CSE 332 Summer 2024
Lecture 14: Graphs

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http://www.cs.uw.edu/332
Undirected Graphs

Definition: $G = (V, E)$

Vertices/Nodes

Edges

$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$E = \{(1, 2), (2, 3), (1, 3), \ldots\}$
Directed Graphs

Definition: \( G = (V, E) \)

Vertices/Nodes

\( V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \)

Edges

\( E = \{(1, 2), (2, 3), (1, 3), \ldots\} \)
Some graphs may have duplicate edges (e.g. here we have the edge (1,2) twice). Some may also have self-edges (e.g. here there is an edge from 1 to 1). Graph with Neither self-edges nor duplicate edges are called simple graphs.
Weighted Graphs

Definition: $G = (V, E)$

$w(e) = \text{weight of edge } e$

$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$E = \{(1, 2), (2, 3), (1, 3), \ldots\}$
For each application below, consider:
• What are the nodes, what are the edges?
• Is the graph directed?
• Is the graph simple?
• Is the graph weighted?

• **Facebook friends**
  • Nodes: Users
  • Undirected: friendship is required to be mutual
  • Simple
  • Depends

• **Twitter/X followers**
  • Nodes: Users
  • Directed
  • Simple
  • Depends (same as above)

• **Java inheritance**
  • Nodes: classes
  • Directed
  • Simple
  • unweighted

• **Airline Routes**
  • Nodes: airports
  • Directed
  • Non-simple
  • Weighted
Some Graph Terms

- **Adjacent/Neighbors**
  - Nodes are adjacent/neighbors if they share an edge

- **Degree**
  - Number of edges “touching” a vertex

- **Indegree**
  - Number of incoming edges

- **Outdegree**
  - Number of outgoing edges
Definition: Complete Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is an edge from $v_1$ to $v_2$
Definition: Path

A sequence of nodes \((v_1, v_2, ..., v_k)\) s.t. \(\forall 1 \leq i \leq k - 1, (v_i, v_{i+1}) \in E\)

Simple Path:
A path in which each node appears at most once

Cycle:
A path which starts and ends in the same place
Definition: (Strongly) Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from $v_1$ to $v_2$
Definition: (Strongly) Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from $v_1$ to $v_2$.
Definition: Weakly Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from $v_1$ to $v_2$ ignoring direction of edges.
The maximum number of edges in a graph is $\Theta(|V|^2)$:
- Undirected and simple: $\frac{|V|(|V|-1)}{2}$
- Directed and simple: $|V|(|V| - 1)$
- Directed and non-simple (but no duplicates): $|V|^2$

If the graph is connected, the minimum number of edges is $|V| - 1$.

If $|E| \in \Theta(|V|^2)$ we say the graph is dense.
If $|E| \in \Theta(|V|)$ we say the graph is sparse.

Because $|E|$ is not always near to $|V|^2$ we do not typically substitute $|V|^2$ for $|E|$ in running times, but leave it as a separate variable.
- However, $\log(|E|) \in \Theta(\log(|V|))$
A Graph $G = (V, E)$ is a tree if it is undirected, connected, and has no cycles (i.e. is acyclic). Often one node is identified as the “root”.

A Tree

A Rooted Tree
Graph Operations

• To represent a Graph (i.e. build a data structure) we need:
  • Add Edge
  • Remove Edge
  • Check if Edge Exists
  • Get Neighbors (incoming)
  • Get Neighbors (outgoing)
Adjacency List

Time/Space Tradeoffs
Space to represent: $\Theta(n + m)$
Add Edge $(v, w)$: $\Theta(\text{deg}(v))$
Remove Edge $(v, w)$: $\Theta(\text{deg}(v))$
Check if Edge $(v, w)$ Exists: $\Theta(\text{deg}(v))$
Get Neighbors (incoming): $\Theta(n + m)$
Get Neighbors (outgoing): $\Theta(\text{deg}(v))$

$|V| = n$
$|E| = m$
Adjacency List (Weighted)

Time/Space Tradeoffs
Space to represent: $\Theta(n + m)$
Add Edge $(v, w)$: $\Theta(\deg(v))$
Remove Edge $(v, w)$: $\Theta(\deg(v))$
Check if Edge $(v, w)$ Exists: $\Theta(\deg(v))$
Get Neighbors (incoming): $\Theta(n + m)$
Get Neighbors (outgoing): $\Theta(\deg(v))$

$|V| = n$
$|E| = m$
Time/Space Tradeoffs

Space to represent: $\Theta(n^2)$
Add Edge $(v, w)$: $\Theta(1)$
Remove Edge $(v, w)$: $\Theta(1)$
Check if Edge $(v, w)$ Exists: $\Theta(1)$
Get Neighbors (incoming): $\Theta(n)$
Get Neighbors (outgoing): $\Theta(n)$

$|V| = n$
$|E| = m$
Adjacency Matrix

Time/Space Tradeoffs
Space to represent: $\Theta(n^2)$
Add Edge $(v, w)$: $\Theta(1)$
Remove Edge $(v, w)$: $\Theta(1)$
Check if Edge $(v, w)$ Exists: $\Theta(1)$
Get Neighbors (incoming): $\Theta(n)$
Get Neighbors (outgoing): $\Theta(n)$

$$|V| = n$$
$$|E| = m$$
Adjacency Matrix (weighted)

Time/Space Tradeoffs
Space to represent: \( \Theta(n^2) \)
Add Edge \((v, w)\): \( \Theta(1) \)
Remove Edge \((v, w)\): \( \Theta(1) \)
Check if Edge \((v, w)\) Exists: \( \Theta(1) \)
Get Neighbors (incoming): \( \Theta(n) \)
Get Neighbors (outgoing): \( \Theta(n) \)

|\( |V| = n \) |\( |E| = m \) |

\[
\begin{array}{cccccccccc}
A & B & C & D & E & F & G & H & I \\
\hline 
A & 1 & 1 & & & & & & & \\
B & 1 & 1 & 1 & & & & & & \\
C & 1 & 1 & 1 & 1 & & & & & \\
D & & 1 & 1 & 1 & & & & & \\
E & 1 & 1 & & & 1 & 1 & & & \\
F & & 1 & 1 & & & 1 & & & \\
G & & & & 1 & 1 & & 1 & 1 & \\
H & 1 & 1 & & & 1 & 1 & & & \\
I & & 1 & 1 & & & & 1 & 1 &
\end{array}
\]
Comparison

• Adjacency List:
  • Less memory when \(|E| < |V|^2\)
  • Operations with running time linear in degree of source node
    • Add an edge
    • Remove an edge
    • Check for edge
    • Get neighbors

• Adjacency Matrix:
  • Similar amount of memory when \(|E| \approx |V|^2\)
  • Constant time operations:
    • Add an edge
    • Remove an edge
    • Check for an edge
  • Operations running with linear time in \(|V|\)
    • Get neighbors

Adjacency List is more common in practice:
• Most graphs have \(|E| \ll |V|^2\)
  • Saves memory
  • Most nodes will have small degree
• Getting neighbors is a common operation
• Adjacency Matrix may be better if the graph is “dense” or if its edges change a lot
Breadth-First Search

• Input: a node $s$
• Behavior: Start with node $s$, visit all neighbors of $s$, then all neighbors of neighbors of $s$, ...
• Visits every node reachable from $s$ in order of distance
• Output:
  • How long is the shortest path?
  • Is the graph connected?
void bfs(graph, s) {
    found = new Queue();
    found.enqueue(s);
    mark s as “visited”;
    While (!found.isEmpty()){
        current = found.dequeue();
        for (v : neighbors(current)){
            if (!v marked “visited”){
                mark v as “visited”;
                found.enqueue(v);
            }
        }
    }
}

Running time: $\Theta(|V| + |E|)$
int shortestPath(graph, s, t){
    found = new Queue();
    layer = 0;
    found.enqueue(s);
    mark s as “visited”;
    While (!found.isEmpty()){
        current = found.dequeue();
        layer = depth of current;
        for (v : neighbors(current)){
            if (!v marked “visited”){
                mark v as “visited”;
                depth of v = layer + 1;
                found.enqueue(v);
            }
        }
    }
    return depth of t;
}
Depth-First Search
Depth-First Search

• Input: a node $s$

• Behavior: Start with node $s$, visit one neighbor of $s$, then all nodes reachable from that neighbor of $s$, then another neighbor of $s$, ...
  • Before moving on to the second neighbor of $s$, visit everything reachable from the first neighbor of $s$

• Output:
  • Does the graph have a cycle?
  • A topological sort of the graph.
DFS (non-recursive)

void dfs(graph, s){
    found = new Stack();
    found.pop(s);
    mark s as “visited”;
    While (!found.isEmpty()){
        current = found.pop();
        for (v : neighbors(current)){
            if (! v marked “visited”){
                mark v as “visited”;
                found.push(v);
            }
        }
    }
}

Running time: $\Theta(|V| + |E|)$
DFS Recursively (more common)

```java
void dfs(graph, curr){
    mark curr as “visited”;  
    for (v : neighbors(current)){
        if (! v marked “visited”){
            dfs(graph, v);
        }
    }

    mark curr as “done”;
}
```
Using DFS

• Consider the “visited times” and “done times”

• Edges can be categorized:
  • Tree Edge
    • \((a, b)\) was followed when pushing
    • \((a, b)\) when \(b\) was unvisited when we were at \(a\)
  • Back Edge
    • \((a, b)\) goes to an “ancestor”
    • \(a\) and \(b\) visited but not done when we saw \((a, b)\)
    • \(t_{\text{visited}}(b) < t_{\text{visited}}(a) < t_{\text{done}}(a) < t_{\text{done}}(b)\)
  • Forward Edge
    • \((a, b)\) goes to a “descendent”
    • \(b\) was visited and done between when \(a\) was visited and done
    • \(t_{\text{visited}}(a) < t_{\text{visited}}(b) < t_{\text{done}}(b) < t_{\text{done}}(a)\)
  • Cross Edge
    • \((a, b)\) goes to a node that doesn’t connect to \(a\)
    • \(b\) was seen and done before \(a\) was ever visited
    • \(t_{\text{done}}(b) < t_{\text{visited}}(a)\)
Back Edges

• Behavior of DFS:
  • “Visit everything reachable from the current node before going back”

• Back Edge:
  • The current node’s neighbor is an “in progress” node
  • Since that other node is “in progress”, the current node is reachable from it
  • The back edge is a path to that other node
  • Cycle!
boolean hasCycle(graph, curr){
    mark curr as “visited”;
    cycleFound = false;
    for (v : neighbors(current)){
        if (v marked “visited” && ! v marked “done”){
            cycleFound=true;
        }
        if (! v marked “visited” && !cycleFound){
            cycleFound = hasCycle(graph, v);
        }
    }
    mark curr as “done”;
    return cycleFound;
}
Topological Sort

• A Topological Sort of a directed acyclic graph $G = (V, E)$ is a permutation of $V$ such that if $(u, v) \in E$ then $u$ is before $v$ in the permutation.
DFS Recursively

void dfs(graph, curr){
    mark curr as “visited”;
    for (v : neighbors(current)){
        if (! v marked “visited”){
            dfs(graph, v);
        }
    }
    mark curr as “done”;
}

Idea: List in reverse order by “done” time
DFS: Topological sort

List topSort(graph){
    List<Nodes> done = new List<>();
    for (Node v : graph.vertices){
        if (!v.visited){
            finishTime(graph, v, finished);
        }
    }
    done.reverse();
    return done;
}

void finishTime(graph, curr, finished){
    curr.visited = true;
    for (Node v : curr.neighbors){
        if (!v.visited){
            finishTime(graph, v, finished);
        }
    }
    done.add(curr)
}
Single-Source Shortest Path

Find the quickest way to get from UVA to each of these other places

Given a graph $G = (V, E)$ and a start node $s \in V$, for each $v \in V$ find the least-weight path from $s \rightarrow v$ (call this weight $\delta(s, v)$)

(assumption: all edge weights are positive)
Dijkstra’s Algorithm

• Input: graph with **no negative edge weights**, start node $s$, end node $t$
• Behavior: Start with node $s$, repeatedly go to the incomplete node “nearest” to $s$, stop when
• Output:
  • Distance from start to end
  • Distance from start to every node
Dijkstra’s Algorithm

Start: 0
End: 8

<table>
<thead>
<tr>
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<th>Done?</th>
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<tbody>
<tr>
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Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path.
Dijkstra’s Algorithm

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End: 8

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Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path
Dijkstra’s Algorithm

```java
int dijkstras(graph, start, end){
    distances = [∞, ∞, ∞,...]; // one index per node
    done = [False,False,False,...]; // one index per node
    PQ = new minheap();
    PQ.insert(0, start); // priority=0, value=start
    distances[start] = 0;
    while (!PQ.isEmpty){
        current = PQ.deleteMin();
        done[current] = true;
        for (neighbor : current.neighbors){
            if (!done[neighbor]){ // new update
                new_dist = distances[current]+weight(current,neighbor);
                if(distances[neighbor] == ∞){
                    distances[neighbor] = new_dist;
                    PQ.insert(new_dist, neighbor);
                }
                if (new_dist < distances[neighbor]){ // update
                    distances[neighbor] = new_dist;
                    PQ.decreaseKey(new_dist, neighbor);
                }
            }
        }
    }
    return distances[end]
}
```
Dijkstra’s Algorithm: Running Time

• How many total priority queue operations are necessary?
  • How many times is each node added to the priority queue?
  • How many times might a node’s priority be changed?
• What’s the running time of each priority queue operation?
• Overall running time:
  • \( \Theta(|E| \log|V|) \)
Dijkstra’s Algorithm: Correctness

• Claim: when a node is removed from the priority queue, we have found its shortest path

• Induction over number of completed nodes

• Base Case:

• Inductive Step:
Dijkstra’s Algorithm: Correctness

• Claim: when a node is removed from the priority queue, its distance is that of the shortest path
• Induction over number of completed nodes
• Base Case: Only the start node removed
  • It is indeed 0 away from itself
• Inductive Step:
  • If we have correctly found shortest paths for the first \( k \) nodes, then when we remove node \( k + 1 \) we have found its shortest path
Dijkstra’s Algorithm: Correctness

• Suppose $a$ is the next node removed from the queue. What do we know bout $a$?
Dijkstra’s Algorithm: Correctness

• Suppose $a$ is the next node removed from the queue.
  • No other node incomplete node has a shorter path discovered so far

• Claim: no undiscovered path to $a$ could be shorter
  • Consider any other incomplete node $b$ that is 1 edge away from a complete node
  • $a$ is the closest node that is one away from a complete node
  • Thus no path that includes $b$ can be a shorter path to $a$
  • Therefore the shortest path to $a$ must use only complete nodes, and therefore we have found it already!
Dijkstra’s Algorithm: Correctness

- Suppose $a$ is the next node removed from the queue.
  - No other node incomplete node has a shorter path discovered so far
- Claim: no undiscovered path to $a$ could be shorter
  - Consider any other incomplete node $b$ that is 1 edge away from a complete node
  - $a$ is the closest node that is one away from a complete node
  - No path from $b$ to $a$ can have negative weight
  - Thus no path that includes $b$ can be a shorter path to $a$
  - Therefore the shortest path to $a$ must use only complete nodes, and therefore we have found it already!
Definition: Tree

A connected graph with no cycles

Note: A tree does not need a root, but they often do!
Definition: Tree

A connected graph with no cycles

Pick some arbitrary root node and rearrange tree
Definition: Spanning Tree

A Tree $T = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$

How many edges does $T$ have? $V - 1$

Any set of $V$-1 edges in the graph that doesn’t have any cycles is guaranteed to be a spanning tree!

Any set of $V$-1 edges that connects all the nodes in the graph is guaranteed to be a spanning tree!
Definition: Minimum Spanning Tree

A Tree $T = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$, that has minimal cost

$$Cost(T) = \sum_{e \in E_T} w(e)$$
Kruskal’s Algorithm

Start with an empty tree $A$
Add to $A$ the lowest-weight edge that does not create a cycle
Kruskal’s Algorithm

Start with an empty tree $A$
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Kruskal’s Algorithm

Start with an empty tree \( A \)
Add to \( A \) the lowest-weight edge that does not create a cycle
Kruskal’s Algorithm

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Start with an empty tree $A$

Add to $A$ the lowest-weight edge that does not create a cycle
Kruskal’s Algorithm

Start with an empty tree $A$
Add to $A$ the lowest-weight edge that does not create a cycle
Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, $S$ and $V - S$.

Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. $(A, C)$.

A set of edges $R$ Respects a cut if no edges cross the cut, e.g. $R = \{(A, B), (E, G), (F, G)\}$. 
Cut Theorem

If a set of edges $A$ is a subset of a minimum spanning tree $T$, let $(S, V - S)$ be any cut which $A$ respects. Let $e$ be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.
Cut Theorem

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Proof of Kruskal’s Algorithm

Start with an empty tree \( A \)

Repeat \( V - 1 \) times:

Add the min-weight edge that doesn’t cause a cycle

**Proof:** Suppose we have some arbitrary set of edges \( A \) that Kruskal’s has already selected to include in the MST. \( e = (F,G) \) is the edge Kruskal’s selects to add next

We know that there cannot exist a path from \( F \) to \( G \) using only edges in \( A \) because \( e \) does not cause a cycle

We can cut the graph therefore into 2 disjoint sets:

- nodes reachable from \( G \) using edges in \( A \)
- All other nodes

\( e \) is the minimum cost edge that crosses this cut, so by the Cut Theorem, Kruskal’s is optimal!
Kruskal’s Algorithm Runtime

Start with an empty tree $A$
Repeat $V - 1$ times:
  Add the min-weight edge that doesn’t cause a cycle

Keep edges in a Disjoint-set data structure (very fancy)
$O(E \log V)$
General MST Algorithm

Start with an empty tree $A$
Repeat $V - 1$ times:
   Pick a cut $(S, V - S)$ which $A$ respects (typically implicitly)
   Add the min-weight edge which crosses $(S, V - S)$
Prim’s Algorithm
Start with an empty tree $A$
Repeat $V - 1$ times:
   Pick a cut $(S, V - S)$ which $A$ respects
   Add the min-weight edge which crosses $(S, V - S)$

$S$ is all endpoint of edges in $A$
ed is the min-weight edge that grows the tree
Prim’s Algorithm
Start with an empty tree $A$
Pick a start node
Repeat $V - 1$ times:
Add the min-weight edge which connects to node in $A$ with a node not in $A$
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Start with an empty tree $A$
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Prim’s Algorithm

Start with an empty tree $A$

Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in $A$ with a node not in $A$
Prim’s Algorithm
Start with an empty tree $A$
Pick a start node
Repeat $V - 1$ times:
  Add the min-weight edge which connects to node in $A$ with a node not in $A$

Keep edges in a Heap
$O(E \log V)$
Dijkstra’s Algorithm

```java
int dijkstras(graph, start, end){
    distances = [∞, ∞, ∞,...]; // one index per node
    done = [False,False,False,...]; // one index per node
    PQ = new minheap();
    PQ.insert(0, start); // priority=0, value=start
    distances[start] = 0;
    while (!PQ.isEmpty){
        current = PQ.deleteMin();
        done[current] = true;
        for (neighbor : current.neighbors){
            if (!done[neighbor]){
                new_dist = distances[current]+weight(current,neighbor);
                if(distances[neighbor] == ∞){
                    distances[neighbor] = new_dist;
                    PQ.insert(new_dist, neighbor);
                }
                if (new_dist < distances[neighbor]){  // key update
                    distances[neighbor] = new_dist;
                    PQ.decreaseKey(new_dist,neighbor);  }
            }
        }
    }
    return distances[end]
}
```
int primss(graph, start, end){
    distances = [\(\infty, \infty, \infty, \ldots\)]; // one index per node
    done = [False, False, False, \ldots]; // one index per node
    PQ = new minheap();
    PQ.insert(0, start); // priority=0, value=start
    distances[start] = 0;
    while (!PQ.isEmpty){
        current = PQ.deleteMin();
        done[current] = true;
        for (neighbor : current.neighbors){
            if (!done[neighbor]){ // neighbors are not visited
                new_dist = weight(current, neighbor);
                if (distances[neighbor] == \(\infty\)){
                    distances[neighbor] = new_dist;
                    PQ.insert(new_dist, neighbor);
                }
                if (new_dist < distances[neighbor]){ // update shortest distance
                    distances[neighbor] = new_dist;
                    PQ.decreaseKey(new_dist, neighbor);
                }
            }
        }
    }
    return distances[end]
}
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        for (neighbor : current.neighbors){
            if (!done[neighbor]){
                new_dist = distances[current]+weight(current,neighbor);
                if(distances[neighbor] == ∞){
                    distances[neighbor] = new_dist;
                    PQ.insert(new_dist, neighbor);
                }
                if(new_dist < distances[neighbor]){ // if new distance is lower
                    distances[neighbor] = new_dist;
                    PQ.decreaseKey(new_dist, neighbor);
                }
            }
        }
    }
    return distances[end]
}
```

int primss(graph, start, end){
    distances = [∞, ∞, ∞,...];  // one index per node
    done = [False,False,False,...];  // one index per node
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                    PQ.insert(new_dist, neighbor);
                }
                if (new_dist < distances[neighbor]){  
                    distances[neighbor] = new_dist;
                    PQ.decreaseKey(new_dist,neighbor);  }
            }
        }
    }
    return distances[end]
}