CSE 332 Summer 2024
Lecture 13: Sorting

Nathan Brunelle

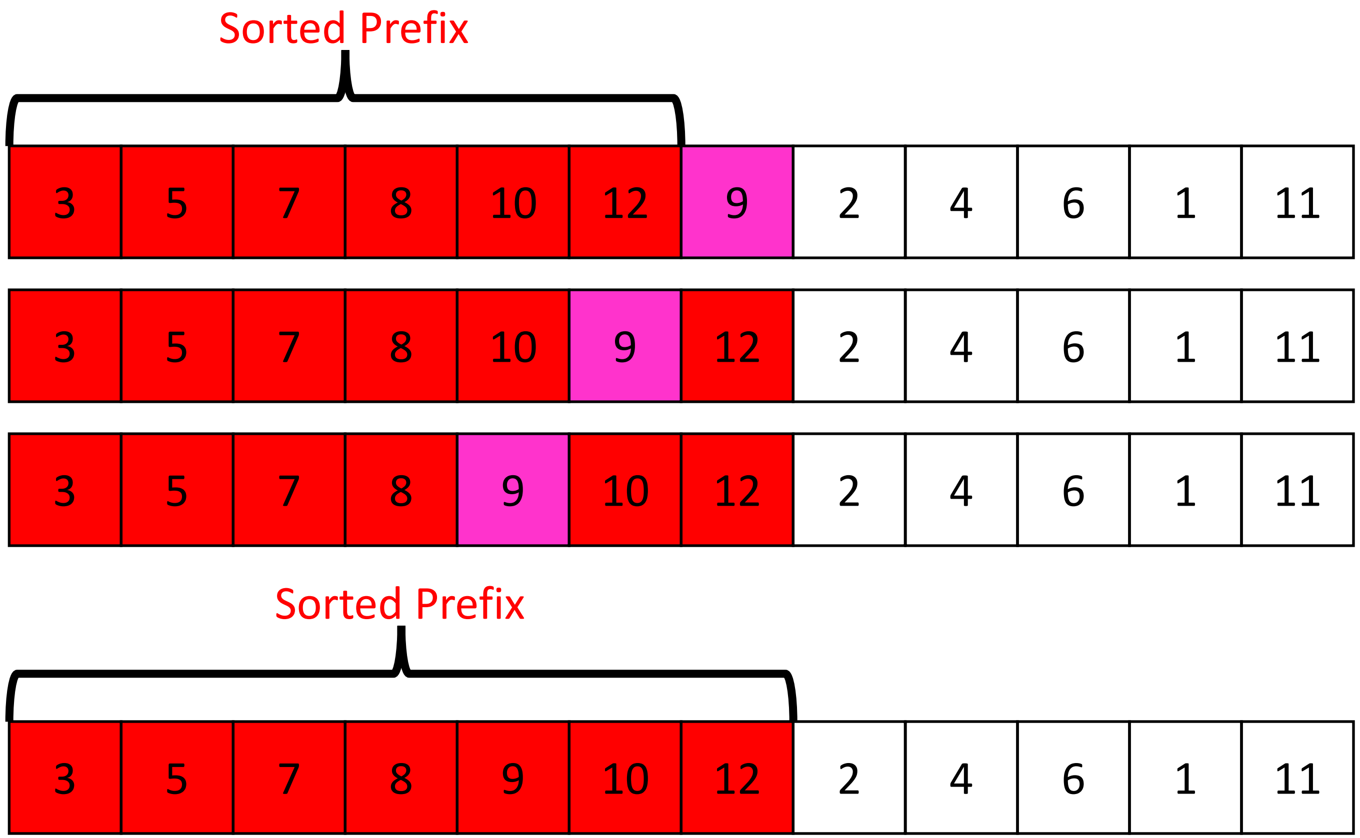
http://www.cs.uw.edu/332

Properties To Consider

- Worst case running time
- In place:
  - We only need to use the pre-existing array to do sorting
  - Constant extra space (only some additional variables needed)
  - *Selection Sort, Insertion Sort, Heap Sort*
- Adaptive
  - The running improves as the given list is closer to being sorted
  - It should be linear time for a pre-sorted list, and nearly linear time if the list is nearly sorted
  - *Insertion Sort*
- Online
  - We can start sorting before we have the entire list.
  - *Insertion Sort*
- Stable
  - “Tied” elements keep their original order

Insertion Sort

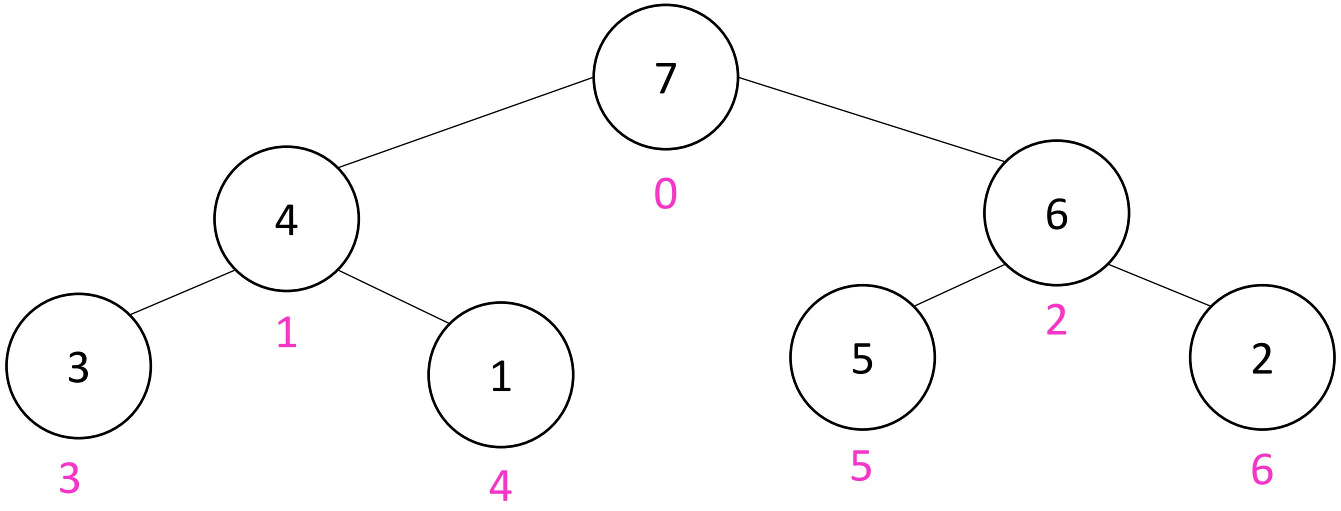
- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



Heap Sort

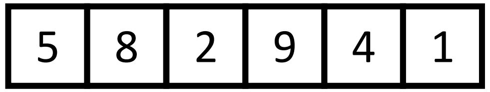
- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

3	8	6	4	7	5	2	8	9	10
0	1	2	3	4	5	6	7	8	9



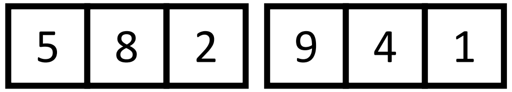
Divide And Conquer Sorting

- Divide and Conquer:
  - Recursive algorithm design technique
  - Solve a large problem by breaking it up into smaller versions of the same problem

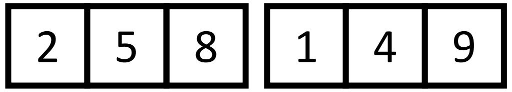


Merge Sort

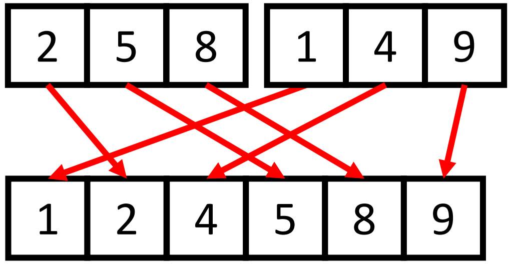
- **Base Case**:
  - If the list is of length 1 or 0, it’s already sorted, so just return it



- **Divide**:
  - Split the list into two “sublists” of (roughly) equal length



- **Conquer**:
  - Sort both lists recursively



- **Combine**:
  - **Merge** sorted sublists into one sorted list

Merge Sort In Action!

Sort between indices low and $high$

5	8	2	9	4	1	3	7

low $high$

Base Case: if low == $high$ then that range is already sorted!

Divide and Conquer: Otherwise call mergesort on ranges $\left(low, \frac{low+high}{2}\right)$ and $\left(\frac{low+high}{2} + 1, high\right)$

5	8	2	9	4	1	3	7

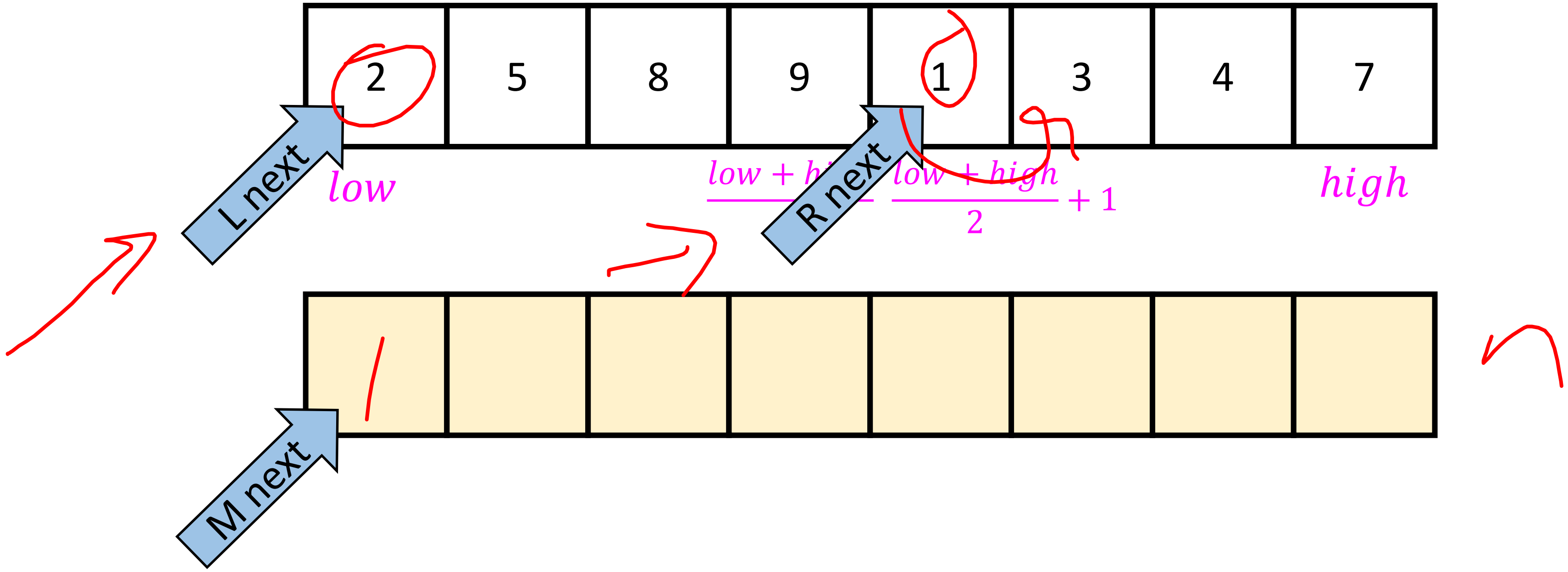
low $\frac{low + high}{2}$ $\frac{low + high}{2} + 1$ $high$

After Recursion:

2	5	8	9	1	3	4	7

low $high$

Merge (the combine part)



Create a new array to merge into, and 3 pointers/indices:

- L_next: the smallest "unmerged" thing on the left
- R_next: the smallest "unmerged" thing on the right
- M_next: where the next smallest thing goes in the merged array

One-by-one: put the smallest of L_next and R_next into M_next, then advance both M_next and whichever of L/R was used.

Properties of Merge Sort

- Worst Case Running time:
  - $\Theta(n \log n)$
- In-Place?
  - No!
- Adaptive?
  - No!
- Stable?
  - Yes!
  - As long as in a tie you always pick l_next

Quicksort

- Like Mergesort:
  - Divide and conquer
  - $O(n \log n)$ run time (kind of…)
- Unlike Mergesort:
  - Divide step is the "hard" part
  - *Typically* faster than Mergesort

Quicksort

Idea: pick a pivot element, recursively sort two sublists around that element

- Divide: select pivot element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Partition (Divide step)

Given: a list, a pivot p

Start: unordered list

8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $> p$ on right

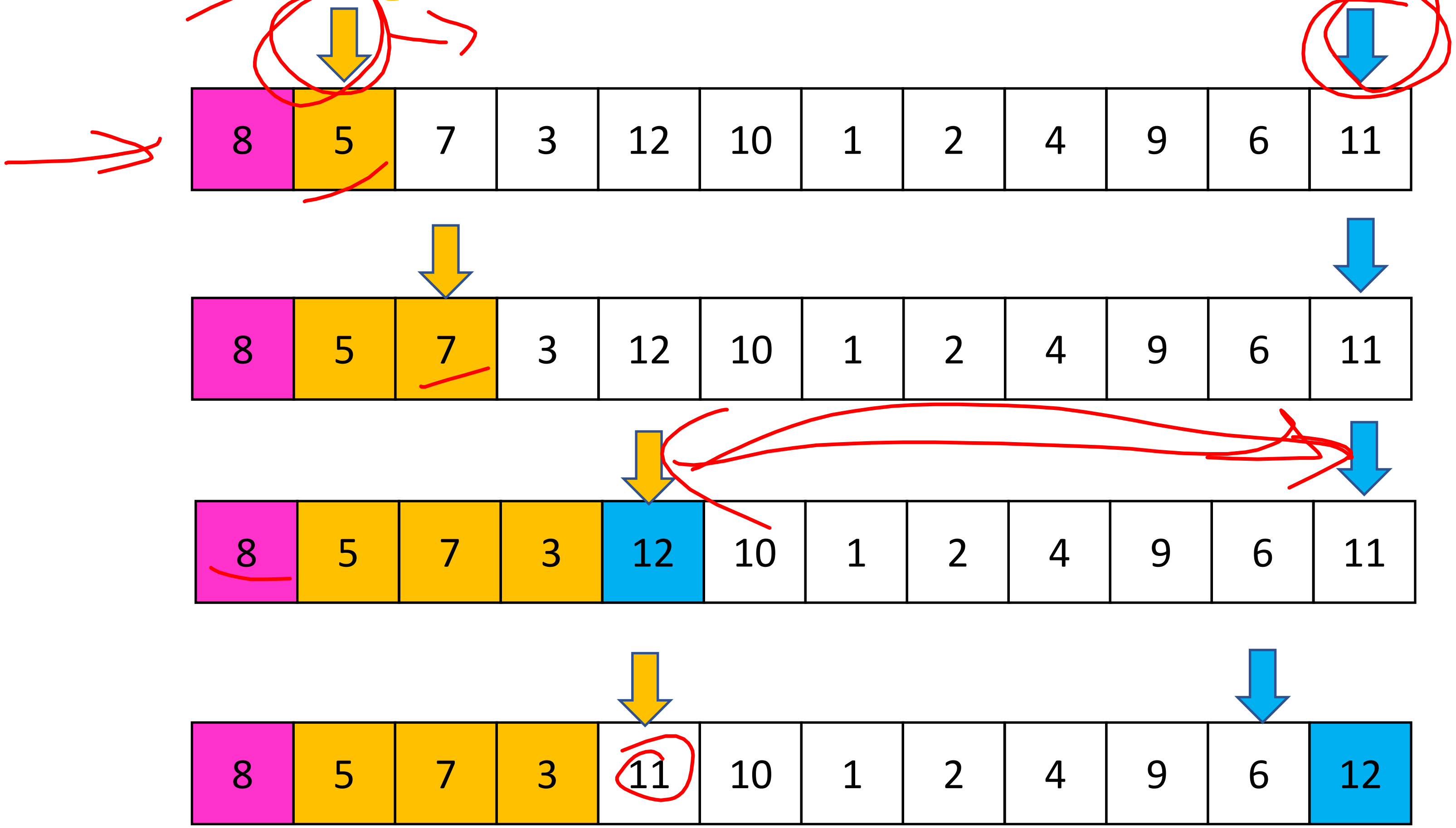
5	7	3	1	2	4	6	8	12	10	9	11

Partition, Procedure

If Begin value $<$ p, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End



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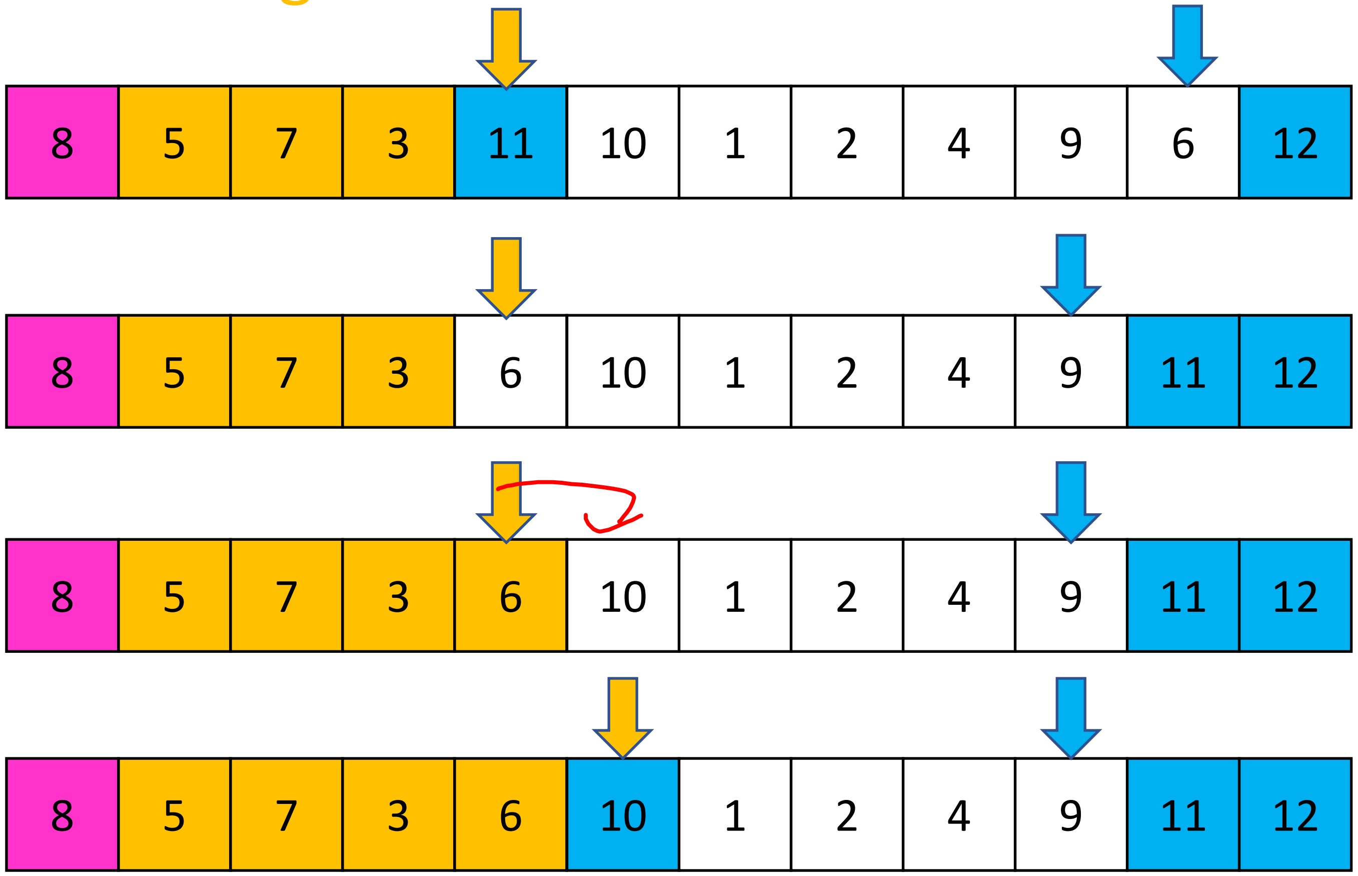

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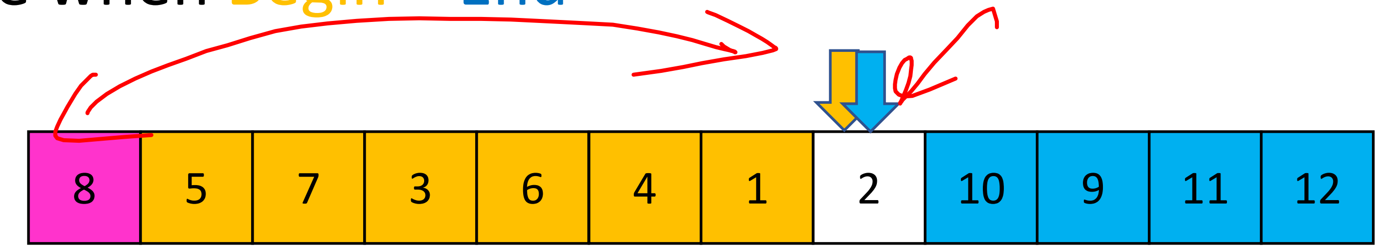
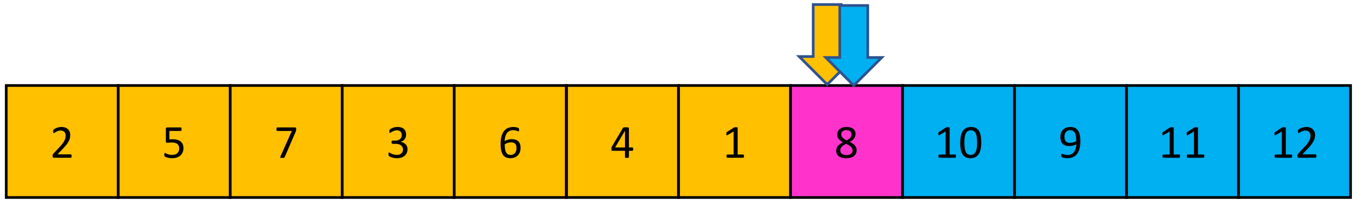


Case 1: meet at element $< p$

Swap p with pointer position (2 in this case)



Partition, Procedure

If Begin value $<$ p, move Begin right

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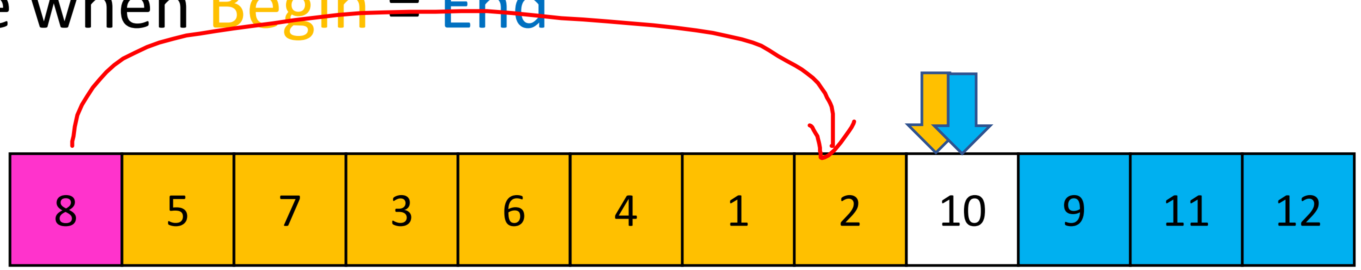
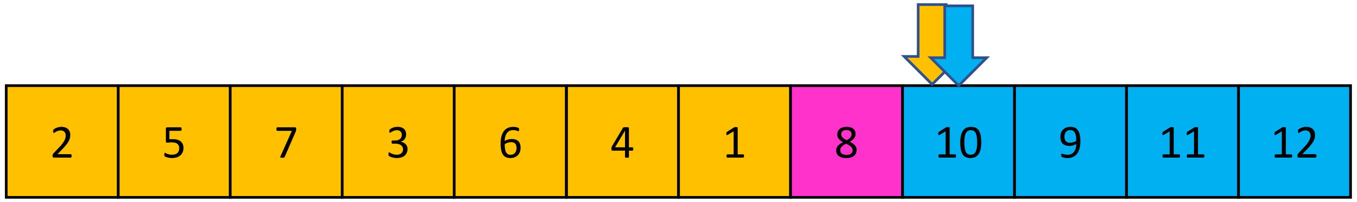
Done when Begin = End



Case 2: meet at element $> p$

Swap p with value to the left (2 in this case)



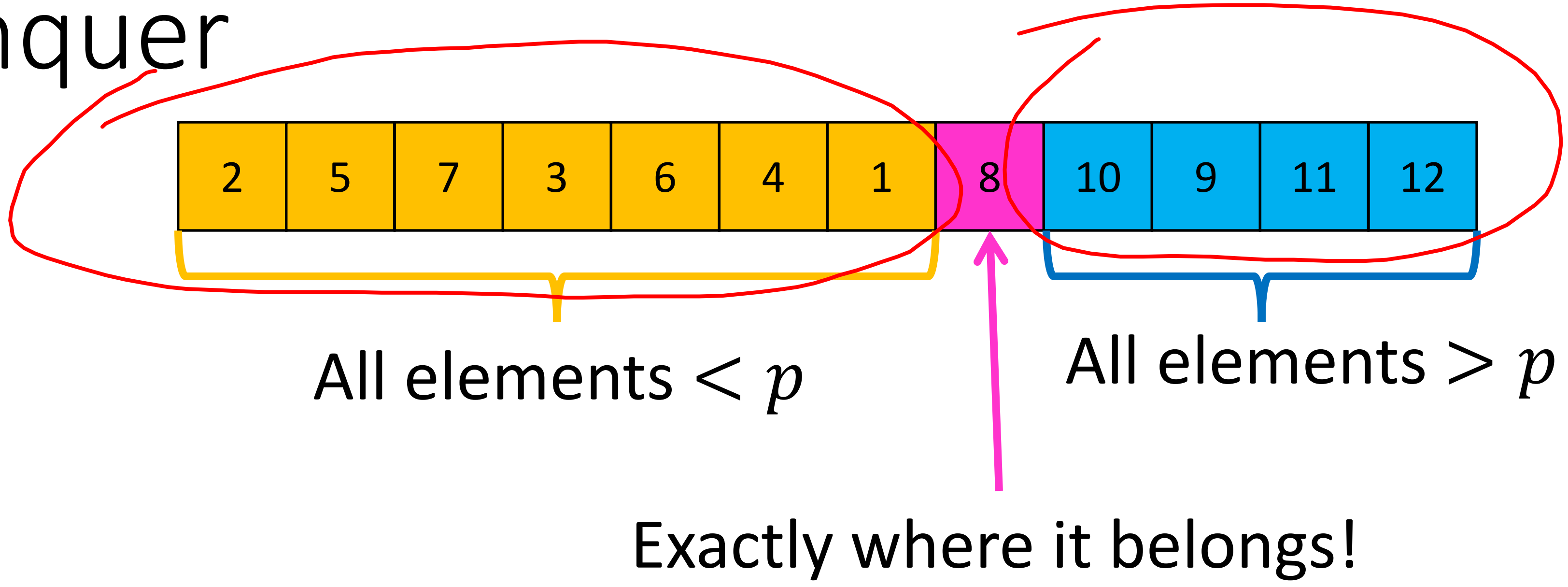
Partition Summary

O(n)

1. Put p at beginning of list
2. Put a pointer (Begin) just after p, and a pointer (End) at the end of the list
3. While Begin < End:
    1. If Begin value $<\ p$, move Begin right
    2. Else swap Begin value with End value, move End Left
4. If pointers meet at element $< p$: Swap p with pointer position
5. Else If pointers meet at element $> p$: Swap p with value to the left

Run time? $O(n)$

Conquer



Recursively sort Left and Right sublists

Quicksort Run Time (Best)

If the pivot is always the median:

2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

Then we divide in half each time

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(n) = O(n \log n)$$

Quicksort Run Time (Worst)

$T(n) = 1T(n-1)+n$

If the pivot is always at the extreme:

1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

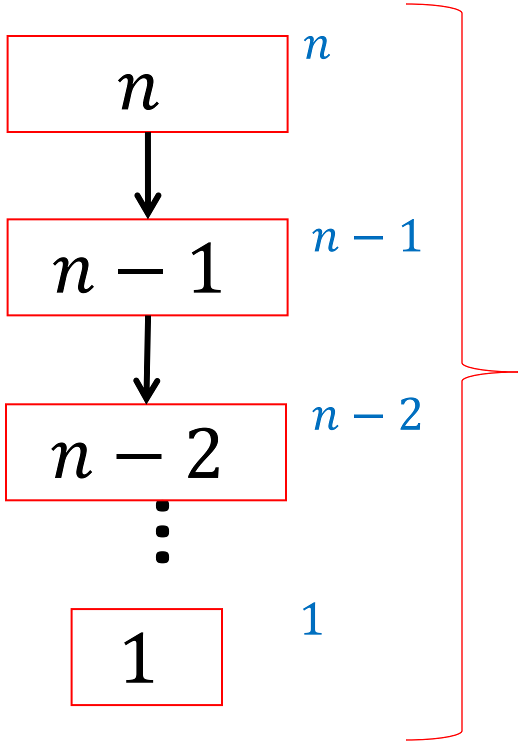
Then we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Quicksort Run Time (Worst)

$$T(n) = T(n-1) + n$$



$$T(n) = 1 + 2 + 3 + \cdots + n$$

$$T(n) = \frac{n(n+1)}{2}$$

$$T(n) = O(n^2)$$

Quicksort on a (nearly) Sorted List

First element always yields unbalanced pivot

1	2	3	4	5	6	7	8	9	10	11	12

1	2	3	4	5	6	7	8	9	10	11	12

So we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Good Pivot

- What makes a good Pivot?
  - Roughly even split between left and right
  - Ideally: median
- There are ways to find the median in linear time, but it’s complicated and slow and you’re better off using mergesort
- In Practice:
  - Pick a random value as a pivot
  - Pick the middle of 3 random values as the pivot

Properties of Quick Sort

- Worst Case Running time:
  - $\Theta(n^2)$
  - But $\Theta(n \log n)$ average! And typically faster than mergesort!
- In-Place?
  - ….Debatable
- Adaptive?
  - No!
- Stable?
  - No!

Improving Running time

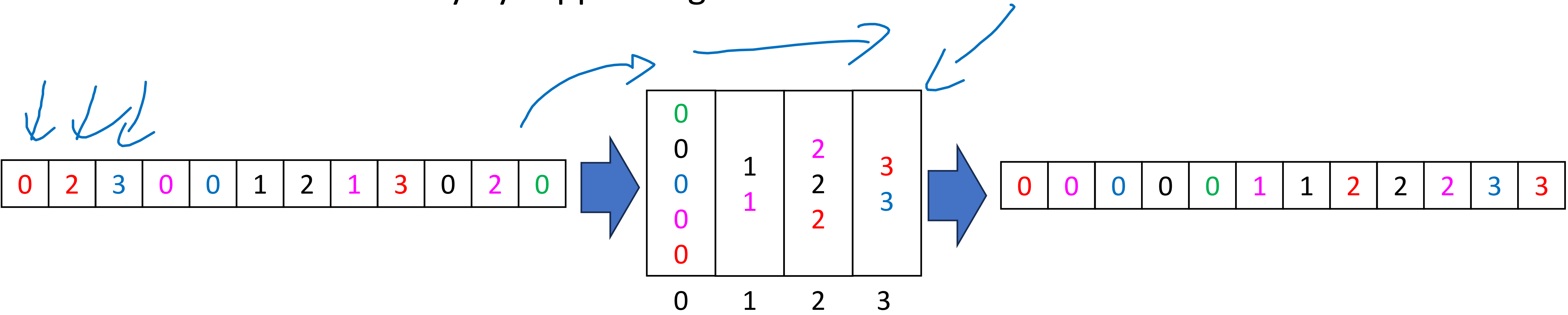
- Recall our definition of the sorting problem:
  - Input:
    - An array A of items
    - A comparison function for these items
      - Given two items x and y, we can determine whether $x < y$, $x > y$, or $x = y$
  - Output:
    - A permutation of A such that if $i \leq j$ then $A[i] \leq A[j]$
- Under this definition, it is impossible to write an algorithm faster than $n \log n$ asymptotically.
- Observation:
  - Sometimes there might be ways to determine the position of values without comparisons!

“Linear Time” Sorting Algorithms

- Useable when you are able to make additional assumptions about the contents of your list (beyond the ability to compare)
  - Examples:
    - The list contains only positive integers less than k
    - The number of distinct values in the list is much smaller than the length of the list
- The running time expression will always have a term other than the list’s length to account for this assumption
  - Examples:
    - Running time might be $\Theta(k \cdot n)$ where k is the range/count of values

BucketSort

- Assumes the array contains integers between 0 and $k - 1$ (or some other small range)
- Idea:
  - Use each value as an index into an array of size k
  - Add the item into the "bucket" at that index (e.g. linked list)
  - Get sorted array by "appending" all the buckets



BucketSort Running Time

- Create array of k buckets
  - Either $\Theta(k)$ or $\Theta(1)$ depending on some things…
- Insert all n things into buckets
  - $\Theta(n)$
- Empty buckets into an array
  - $\Theta(n+k)$
- Overall:
  - $\Theta(n+k)$
- When is this better than mergesort?

Properties of BucketSort

- In-Place?
  - No
- Adaptive?
  - No
- Stable?
  - Yes!

RadixSort

- Radix: The base of a number system
  - We'll use base 10, most implementations will use larger bases
- Idea:
  - BucketSort by each digit, one at a time, from least significant to most significant

103	801	401	323	255	823	999	101	113	901	555	512	245	800	018	121
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Place each element into a "bucket" according to its 1's place

800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999
0	1	2	3	4	5	6	7	8	9

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0	1	2	3	4	5	6	7	8	9
800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999

Place each element into a "bucket" according to its 10's place

0	1	2	3	4	5	6	7	8	9
800 801 401 101 901 103	512 113 018	121 323 823		245	255 555				999

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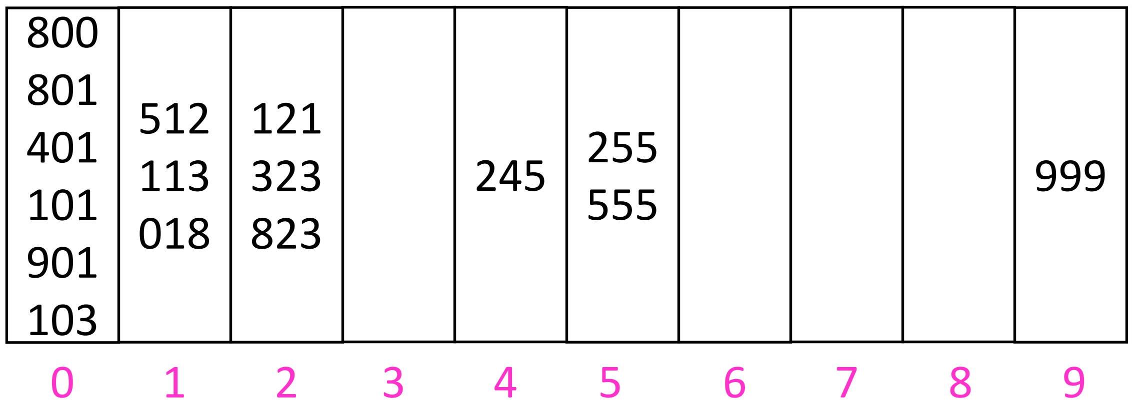
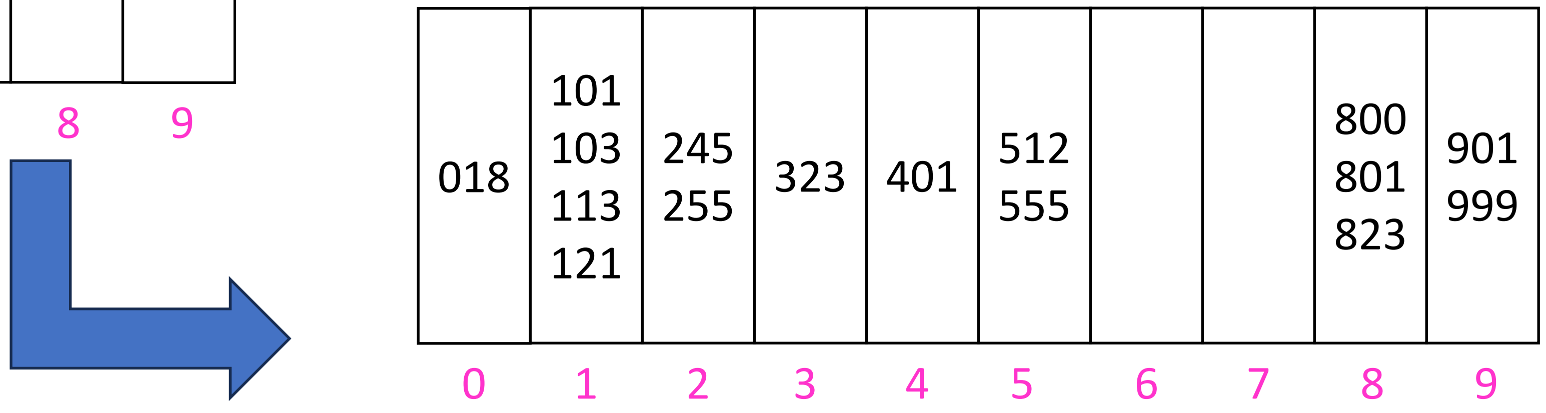


Place each element into a "bucket" according to its 100's place



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0	1	2	3	4	5	6	7	8	9
018	101 103 113 121	245 255	323	401	512 555			800 801 823	901 999

Convert back into an array

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
018	811	103	113	121	245	255	323	401	512	555	800	801	823	901	999

RadixSort Running Time

$n + b$

$(n + b) \log_b m$

- Suppose largest value is m
- Choose a radix (base of representation) b
- BucketSort all n things using b buckets
  - $\Theta(n + k)$
- Repeat once per each digit
  - $\log_b m$ iterations
- Overall:
  - $\Theta(n \log_b m + b \log_b m)$
- In practice, you can select the value of b to optimize running time
- When is this better than mergesort?