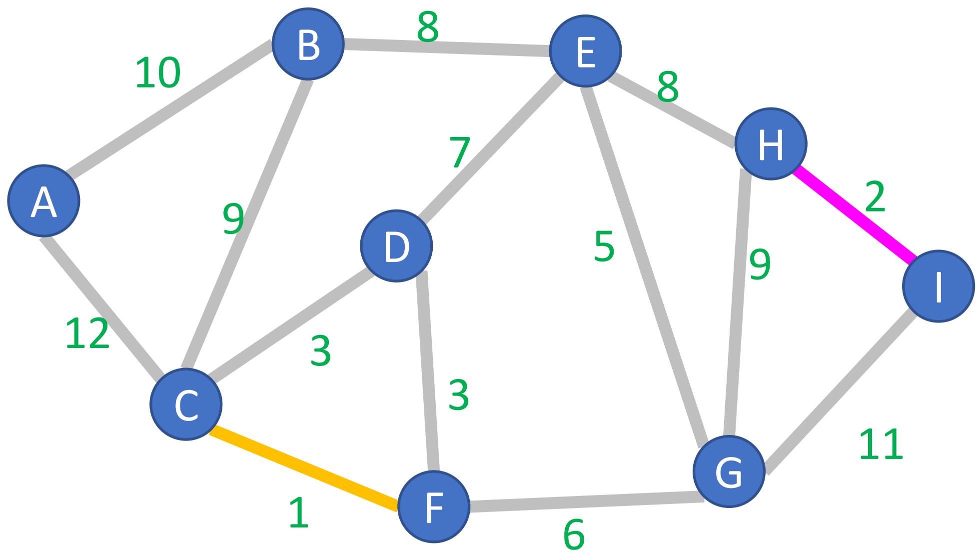

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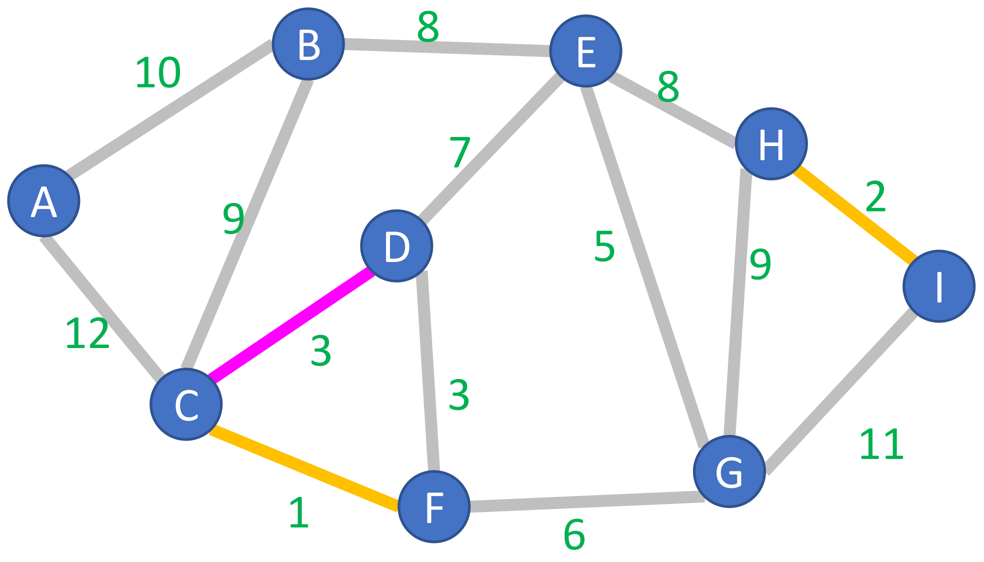

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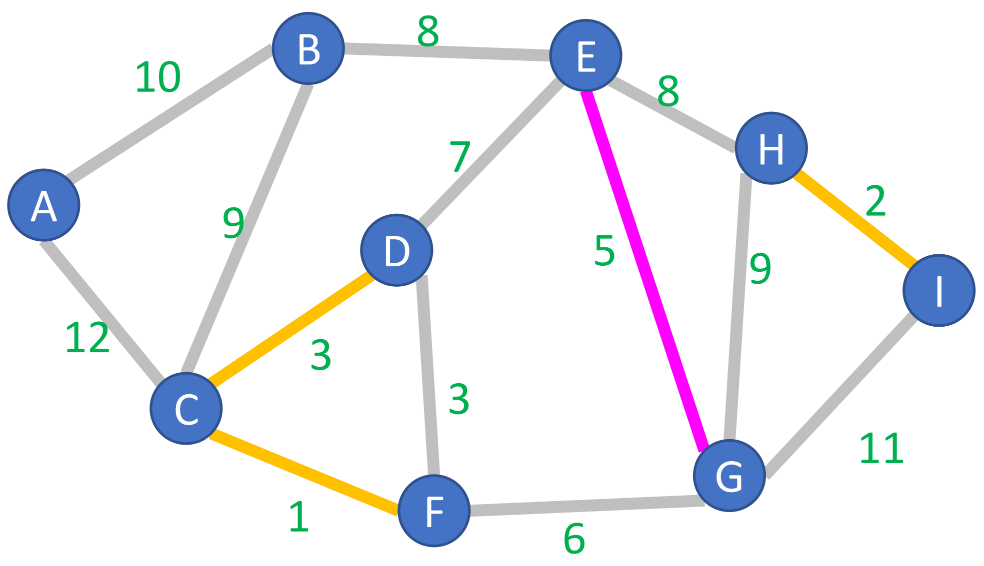

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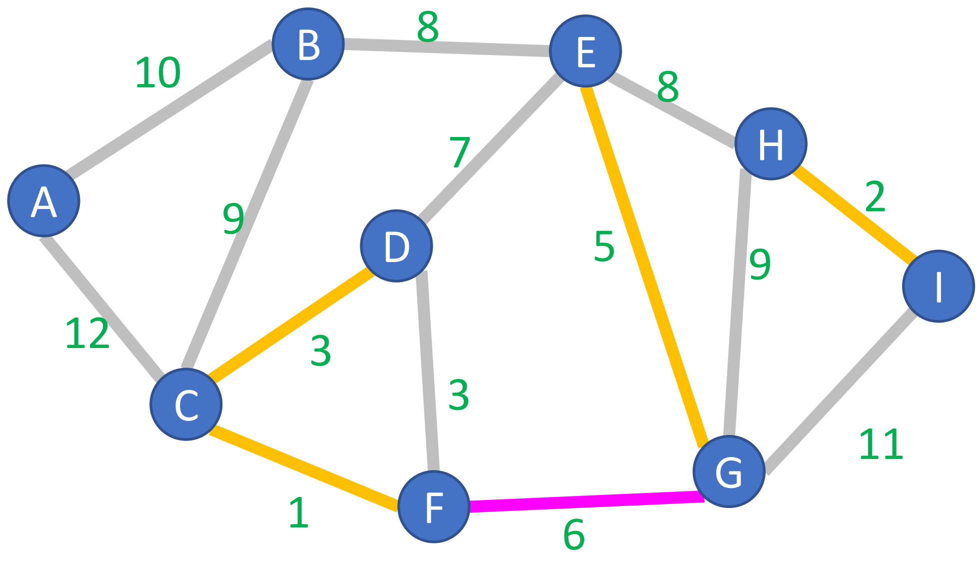
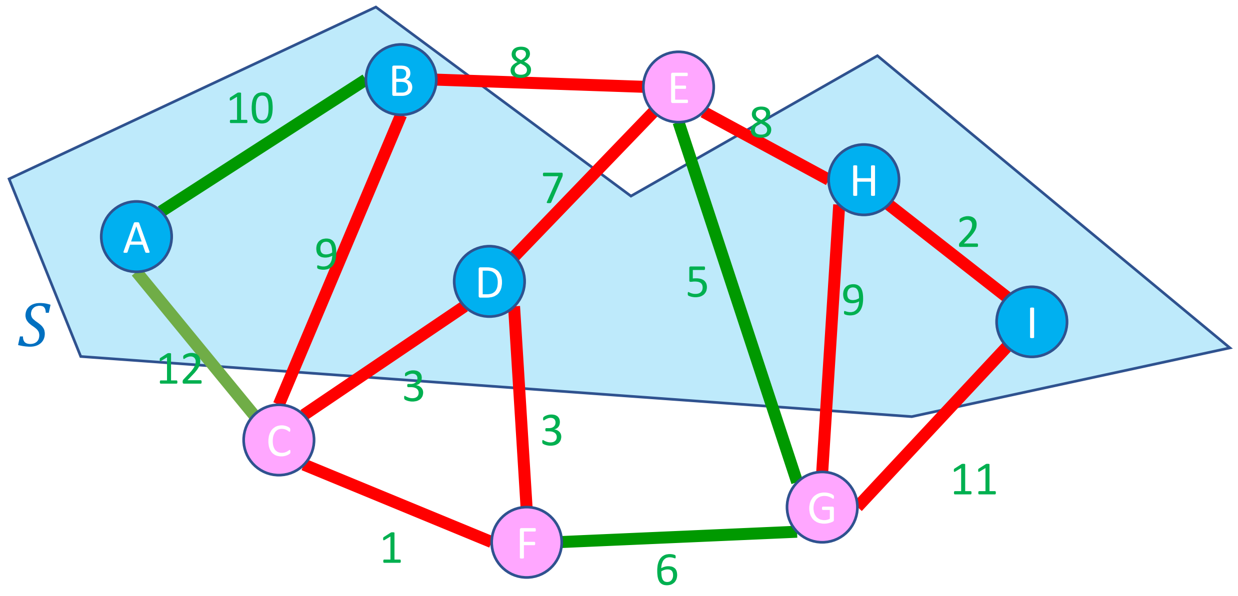


Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, S and $V - S$



Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. (A, C)

A set of edges R Respects a cut if no edges cross the cut e.g. $R = \{(A, B), (E, G), (F, G)\}$

Cut Theorem

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.

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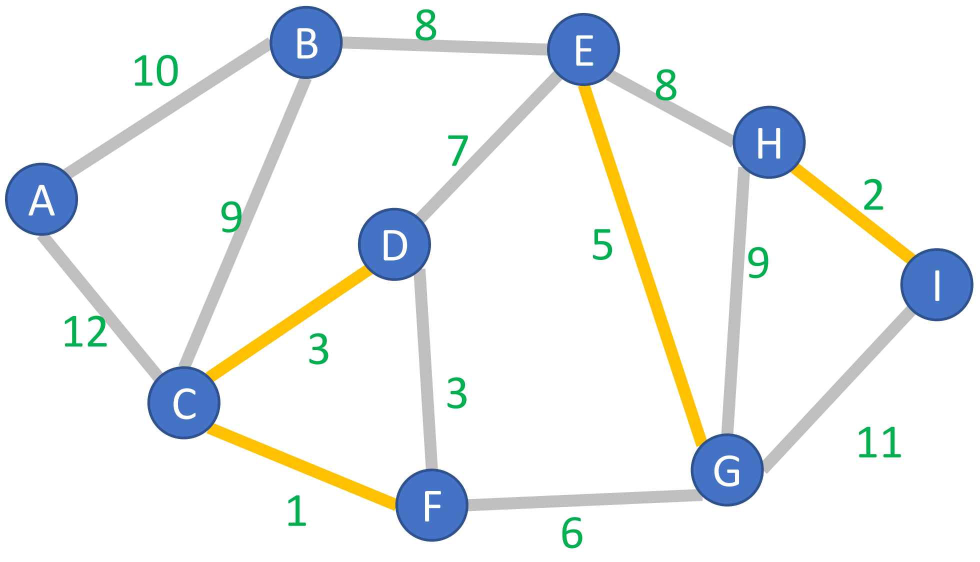

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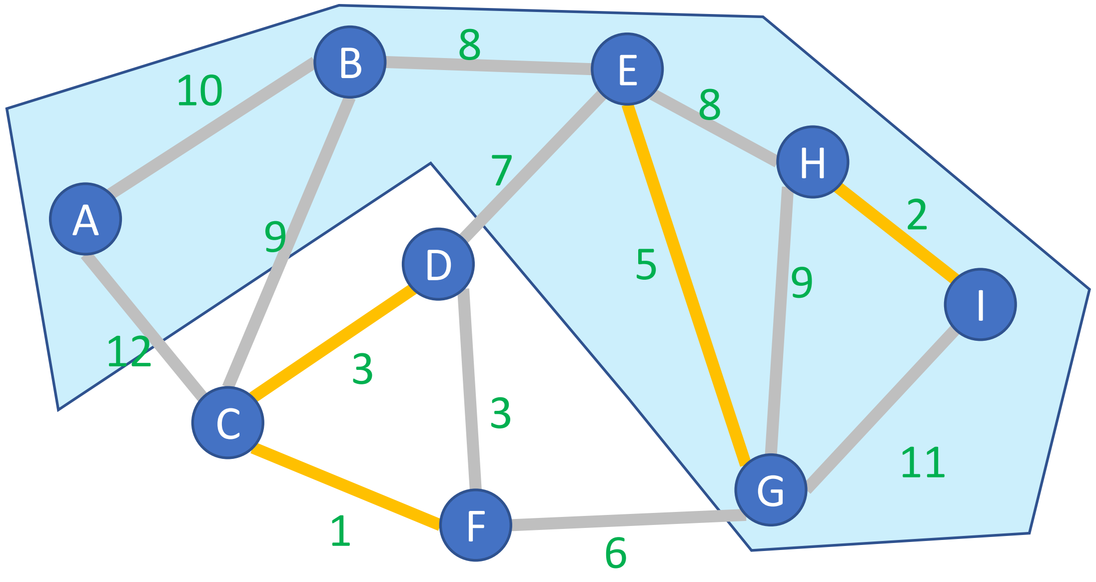

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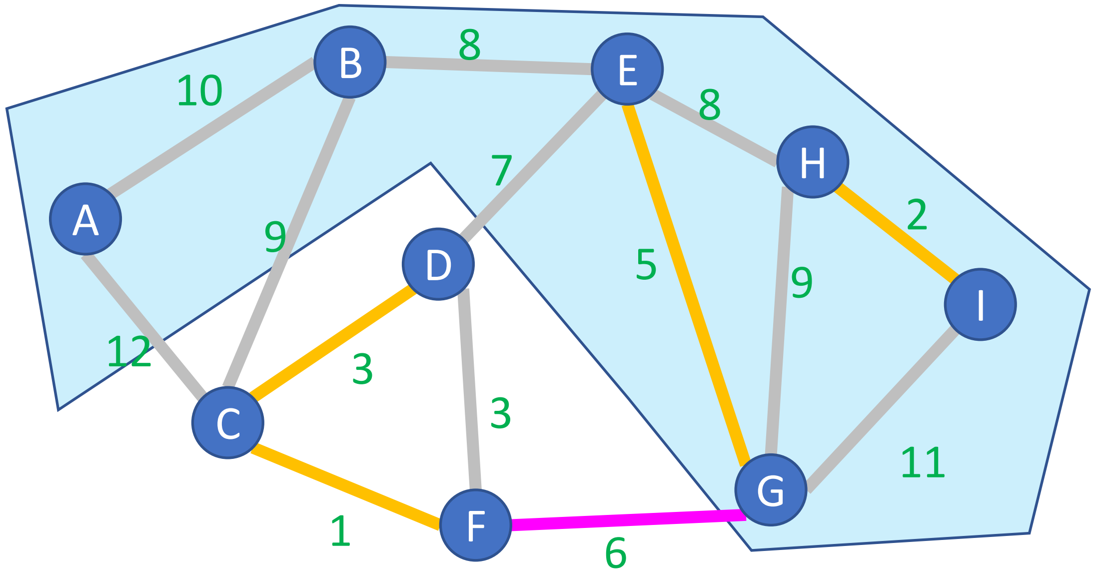

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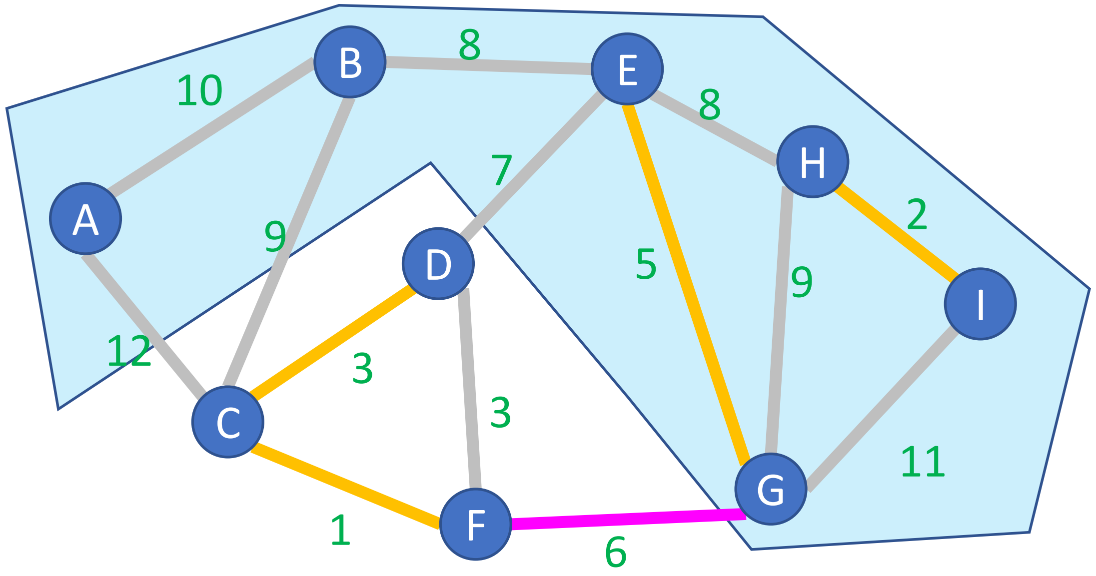
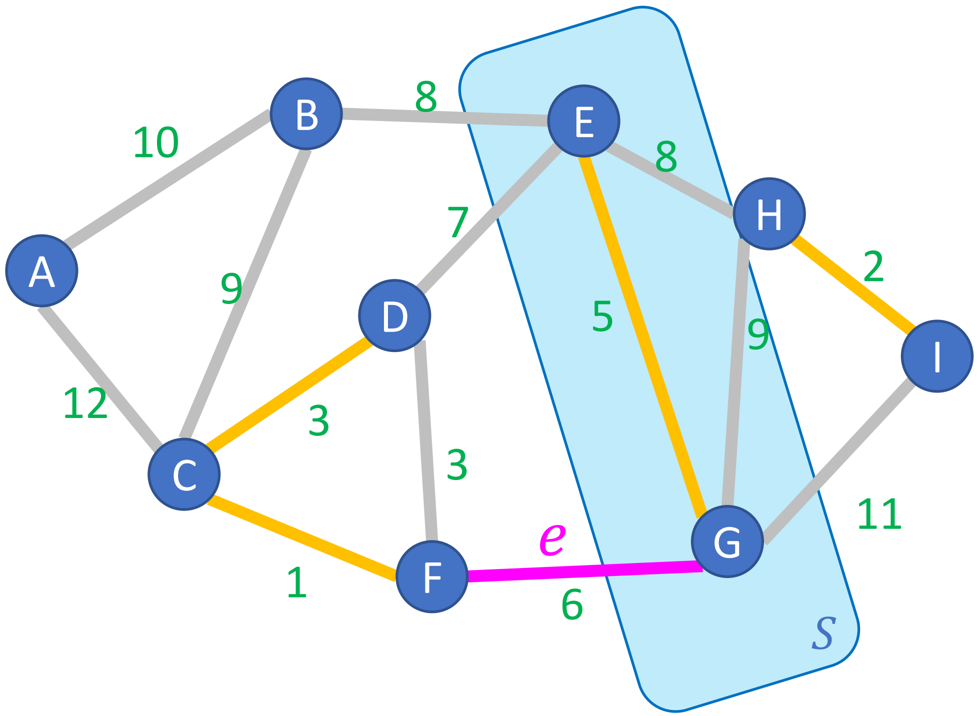


Proof of Kruskal's Algorithm

Start with an empty tree A
Repeat $V - 1$ times:
Add the min-weight edge that doesn't cause a cycle



Proof: Suppose we have some arbitrary set of edges A that Kruskal's has already selected to include in the MST. $e = (F, G)$ is the edge Kruskal's selects to add next

We know that there cannot exist a path from F to G using only edges in A because e does not cause a cycle

We can cut the graph therefore into 2 disjoint sets:

- nodes reachable from G using edges in A
- All other nodes

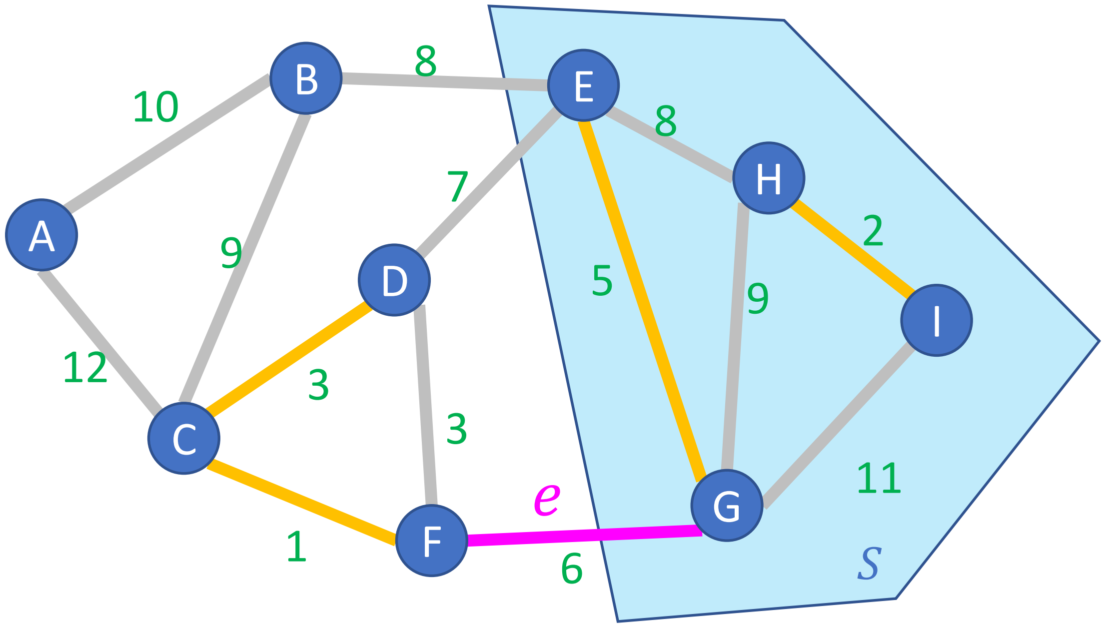
e is the minimum cost edge that crosses this cut, so by the Cut Theorem, Kruskal's is optimal!

Kruskal's Algorithm Runtime

Start with an empty tree A

Repeat $V-1$ times:

Add the min-weight edge that doesn't cause a cycle



Keep edges in a Disjoint-set data structure (very fancy)

$O(E \log V)$

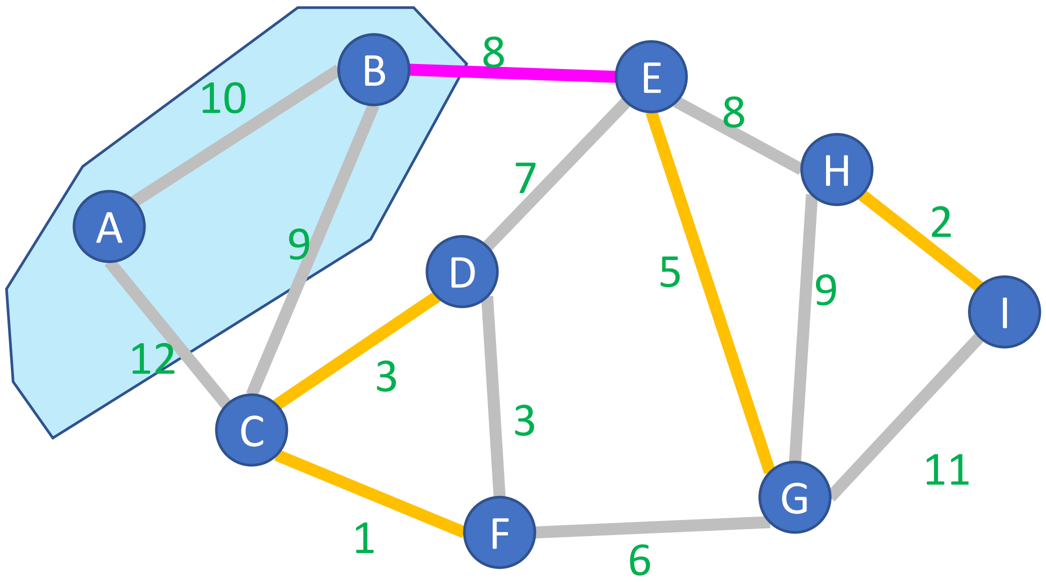
General MST Algorithm

Start with an empty tree A

Repeat $V - 1$ times:

Pick a cut $(S, V - S)$ which A respects (typically implicitly)

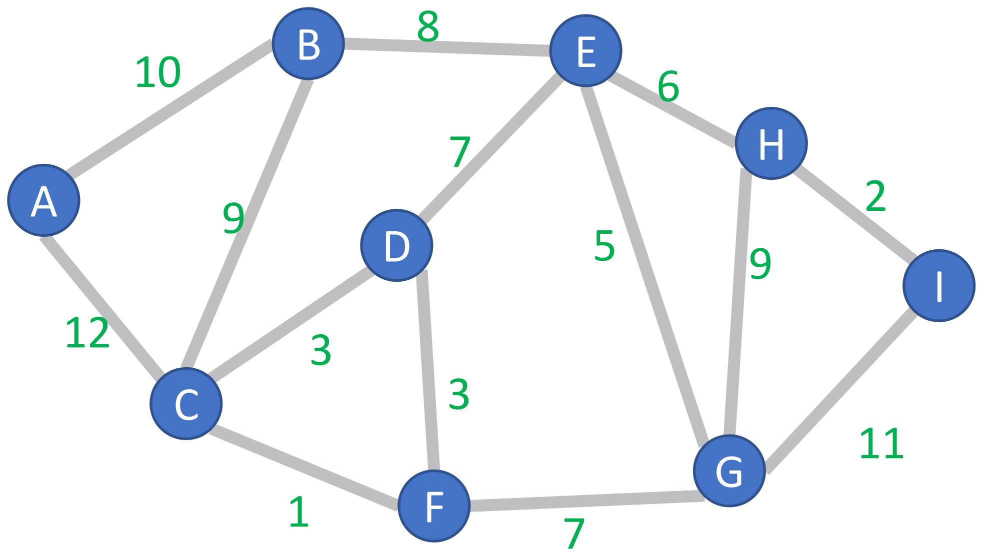
Add the min-weight edge which crosses $(S, V - S)$



Prim's Algorithm

Start with an empty tree A
Repeat $V - 1$ times:
  Pick a cut $(S, V - S)$ which A respects
  Add the min-weight edge which crosses $(S, V - S)$

S is all endpoint of edges in A
e is the min-weight edge that grows the tree



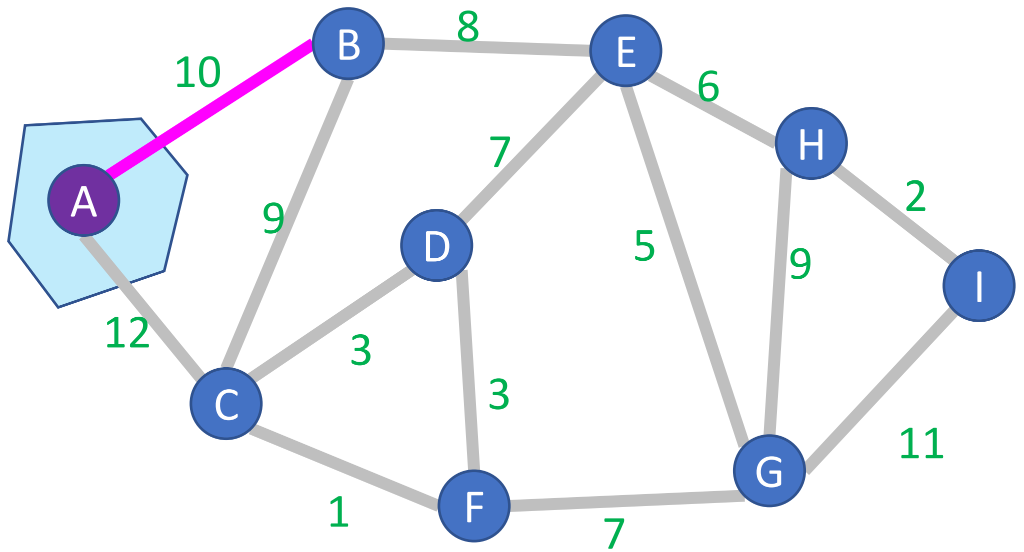
Prim's Algorithm

Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A



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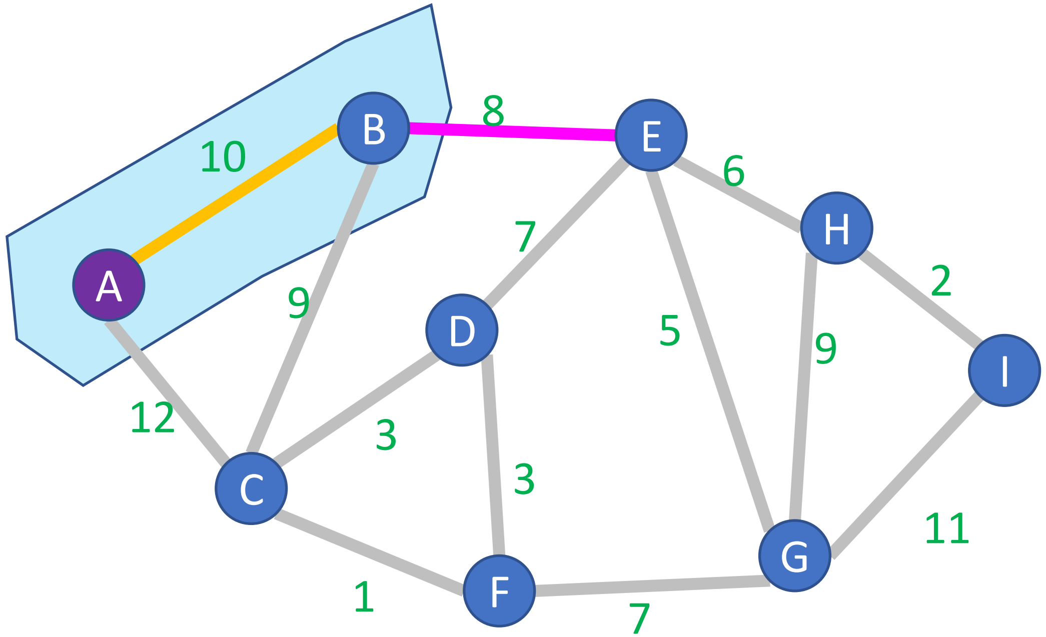

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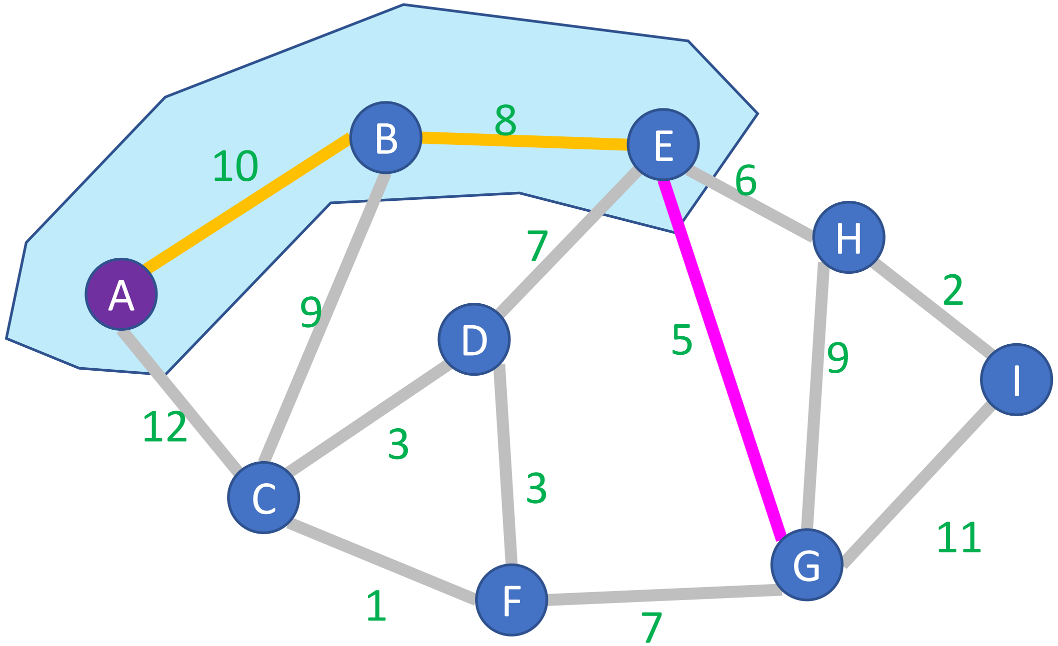

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Repeat $V - 1$ times:

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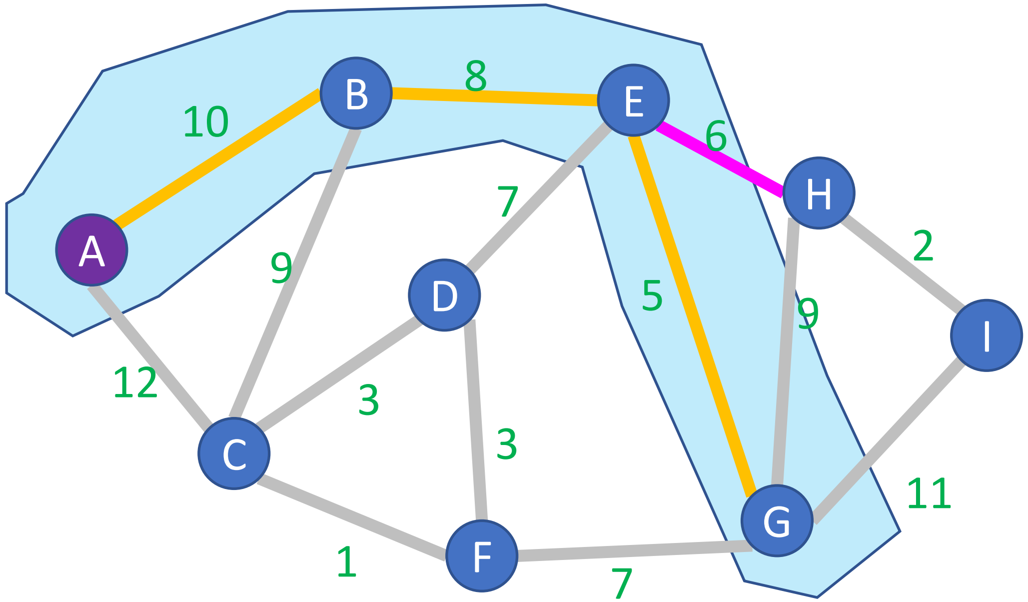


Prim's Algorithm

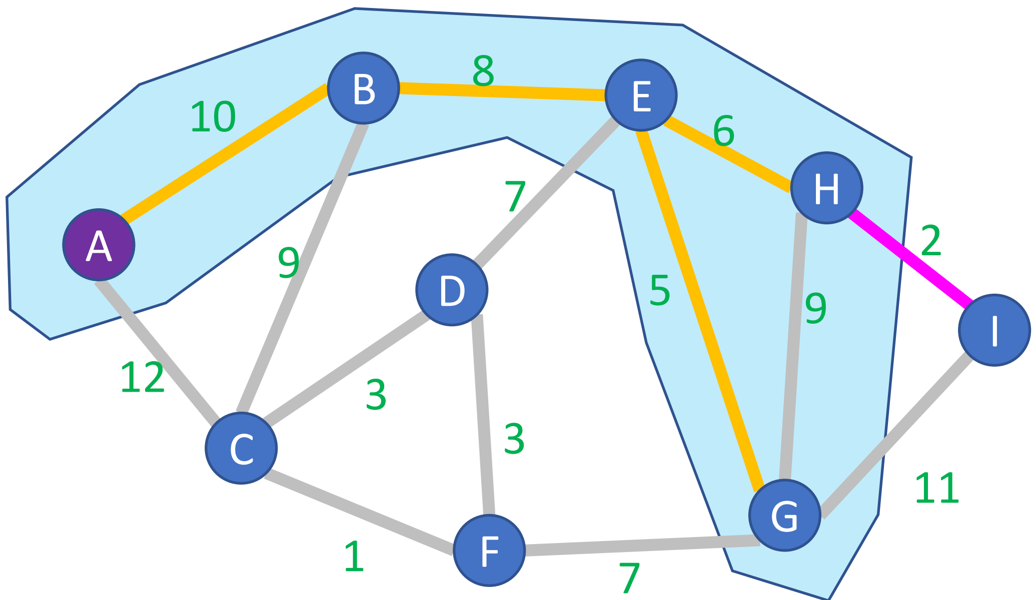
Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

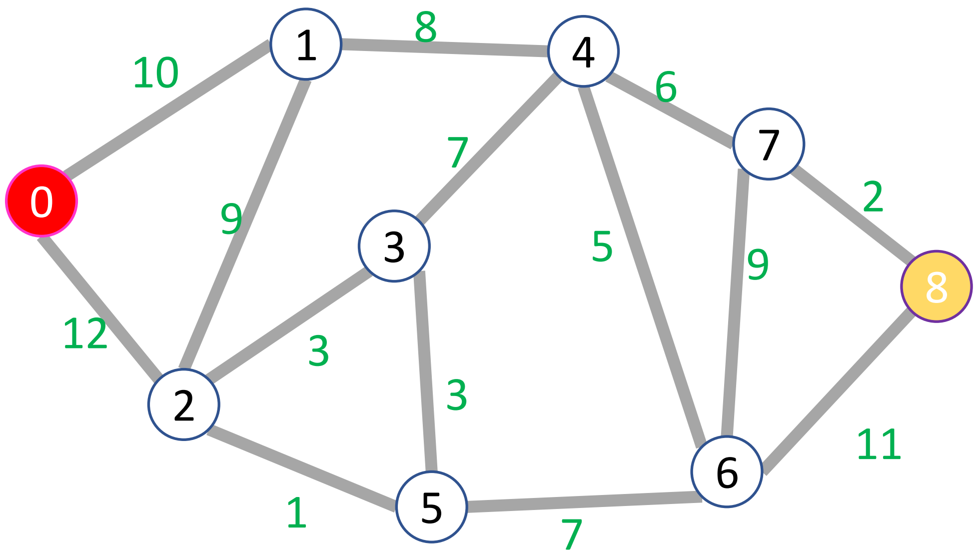
Add the min-weight edge which connects to node in A with a node not in A

Keep edges in a Heap
$O(E \log V)$



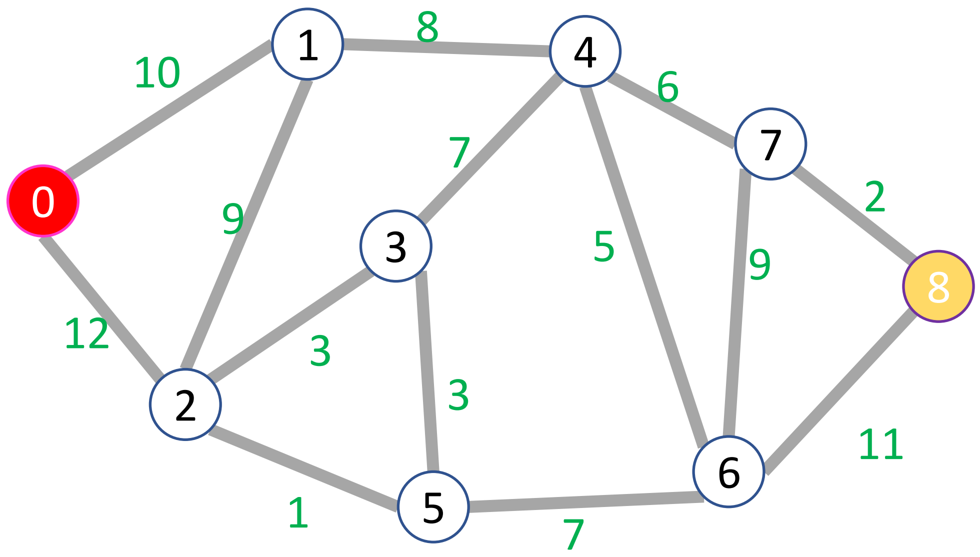
Dijkstra's Algorithm

```
int dijkstras(graph, start, end){
    distances = [∞, ∞, ∞,…];  // one index per node
    done = [False,False,False,…];  // one index per node
    PQ = new minheap();
    PQ.insert(0, start);  // priority=0, value=start
    distances[start] = 0;
    while (!PQ.isEmpty){
        current = PQ.deleteMin();
        done[current] = true;
        for (neighbor : current.neighbors){
            if (!done[neighbor]){
                new_dist = distances[current]+weight(current,neighbor);
                if(distances[neighbor] == ∞){
                    distances[neighbor] = new_dist;
                    PQ.insert(new_dist, neighbor);
                }
                if (new_dist < distances[neighbor]){
                    distances[neighbor] = new_dist;
                    PQ.decreaseKey(new_dist,neighbor); }
            }
        }
    }
    return distances[end]
}
```



Prims's Algorithm

```
int primss(graph, start, end){
        distances = [∞, ∞, ∞,…];  // one index per node
        done = [False,False,False,…];  // one index per node
        PQ = new minheap();
        PQ.insert(0, start);  // priority=0, value=start
        distances[start] = 0;
        while (!PQ.isEmpty){
                current = PQ.deleteMin();
                done[current] = true;
                for (neighbor : current.neighbors){
                        if (!done[neighbor]){
                                new_dist = weight(current,neighbor);
                                if(distances[neighbor] == ∞){
                                        distances[neighbor] = new_dist;
                                        PQ.insert(new_dist, neighbor);
                                }
                                if (new_dist < distances[neighbor]){
                                        distances[neighbor] = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return distances[end]
}
```



Dijkstra's Algorithm

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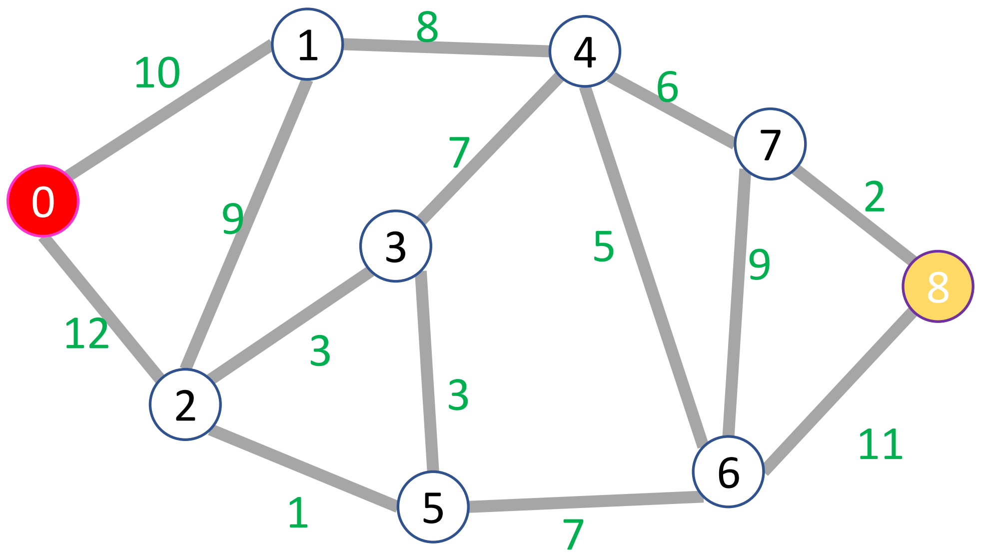
                                        distances[neighbor] = new_dist;
                                        PQ.insert(new_dist, neighbor);
                                }
                                if (new_dist < distances[neighbor]){
                                        distances[neighbor] = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return distances[end]
}
```



Prims's Algorithm

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	PQ = new minheap();
	PQ.insert(0, start);  // priority=0, value=start
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		current = PQ.deleteMin();
		done[current] = true;
		for (neighbor : current.neighbors){
			if (!done[neighbor]){
				new_dist = weight(current,neighbor);
				if(distances[neighbor] == ∞){
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				}
				if (new_dist < distances[neighbor]){
					distances[neighbor] = new_dist;
					PQ.decreaseKey(new_dist,neighbor); }
			}
		}
	}
	return distances[end]
}
```