CSE 332 Summer 2024
Lecture 12: Sorting

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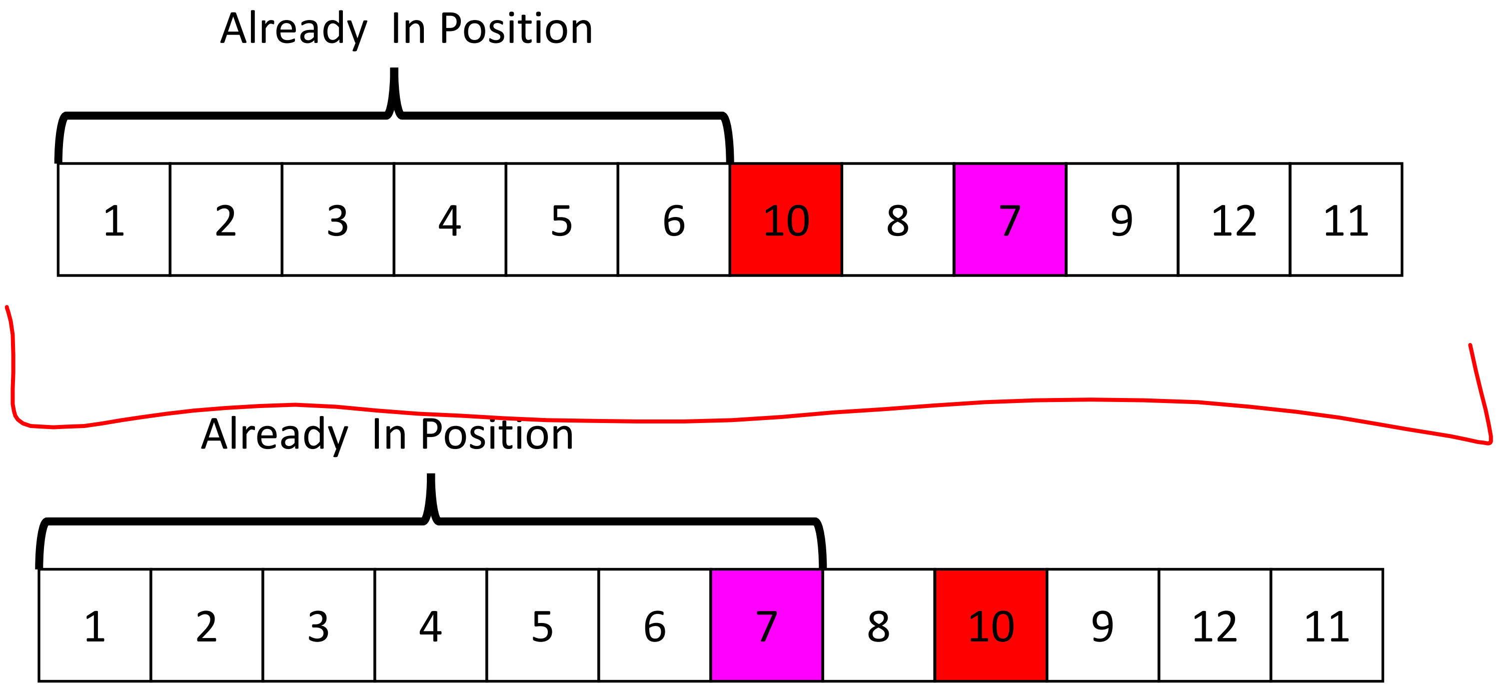
http://www.cs.uw.edu/332

Selection Sort

in place

- Idea: Find the next smallest element, swap it into the next index in the array



Already In Position

1	2	3	4	5	6	10	8	7	9	12	11

Already In Position

1	2	3	4	5	6	7	8	10	9	12	11

Selection Sort

- Swap the thing at index 0 with the smallest thing in the array
- Swap the thing at index 1 with the smallest thing after index 0
- …
- Swap the thing at index i with the smallest thing after index $i-1$

```
for (i=0; i<a.length; i++){
    smallest = i;
    for (j=i; j<a.length; j++){
        if (a[j]<a[smallest]){ smallest=j;}
    }
    temp = a[i];
    a[i] = a[smallest];
    a[smallest] = a[i];
}
```

Running Time:

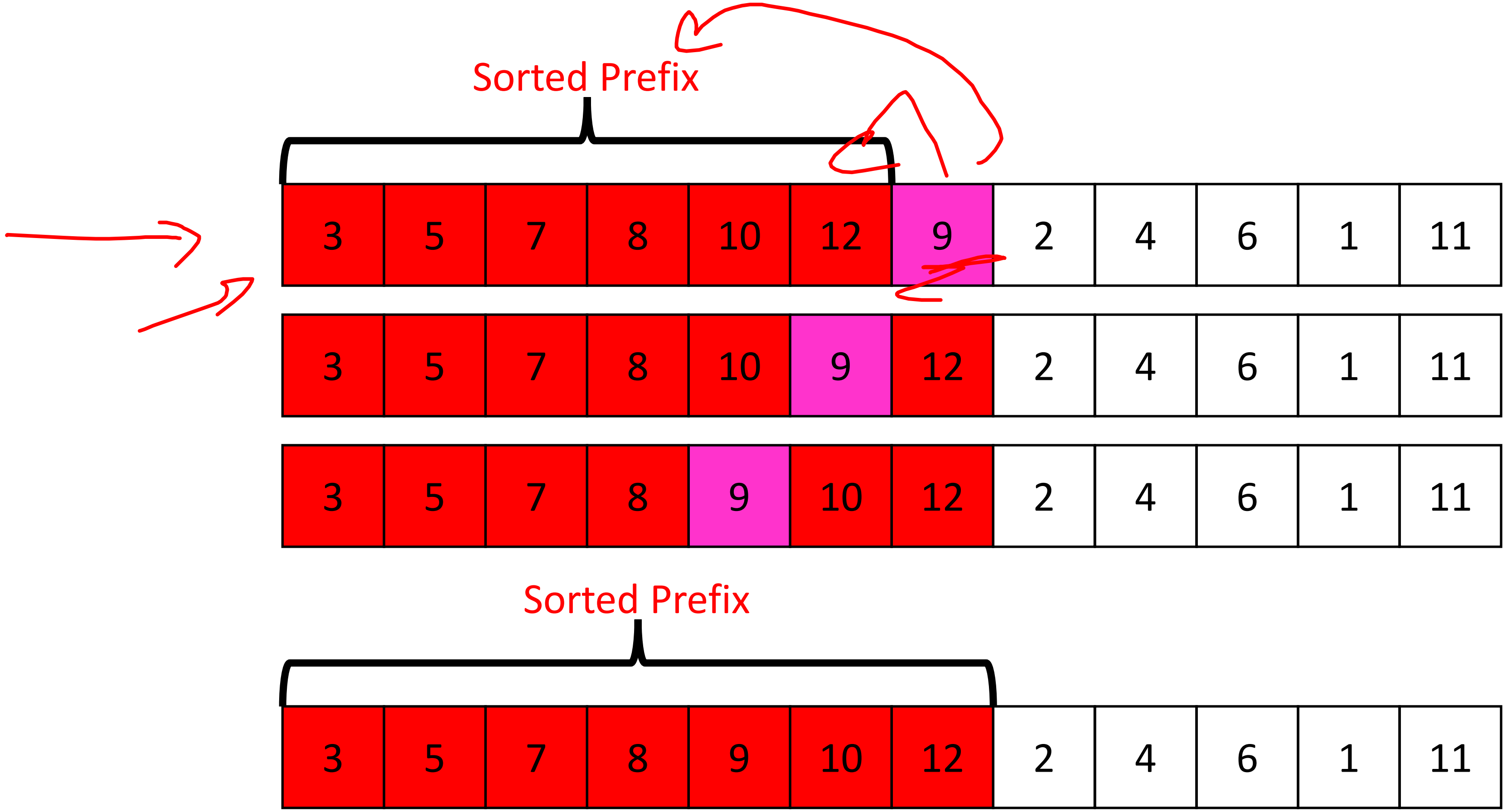
Worst Case: $\Theta(\cdot)$ n^2

Best Case: $\Theta(\cdot)$ n^2

10	77	5	15	2	22	64	41	18	19	30	21	3	24	23	33
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



Sorted Prefix

3	5	7	8	10	12	9	2	4	6	1	11
3	5	7	8	10	9	12	2	4	6	1	11
3	5	7	8	9	10	12	2	4	6	1	11

Sorted Prefix

3	5	7	8	9	10	12	2	4	6	1	11

Insertion Sort

adaptive

- If the items at index 0 and 1 are out of order, swap them
- Keep swapping the item at index 2 with the thing to its left as long as the left thing is larger
- …
- Keep swapping the item at index i with the thing to its left as long as the left thing is larger

```
for (i=1; i<a.length; i++){
    prev = i-1;
    while(a[i] < a[prev] && prev > -1){
        temp = a[i];
        a[i] = a[prev];
        a[prev] = a[i];
        i--;
        prev--;
    }
}
```

Running Time:

Worst Case: $\Theta(\cdot)$ n^2

Best Case: $\Theta(\cdot)$ n

10	77	5	15	2	22	64	41	18	19	30	21	3	24	23	33
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Aside: Bubble Sort – we won’t cover it

"the bubble sort seems to have nothing to recommend it, except a catchy name and the fact that it leads to some interesting theoretical problems” –Donald Knuth, The Art of Computer Programming

Properties

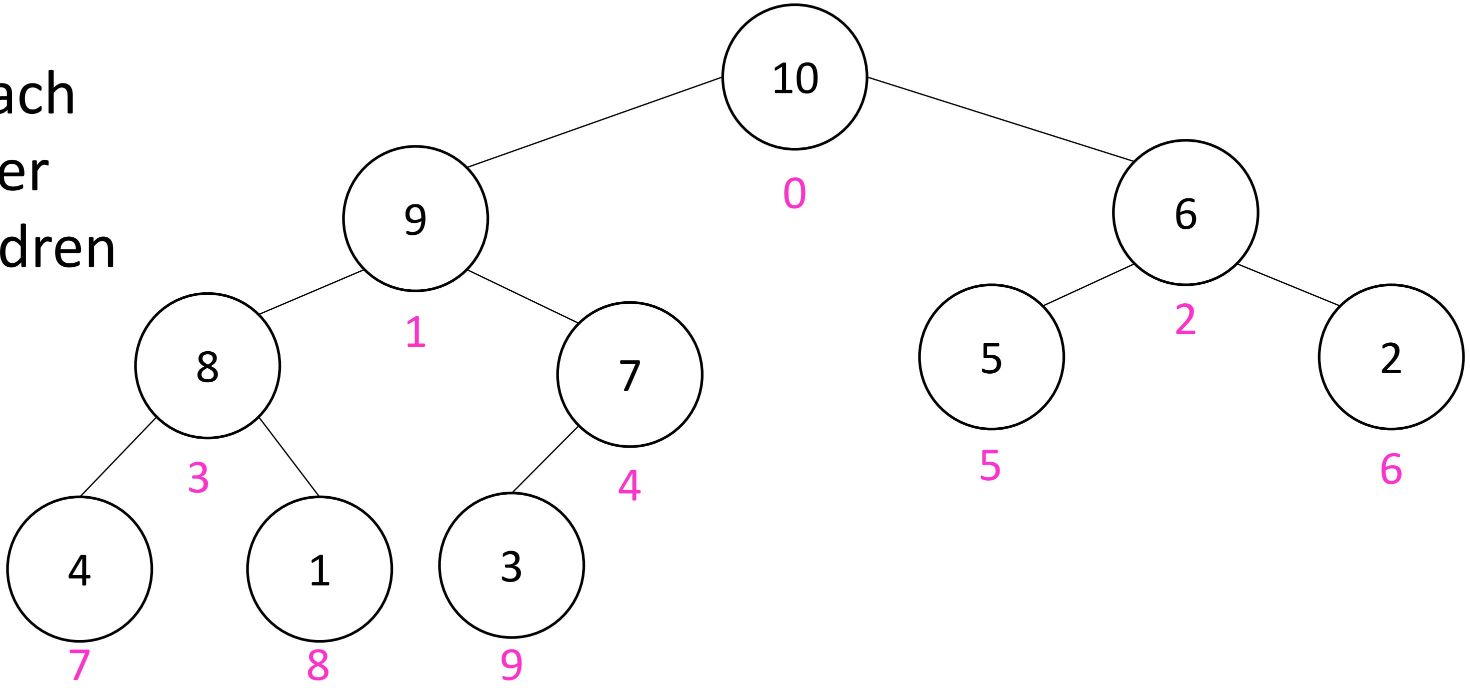
- Worst case running time
    - n^2
- In place:
    - We only need to use the pre-existing array to do sorting
    - Constant extra space (only some additional variables needed)
- Adaptive
    - The running improves as the given list is closer to being sorted
    - It should be linear time for a pre-sorted list, and nearly linear time if the list is nearly sorted
- Online
    - We can start sorting before we have the entire list.

Heap Sort

- Idea: Build a maxHeap, repeatedly delete the max element from the heap to build sorted list Right-to-Left

10	9	6	8	7	5	2	4	1	3
0	1	2	3	4	5	6	7	8	9

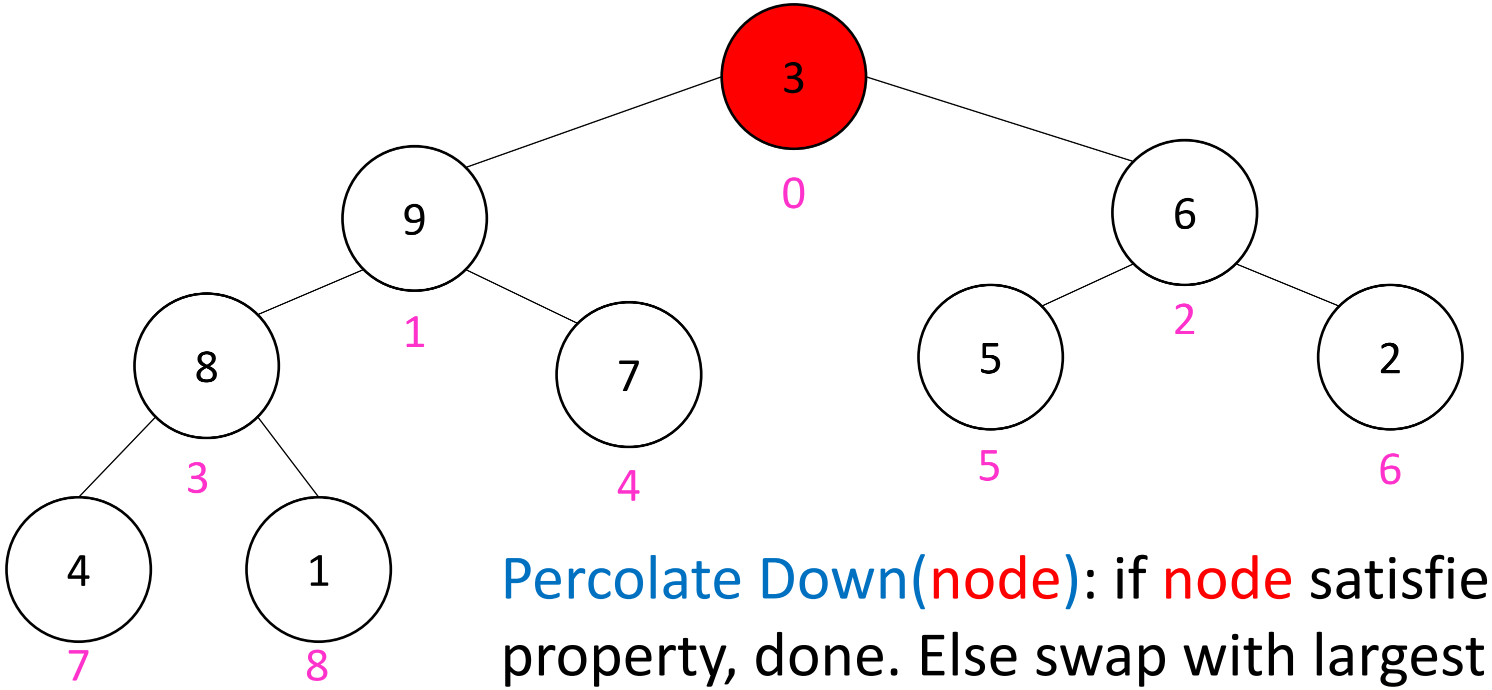
Max Heap Property: Each node is larger than its children



Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call percolateDown(root)

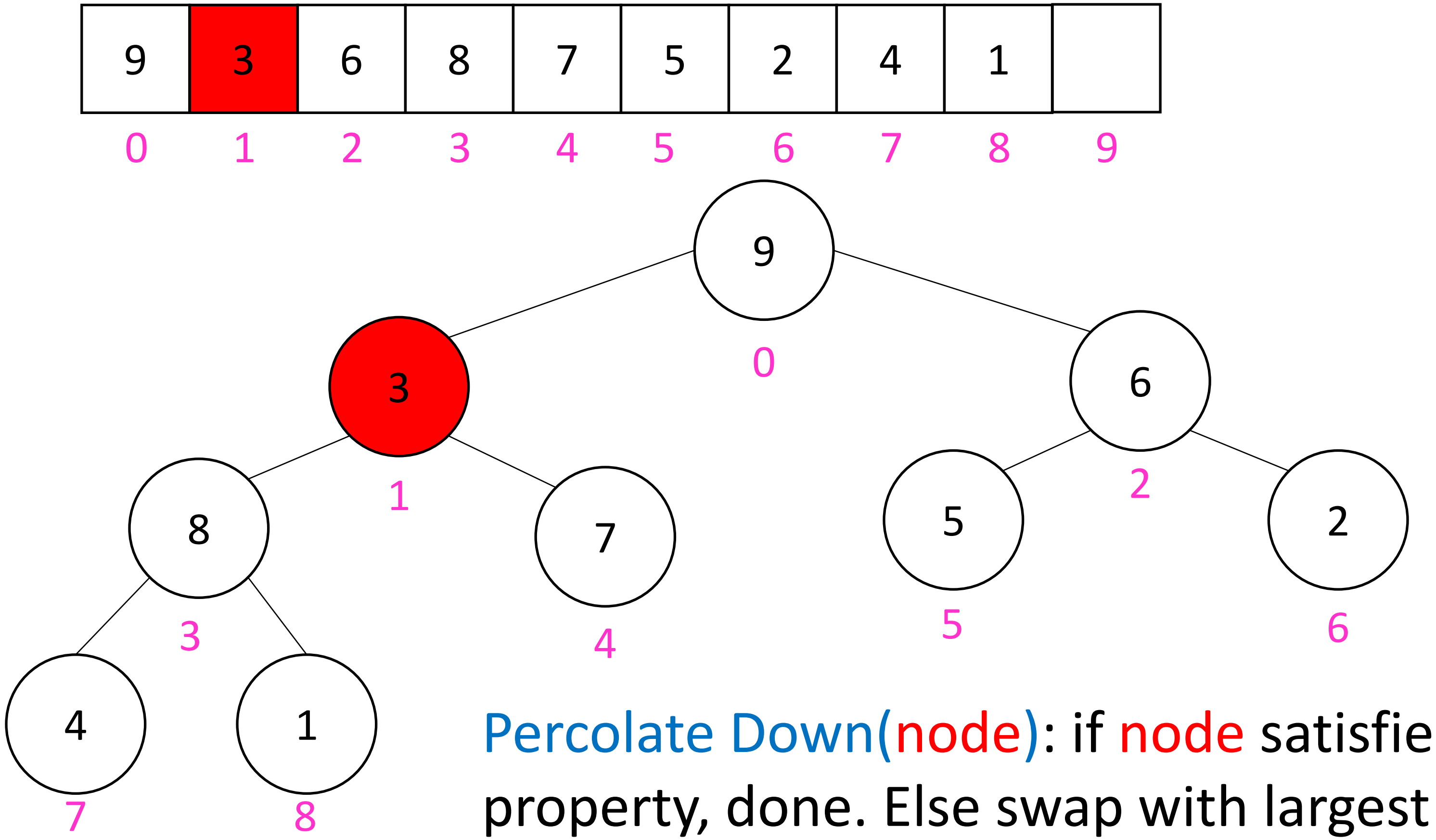
3	9	6	8	7	5	2	4	1	
0	1	2	3	4	5	6	7	8	9



Percolate Down(node): if node satisfies heap property, done. Else swap with largest child and repeat on that subtree

Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call percolateDown(root)



9	3	6	8	7	5	2	4	1	
0	1	2	3	4	5	6	7	8	9

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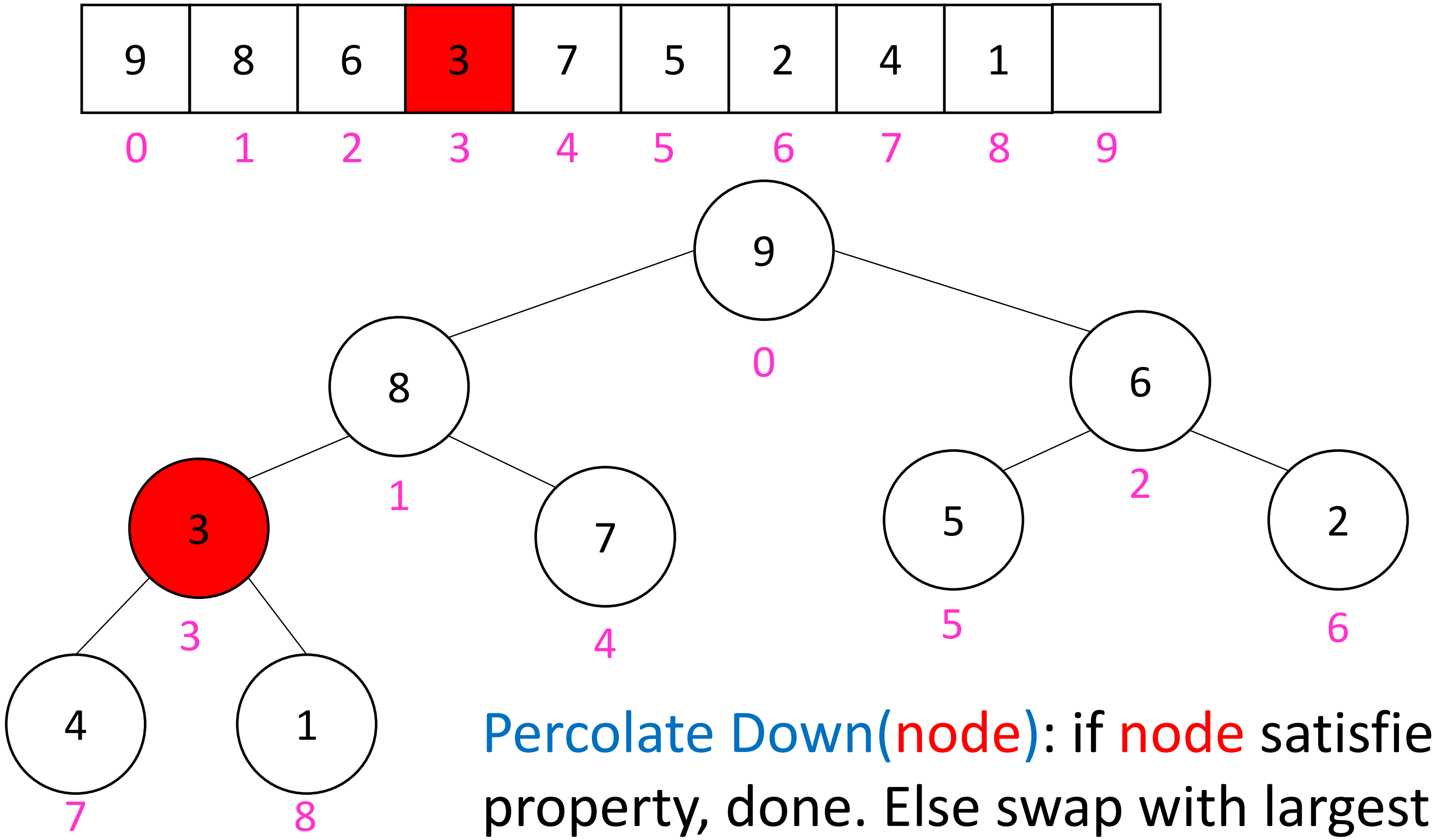

9	8	6	3	7	5	2	4	1	
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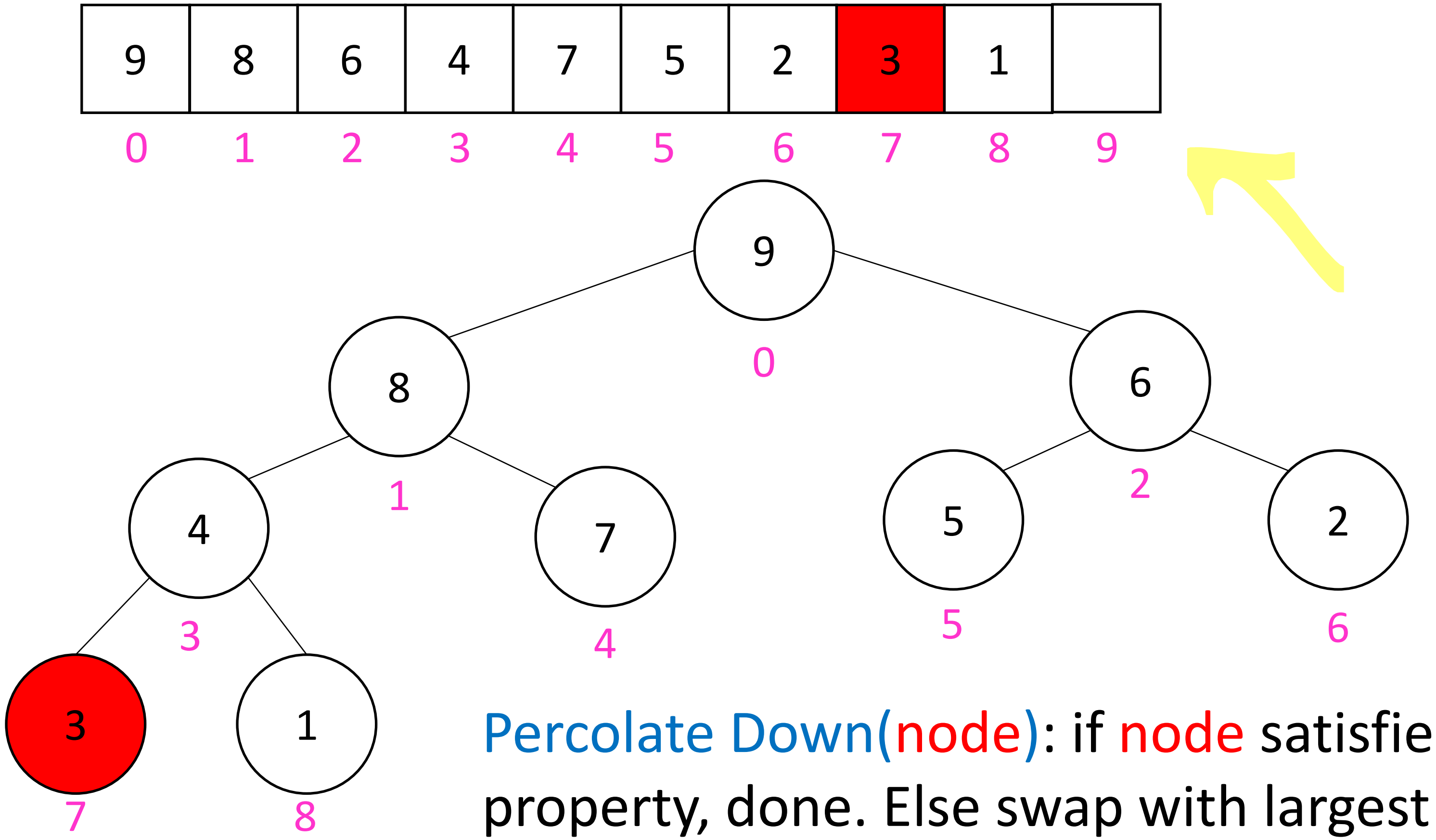


0	1	2	3	4	5	6	7	8	9
9	8	6	4	7	5	2	🟥 3	1	

Percolate Down(node): if node satisfies heap property, done. Else swap with largest child and repeat on that subtree

Heap Sort

- Build a heap
  - $O(n)$
- Call extract
  - $O(\log n)$ each
- Put that at the end of the array
  - $O(1)$

```
myHeap = buildHeap(a);
for (int i = a.length-1; i>=0; i--){
	item = myHeap.deleteMax();
	a[i] = item;
}
```

Running Time:

Worst Case: $\Theta(n \log n)$

Best Case: $\Theta(n \log n)$

“In Place” Sorting Algorithm

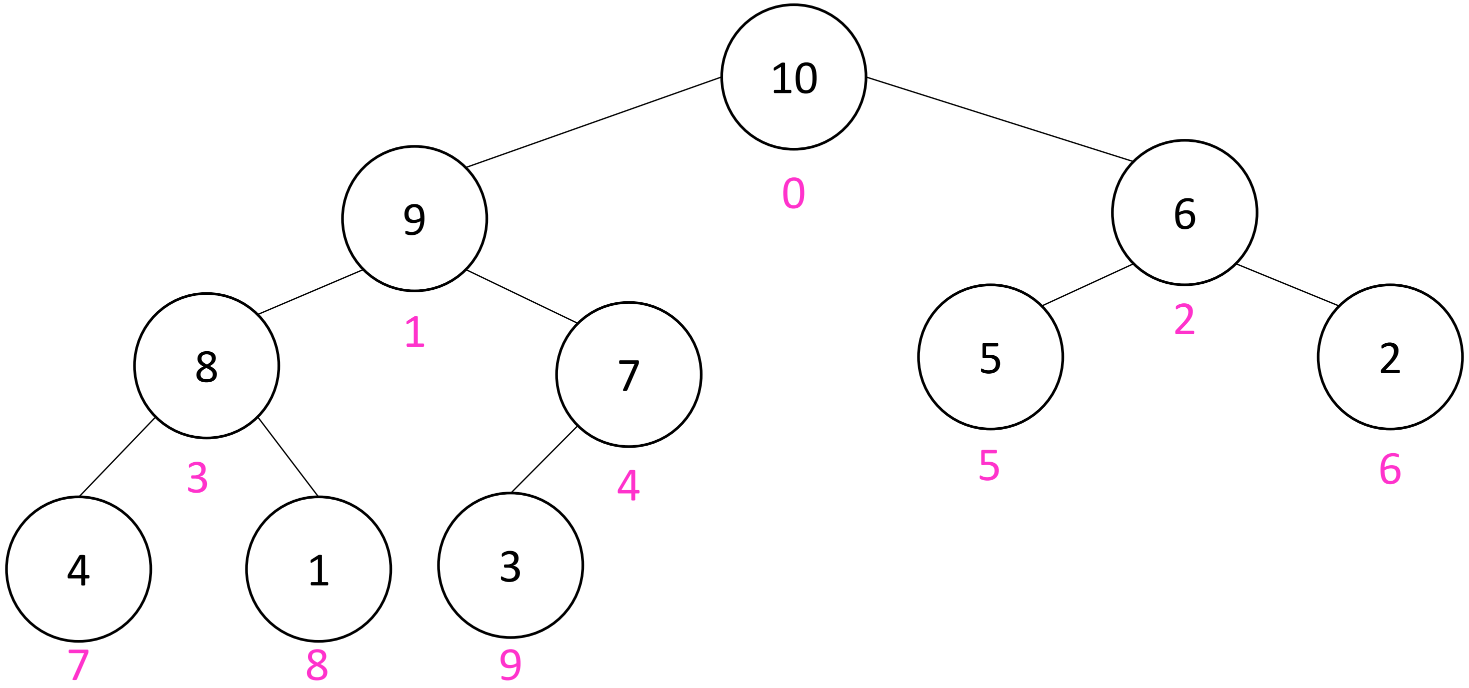
- A sorting algorithm which requires no extra data structures
- Idea: It sorts items just by swapping things in the same array given
- Definition: it only uses $\Theta(1)$ extra space

- Selection sort: In Place!
- Insertion sort: In Place!
- Heap sort: Not In Place!
    - But we can fix that!

In Place Heap Sort

- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

10	9	6	8	7	5	2	4	1	3
0	1	2	3	4	5	6	7	8	9



Heap Sort

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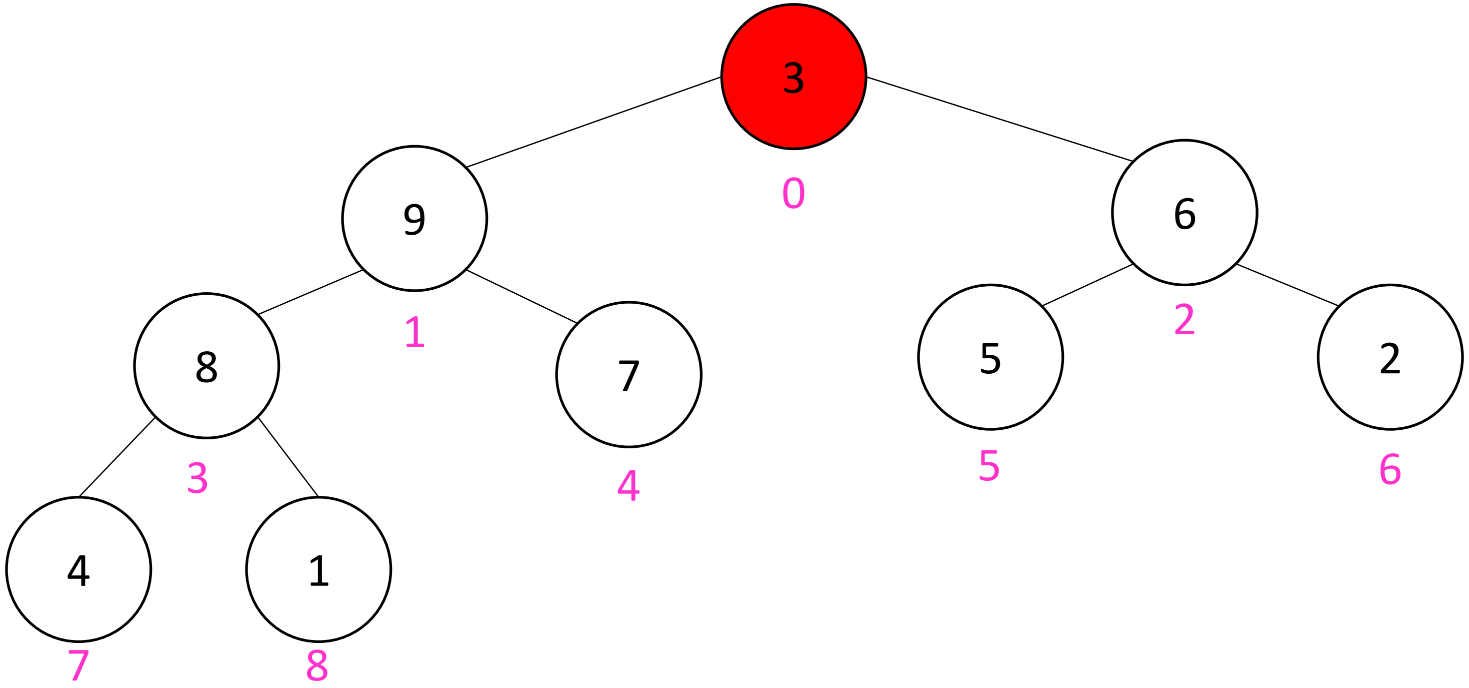

Heap Sort

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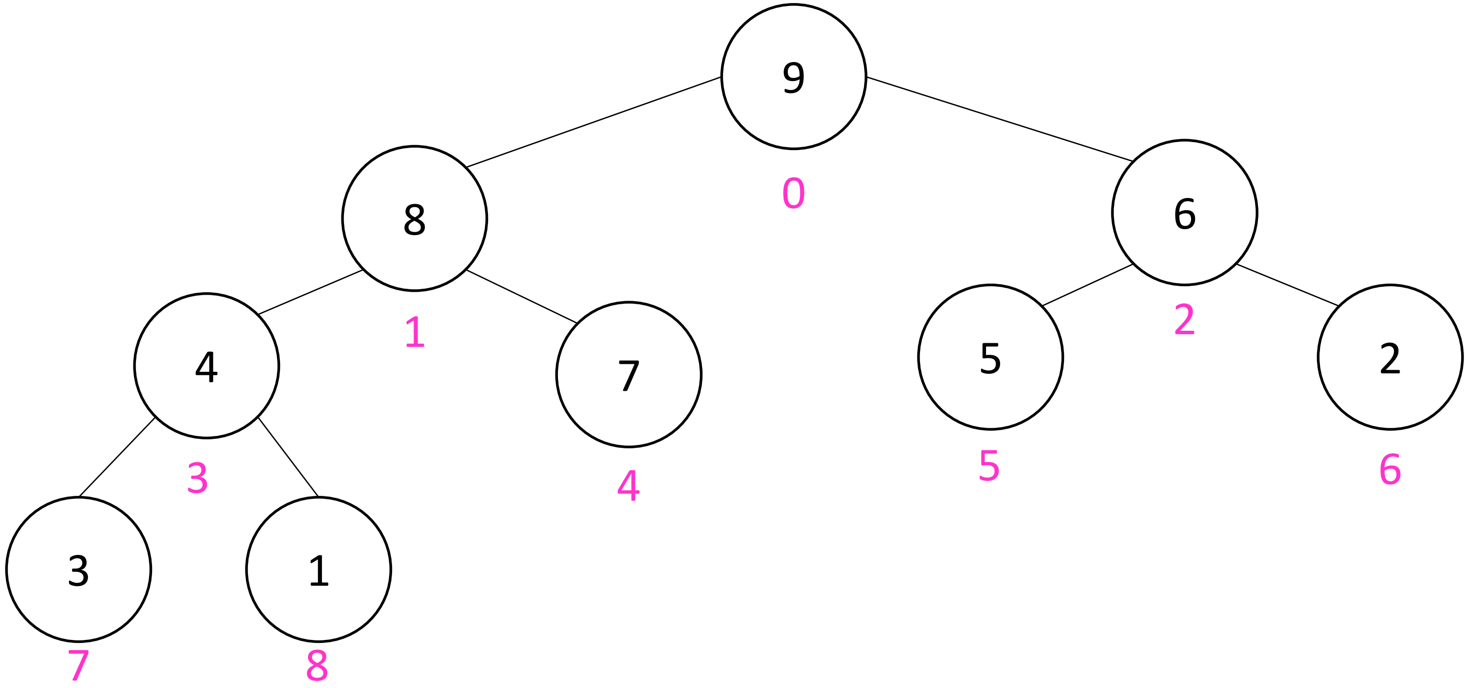

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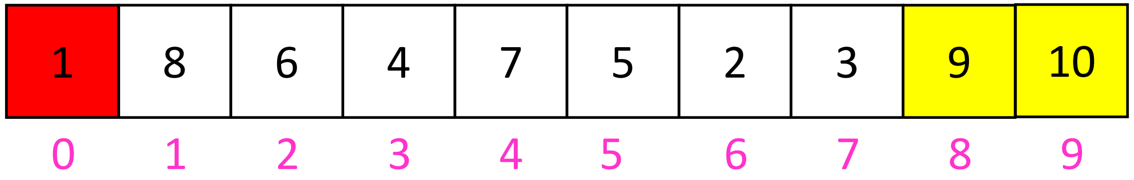
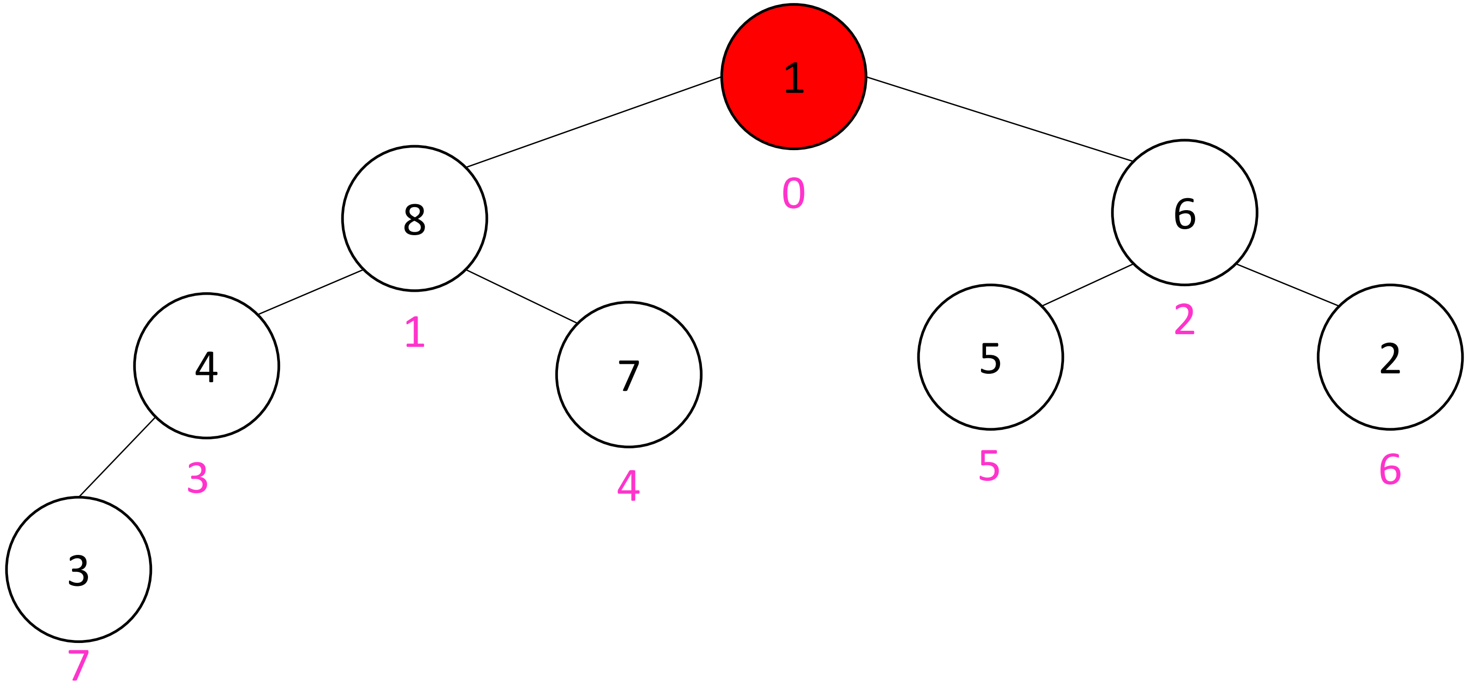


1	8	6	4	7	5	2	3	9	10
0	1	2	3	4	5	6	7	8	9



Heap Sort

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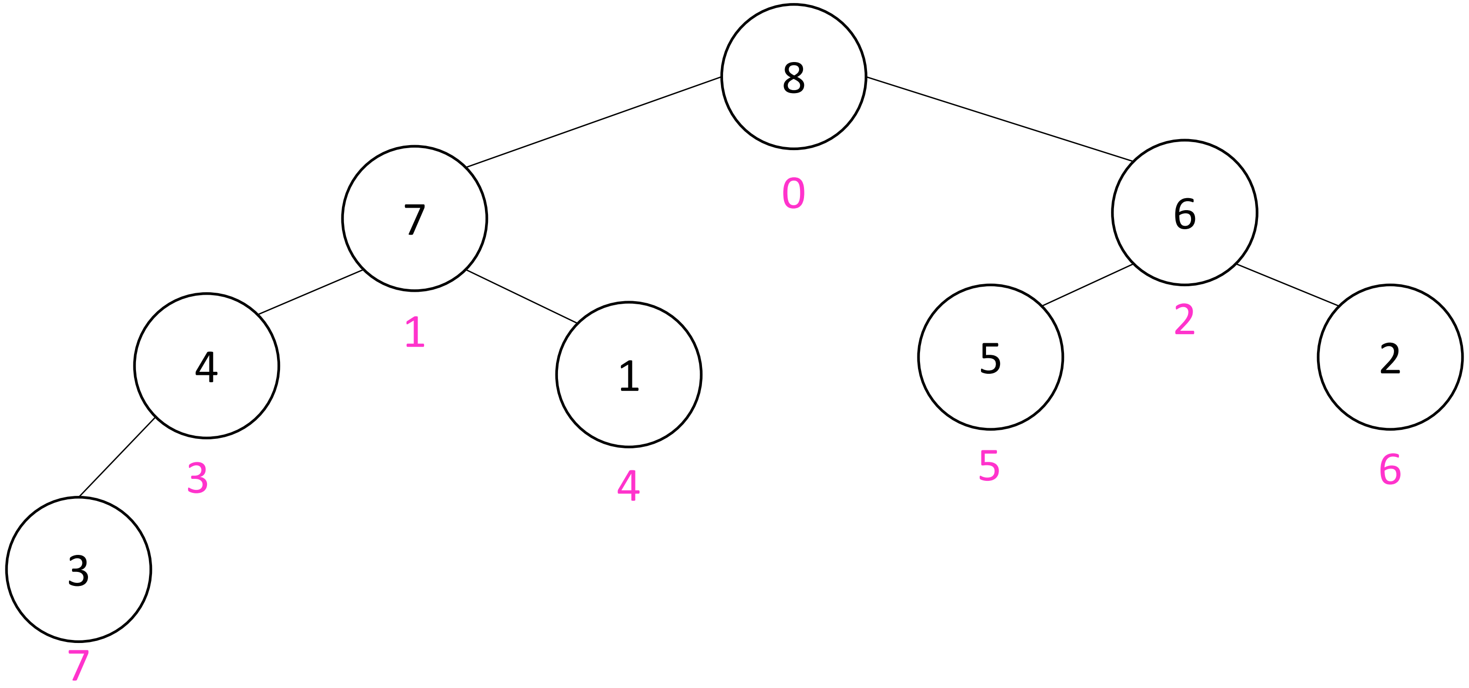
1	8	6	4	7	5	2	3	9	10
0	1	2	3	4	5	6	7	8	9



Heap Sort

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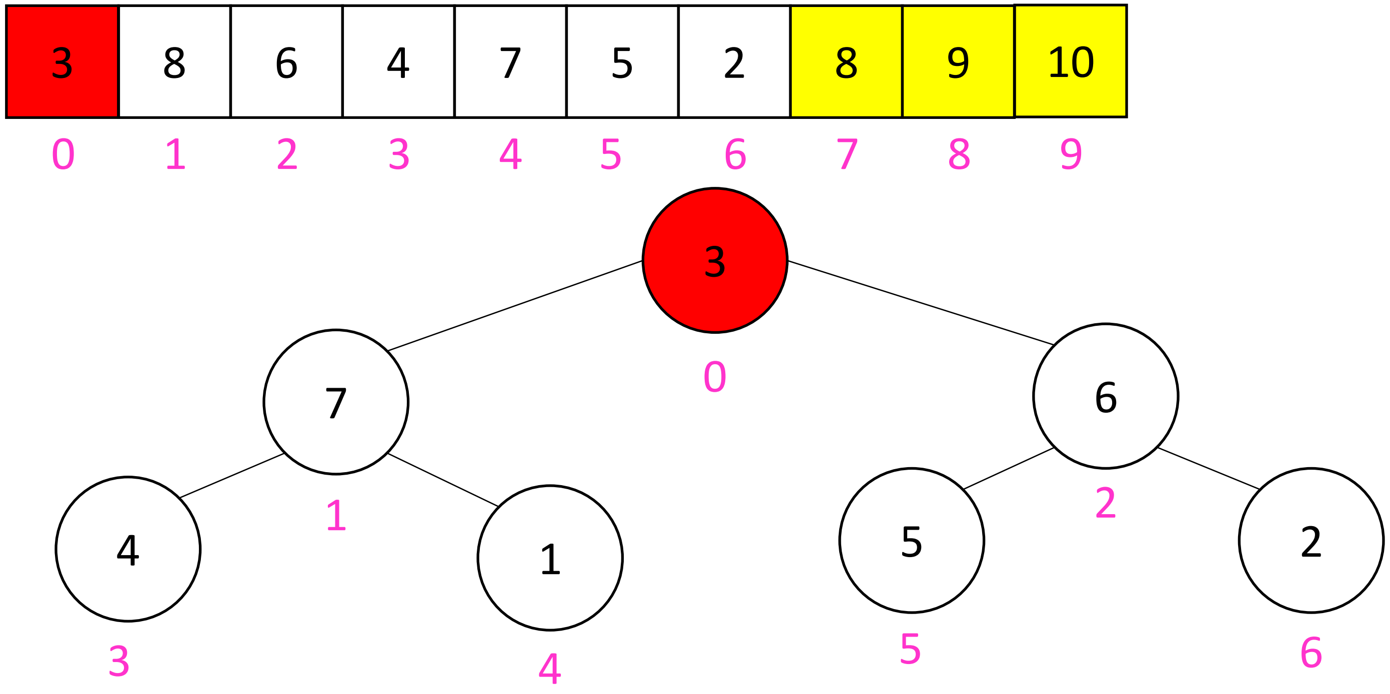

3	8	6	4	7	5	2	8	9	10
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Heap Sort

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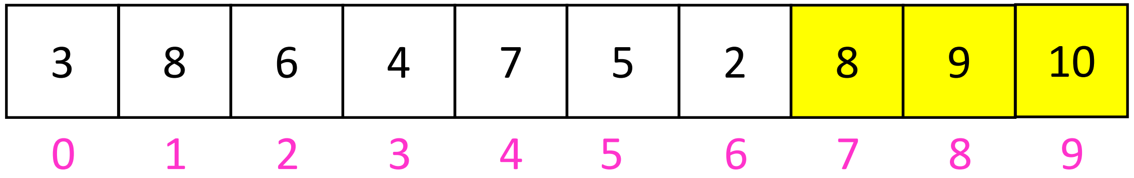

3	8	6	4	7	5	2	8	9	10
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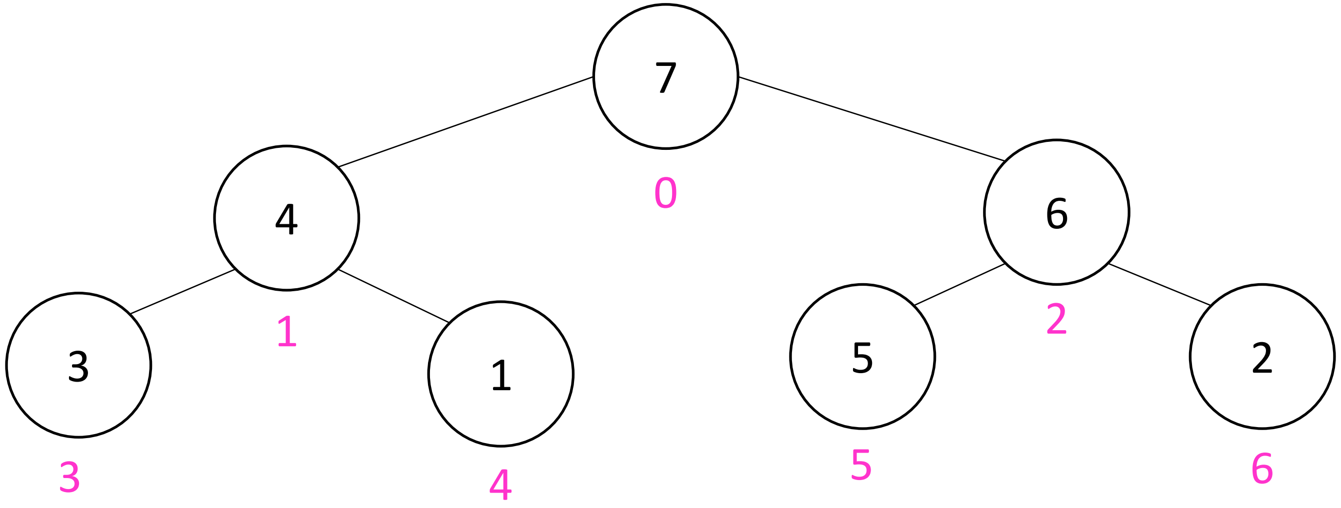


In Place Heap Sort

- Build a heap using the same array (Floyd's build heap algorithm works)
- Call deleteMax
- Put that at the end of the array

```
buildHeap(a);
for (int i = a.length-1; i>=0; i--){
    temp=a[i]
    a[i] = a[0];
    a[0] = temp;
    percolateDown(0);
}
```

Running Time:

Worst Case: $\Theta(n \log n)$

Best Case: $\Theta(n \log n)$

Floyd’s buildHeap method

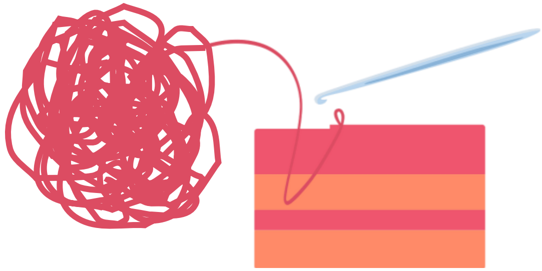
- Working towards the root, one row at a time, percolate down

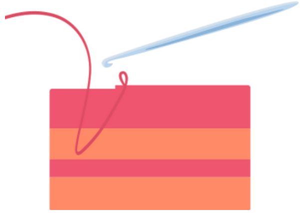
```
buildHeap(){
    for(int i = size; i>0; i--){
        percolateDown(i);
    }
}
```

Divide And Conquer Sorting

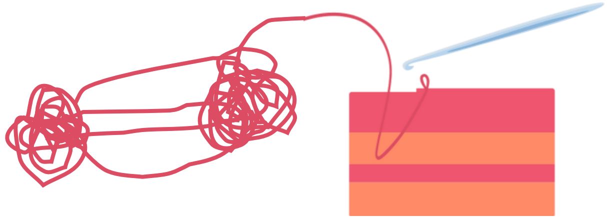
- Divide and Conquer:
  - Recursive algorithm design technique
  - Solve a large problem by breaking it up into smaller versions of the same problem

Divide and Conquer

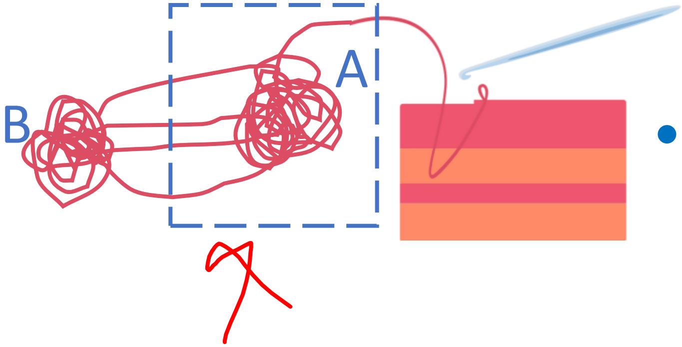




- **Base Case**:
  - If the problem is "small" then solve directly and return



- **Divide**:
  - Break the problem into subproblem(s), each smaller instances



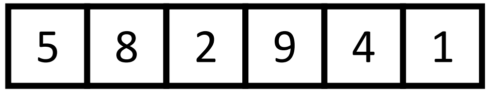
- **Conquer**:
  - Solve subproblem(s) recursively

- **Combine**:
  - Use solutions to subproblems to solve original problem

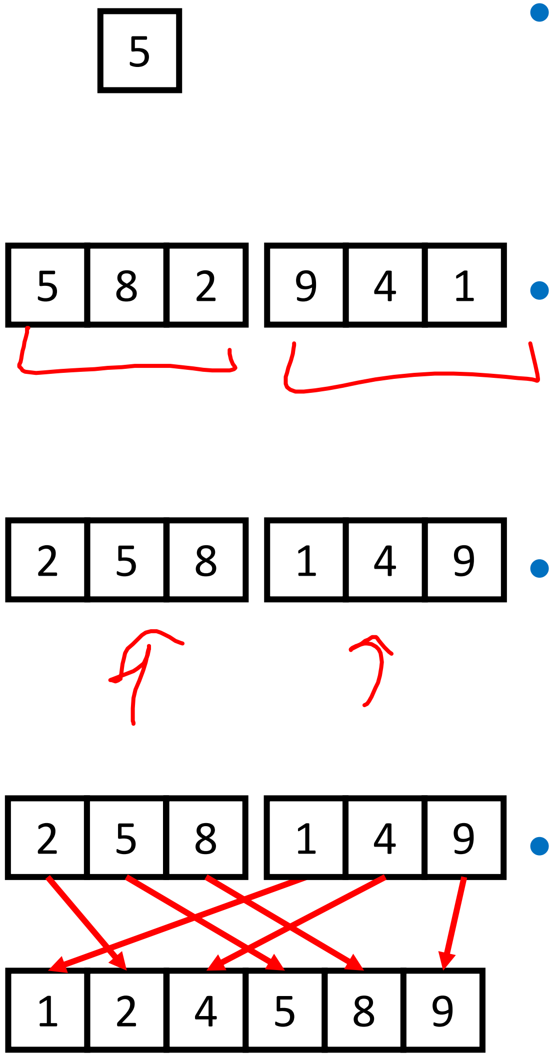
Divide and Conquer Template Pseudocode

```
def my_DandC(problem){
   // Base Case
   if (problem.size() <= small_value){
      return solve(problem);  // directly solve (e.g., brute force)
   }
   // Divide
   List subproblems = divide(problem);

   // Conquer
   solutions = new List();
   for (sub : subproblems){
      subsolution = my_DandC(sub);
      solutions.add(subsolution);
   }
   // Combine
   return combine(solutions);
}
```



Merge Sort



- **Base Case:**
  - If the list is of length 1 or 0, it's already sorted, so just return it
- **Divide:**
  - Split the list into two "sublists" of (roughly) equal length
- **Conquer:**
  - Sort both lists recursively
- **Combine:**
  - **Merge** sorted sublists into one sorted list

Merge Sort In Action!

Sort between indices low and $high$

5	8	2	9	4	1	3	7

low $\qquad\qquad$ $high$

Base Case: if $low == high$ then that range is already sorted!

Divide and Conquer: Otherwise call mergesort on ranges $\left(low, \frac{low+high}{2}\right)$ and $\left(\frac{low+high}{2}+1, high\right)$

5	8	2	9	4	1	3	7

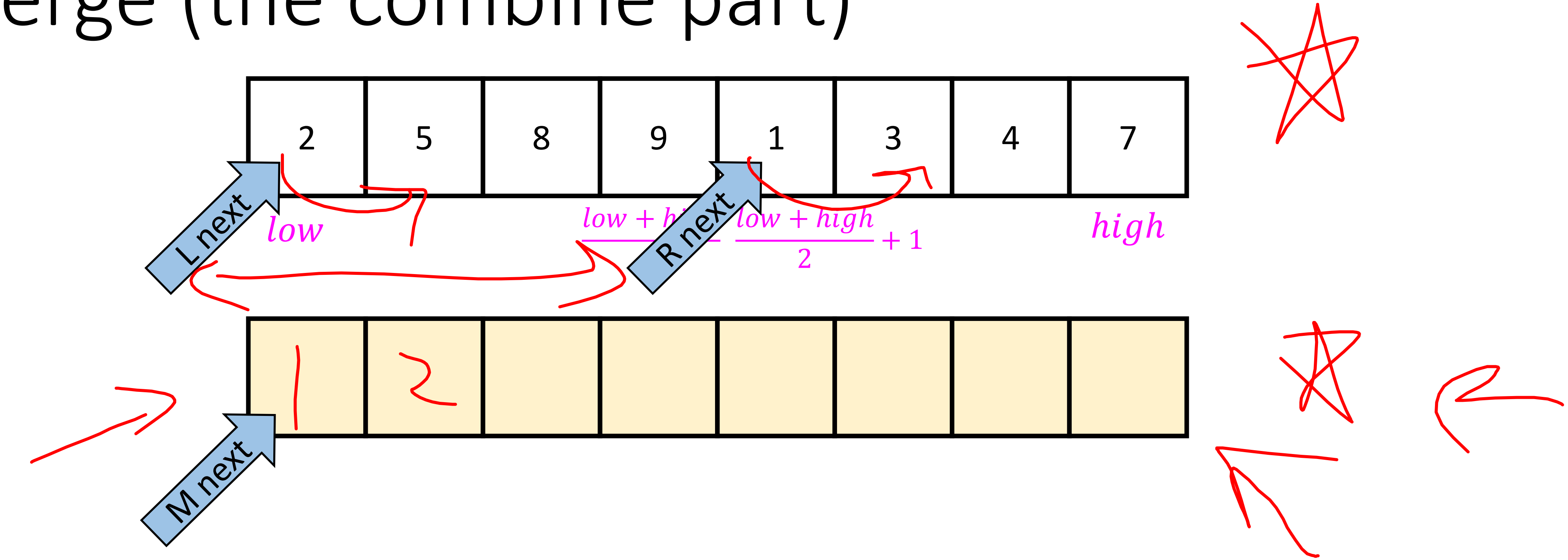
low $\qquad$ $\frac{low + high}{2}$ $\quad$ $\frac{low + high}{2} + 1$ $\qquad$ $high$

After Recursion:

2	5	8	9	1	3	4	7

low $\qquad\qquad$ $high$

Merge (the combine part)



Create a new array to merge into, and 3 pointers/indices:

- L_next: the smallest “unmerged” thing on the left
- R_next: the smallest “unmerged” thing on the right
- M_next: where the next smallest thing goes in the merged array

One-by-one: put the smallest of L_next and R_next into M_next, then advance both M_next and whichever of L/R was used.

Merge Sort Pseudocode

```
void mergesort(myArray){
        ms_helper(myArray, 0, myArray.length());
}
void mshelper(myArray, low, high){
        if (low == high){return;}  // Base Case
        mid = (low+high)/2;
        ms_helper(low, mid);
        ms_helper(mid+1, high);
        merge(myArray, low, mid, high);
}
```

Merge Pseudocode

```
void merge(myArray, low, mid, high){
        merged = new int[high-low+1]; // or whatever type is in myArray
        l_next = low;
        r_next = high;
        m_next = 0;
        while (l_next <= mid && r_next <= high){
                if (myArray[l_next] <= myArray[r_next]){
                        merged[m_next++] = myArray[l_next++];
                }
                else{
                        merged[m_next++] = myArray[r_next++];
                }
        }
        while (l_next <= mid){ merged[m_next++] = myArray[l_next++]; }
        while (r_next <= high){  merged[m_next++] = myArray[r_next++]; }
        for(i=0; i<=merged.length; i++){ myArray[i+low] = merged[i];}
}
```

Analyzing Merge Sort

$T(n) = aT(\frac{n}{b}) + f(n)$

$= 2T(\frac{n}{2}) +$

1. Identify time required to Divide and Combine
2. Identify all subproblems and their sizes
3. Use recurrence relation to express recursive running time
4. Solve and express running time asymptotically

- **Divide**: 0 comparisons
- **Conquer**: recursively sort two lists of size $\frac{n}{2}$
- **Combine**: n comparisons
- **Recurrence:**

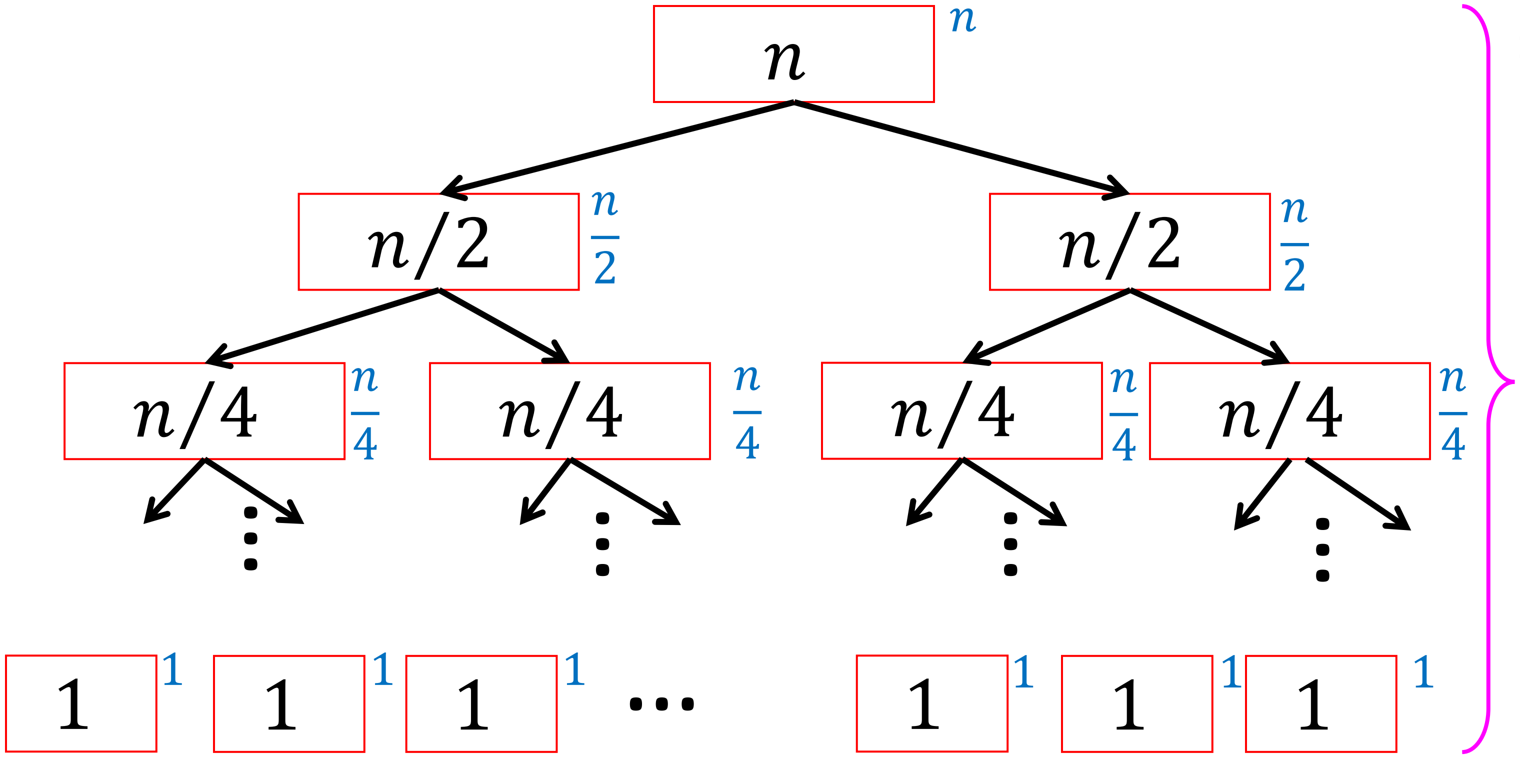
$$T(n) = 0 + T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n$$

$$\rightarrow T(n) = 2T\left(\frac{n}{2}\right) + n$$

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T(\frac{n}{2}) + n$$



$\Rightarrow n$ comparisons / level

$\log_2 n$ levels of recursion

$$T(n) = \sum_{i=1}^{\log_2 n} n = n \log_2 n$$

Properties of Merge Sort

- Worst Case Running time:
  - $\Theta(n \log n)$
- In-Place?
  - No!
- Adaptive?
  - No!
- Stable?
  - Yes!
  - As long as in a tie you always pick l_next

Quicksort

- Like Mergesort:
  - Divide and conquer
  - $O(n \log n)$ run time (kind of…)
- Unlike Mergesort:
  - Divide step is the “hard” part
  - *Typically* faster than Mergesort

Quicksort

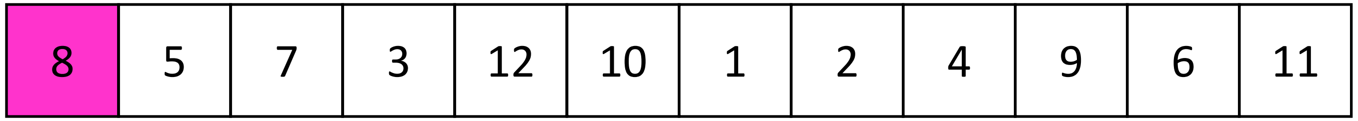
Idea: pick a pivot element, recursively sort two sublists around that element

- Divide: select pivot element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Partition (Divide step)

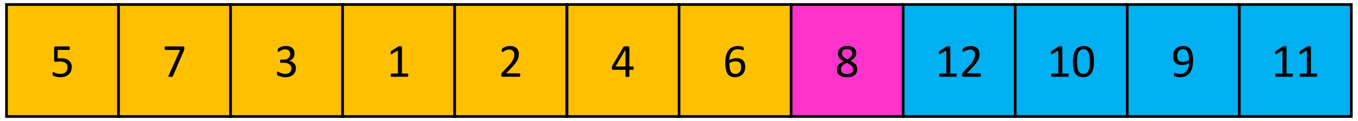
Given: a list, a pivot p

Start: unordered list



8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $> p$ on right



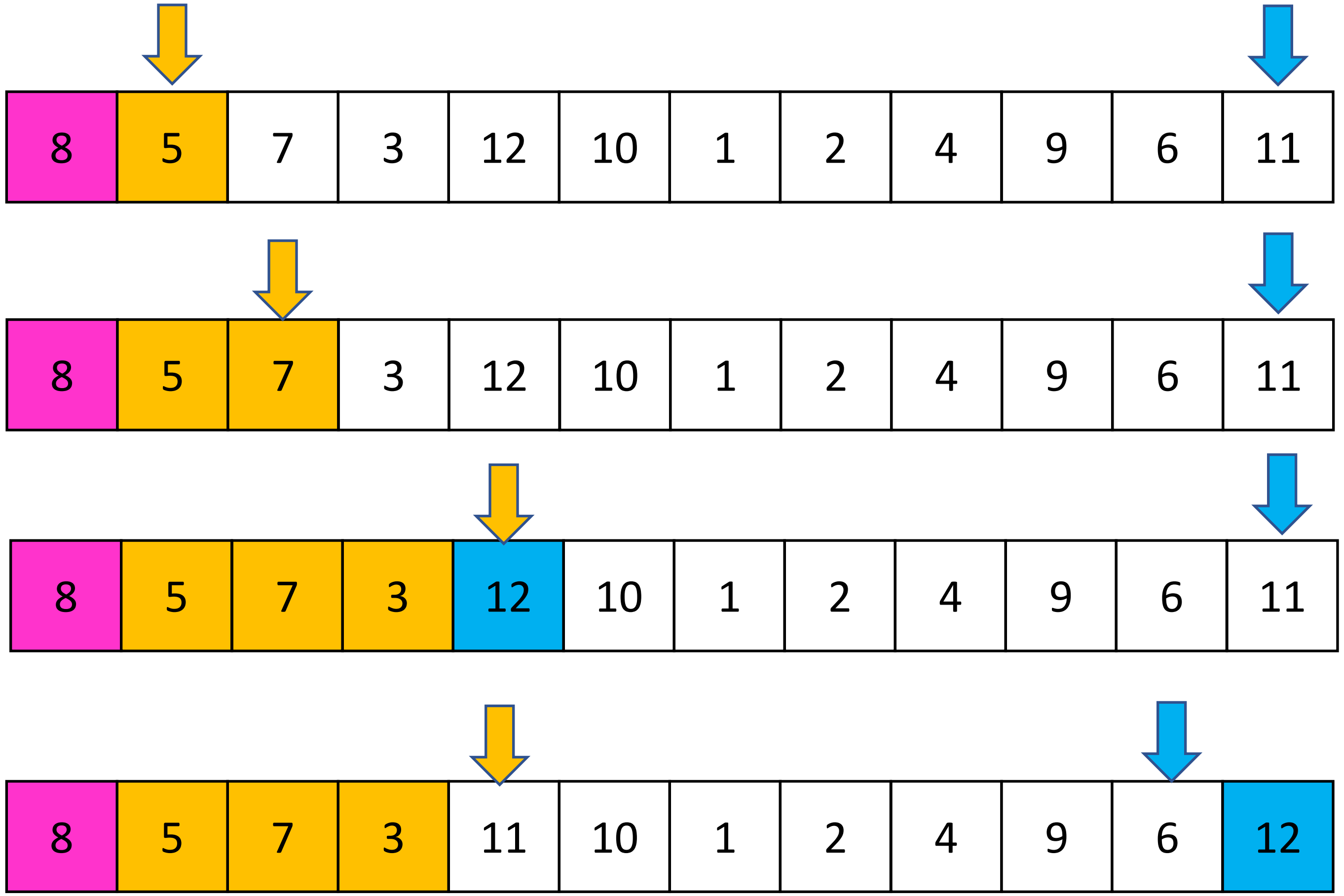
5	7	3	1	2	4	6	8	12	10	9	11

Partition, Procedure

If Begin value $< \; p$, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End



8	5	7	3	12	10	1	2	4	9	6	11

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Partition, Procedure

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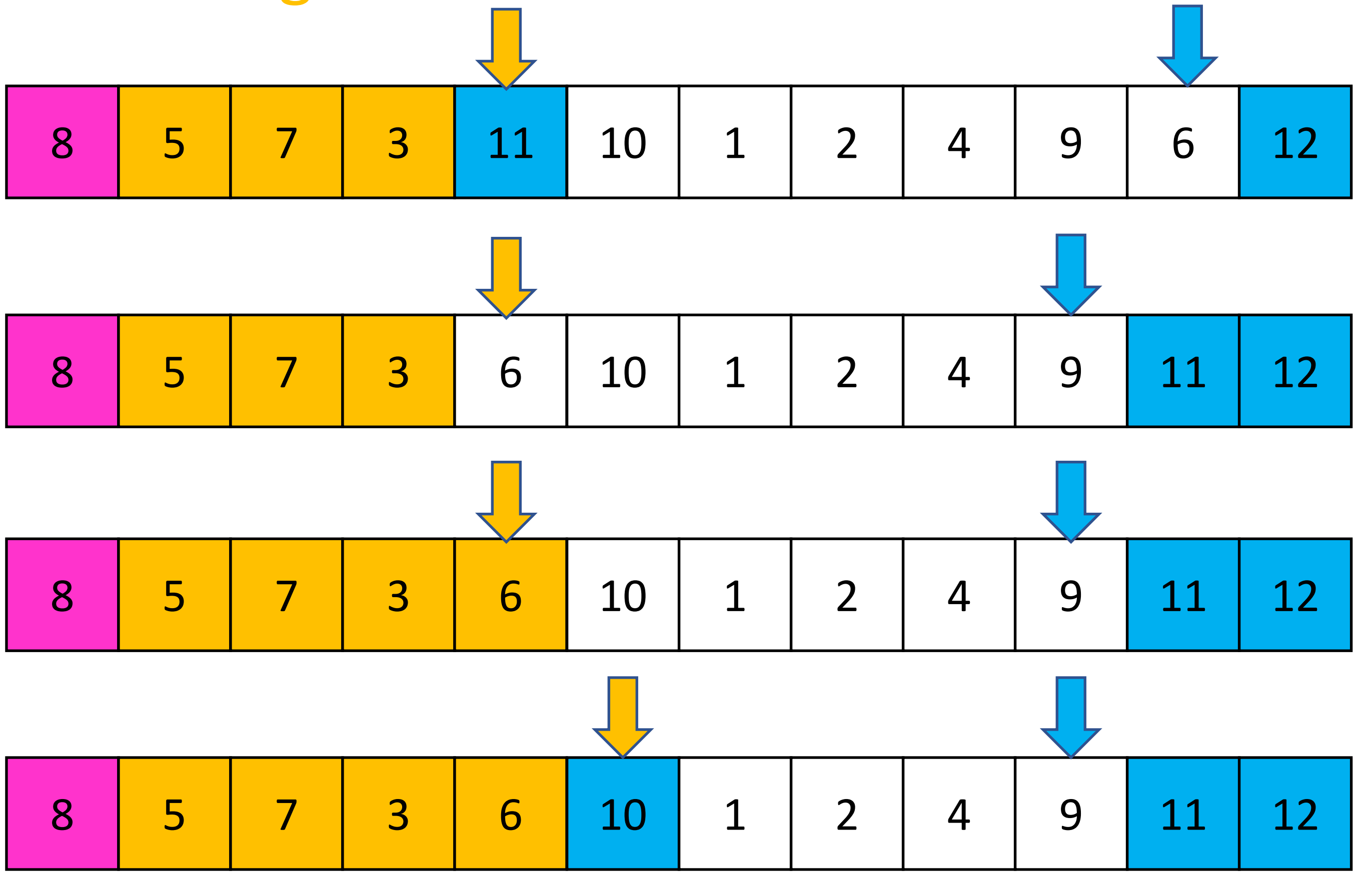
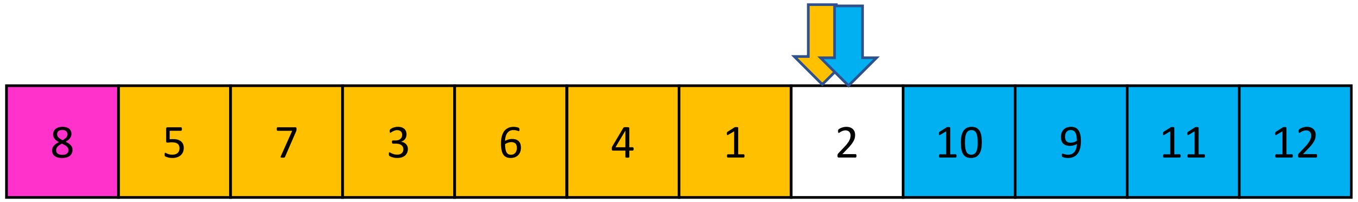


Partition, Procedure

If Begin value $<\ p$, move Begin right

Else swap Begin value with End value, move End Left

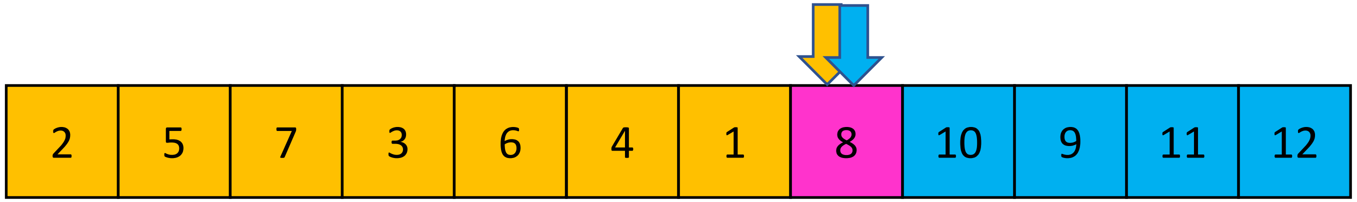
Done when Begin = End



8	5	7	3	6	4	1	2	10	9	11	12

Case 1: meet at element $< p$

Swap p with pointer position (2 in this case)



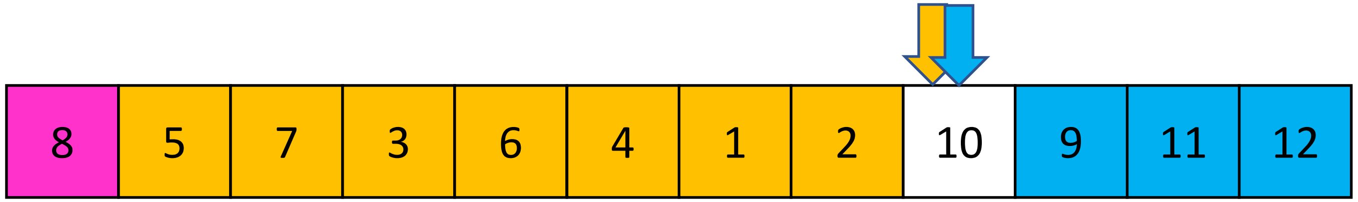
2	5	7	3	6	4	1	8	10	9	11	12

Partition, Procedure

If Begin value $<$ p, move Begin right

Else swap Begin value with End value, move End Left

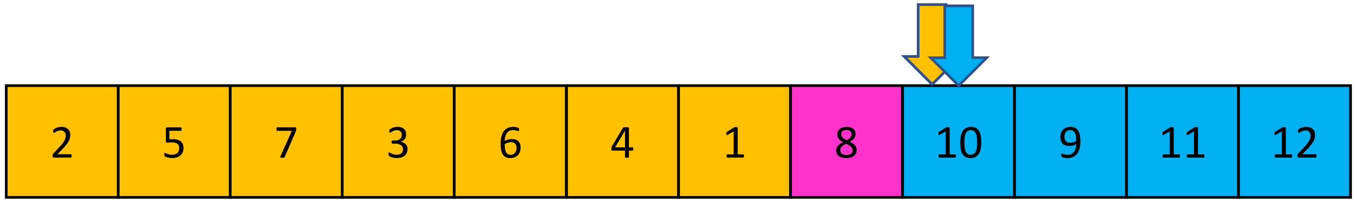
Done when Begin = End



8	5	7	3	6	4	1	2	10	9	11	12

Case 2: meet at element $> p$

Swap p with value to the left (2 in this case)



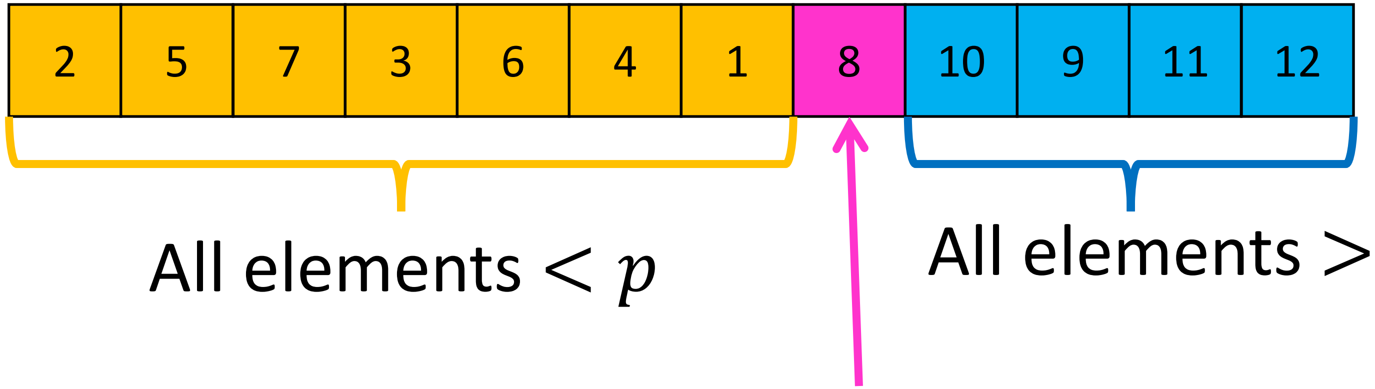
2	5	7	3	6	4	1	8	10	9	11	12

Partition Summary

1. Put p at beginning of list
2. Put a pointer (Begin) just after p, and a pointer (End) at the end of the list
3. While Begin < End:
    1. If Begin value $< p$, move Begin right
    2. Else swap Begin value with End value, move End Left
4. If pointers meet at element $< p$: Swap p with pointer position
5. Else If pointers meet at element $> p$: Swap p with value to the left

Run time? $O(n)$

Conquer



2	5	7	3	6	4	1	8	10	9	11	12

All elements $< p$

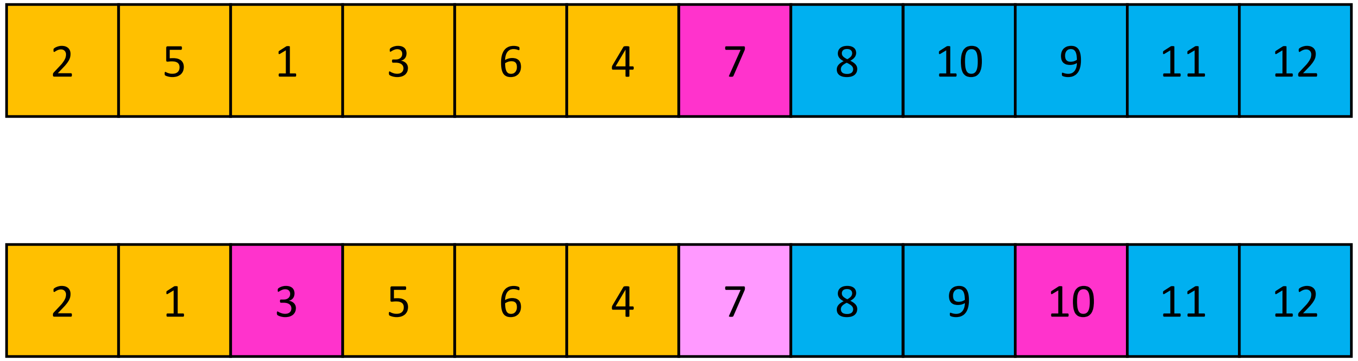
All elements $> p$

Exactly where it belongs!

Recursively sort Left and Right sublists

Quicksort Run Time (Best)

If the pivot is always the median:



2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

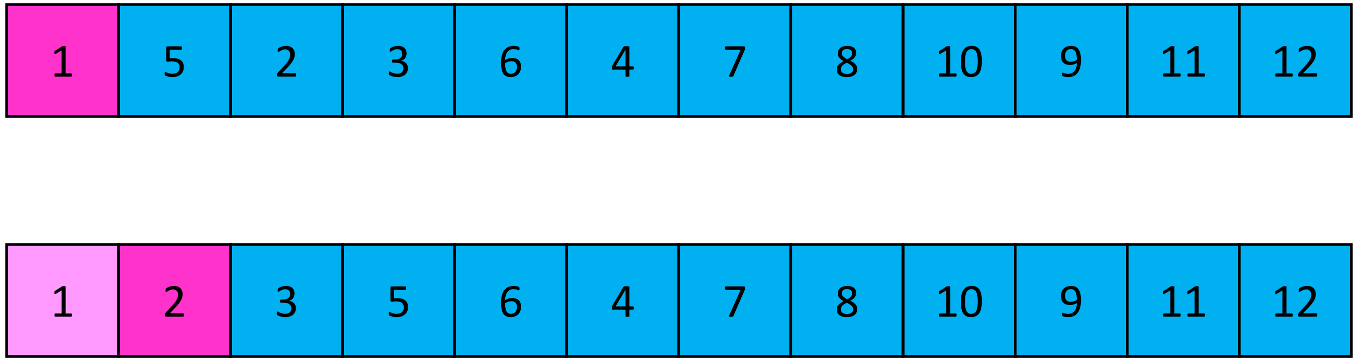
Then we divide in half each time

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(n) = O(n \log n)$$

Quicksort Run Time (Worst)

If the pivot is always at the extreme:



1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

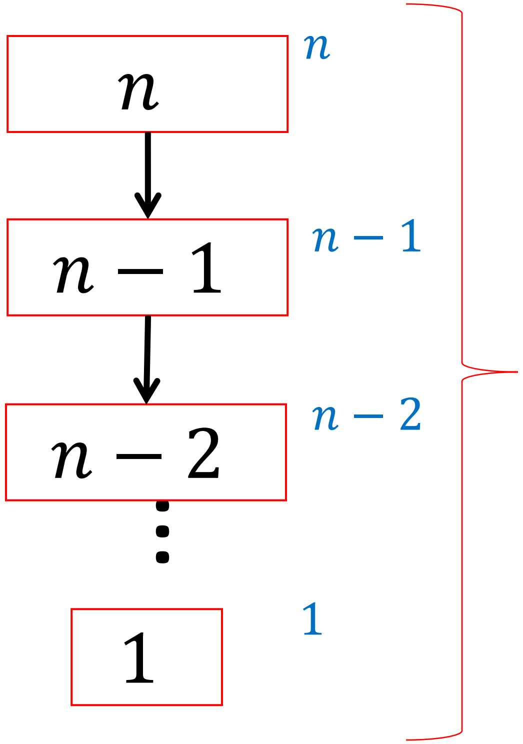
Then we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Quicksort Run Time (Worst)

$$T(n) = T(n-1) + n$$



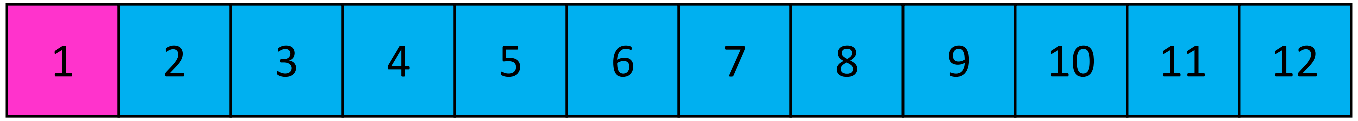
$$T(n) = 1 + 2 + 3 + \cdots + n$$

$$T(n) = \frac{n(n+1)}{2}$$

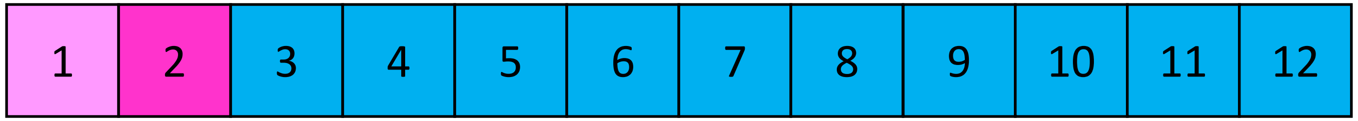
$$T(n) = O(n^2)$$

Quicksort on a (nearly) Sorted List

First element always yields unbalanced pivot



1	2	3	4	5	6	7	8	9	10	11	12



1	2	3	4	5	6	7	8	9	10	11	12

So we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Good Pivot

- What makes a good Pivot?
  - Roughly even split between left and right
  - Ideally: median
- There are ways to find the median in linear time, but it’s complicated and slow and you’re better off using mergesort
- In Practice:
  - Pick a random value as a pivot
  - Pick the middle of 3 random values as the pivot

Properties of Quick Sort

- Worst Case Running time:
  - $\Theta(n^2)$
  - But $\Theta(n \log n)$ average! And typically faster than mergesort!
- In-Place?
  - ….Debatable
- Adaptive?
  - No!
- Stable?
  - No!

Improving Running time

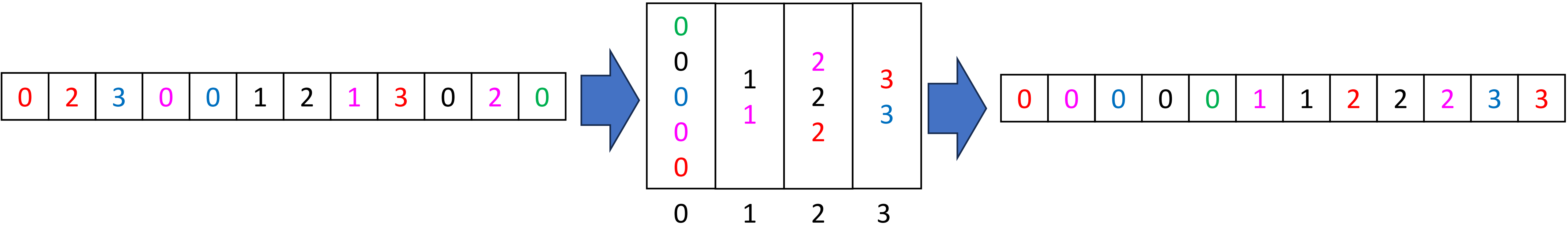
- Recall our definition of the sorting problem:
  - Input:
    - An array A of items
    - A comparison function for these items
      - Given two items x and y, we can determine whether $x < y$, $x > y$, or $x = y$
  - Output:
    - A permutation of A such that if $i \leq j$ then $A[i] \leq A[j]$
- Under this definition, it is impossible to write an algorithm faster than $n \log n$ asymptotically.
- Observation:
  - Sometimes there might be ways to determine the position of values without comparisons!

“Linear Time” Sorting Algorithms

- Useable when you are able to make additional assumptions about the contents of your list (beyond the ability to compare)
  - Examples:
    - The list contains only positive integers less than k
    - The number of distinct values in the list is much smaller than the length of the list
- The running time expression will always have a term other than the list’s length to account for this assumption
  - Examples:
    - Running time might be $\Theta(k \cdot n)$ where k is the range/count of values

BucketSort

- Assumes the array contains integers between 0 and $k-1$ (or some other small range)
- Idea:
  - Use each value as an index into an array of size k
  - Add the item into the "bucket" at that index (e.g. linked list)
  - Get sorted array by "appending" all the buckets



0	2	3	0	0	1	2	1	3	0	2	0

0	1	2	3
0 0 0 0 0	1 1	2 2 2	3 3

0	0	0	0	0	1	1	2	2	2	3	3

BucketSort Running Time

- Create array of k buckets
    - Either $\Theta(k)$ or $\Theta(1)$ depending on some things…
- Insert all n things into buckets
    - $\Theta(n)$
- Empty buckets into an array
    - $\Theta(n+k)$
- Overall:
    - $\Theta(n+k)$
- When is this better than mergesort?

Properties of BucketSort

- In-Place?
    - No
- Adaptive?
    - No
- Stable?
    - Yes!

RadixSort

- Radix: The base of a number system
    - We’ll use base 10, most implementations will use larger bases
- Idea:
    - BucketSort by each digit, one at a time, from least significant to most significant

103	801	401	323	255	823	999	101	113	901	555	512	245	800	018	121
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Place each element into a “bucket” according to its 1’s place

800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999
0	1	2	3	4	5	6	7	8	9

RadixSort

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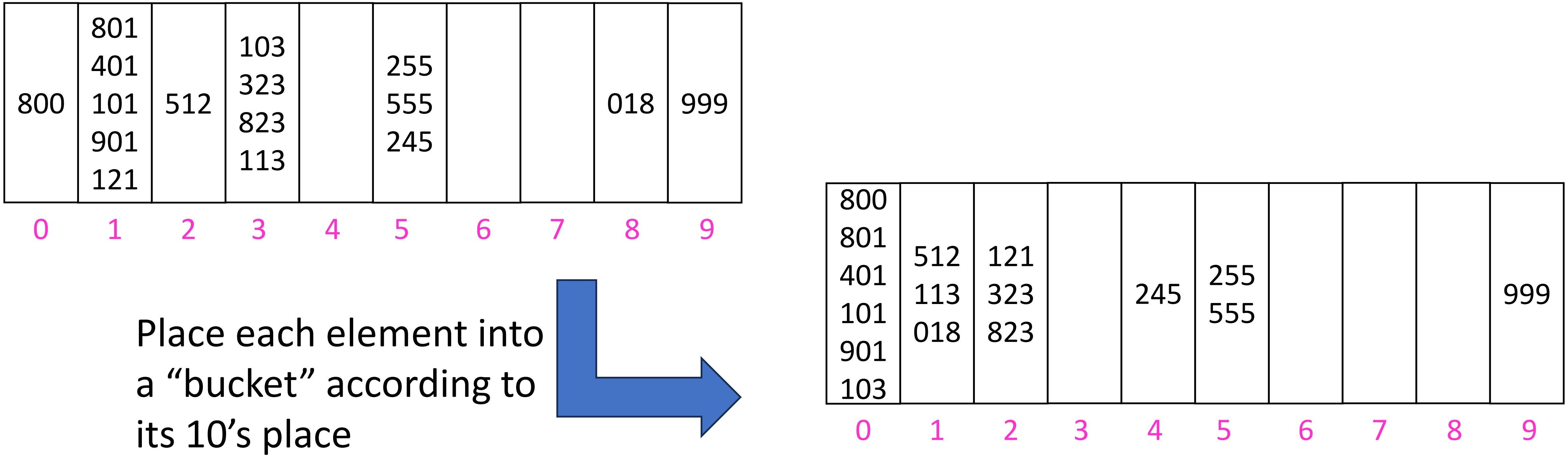


800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999
0	1	2	3	4	5	6	7	8	9

Place each element into a “bucket” according to its 10’s place

800 801 401 101 901 103	512 113 018	121 323 823		245	255 555				999
0	1	2	3	4	5	6	7	8	9

RadixSort

- Radix: The base of a number system
  - We'll use base 10, most implementations will use larger bases
- Idea:
  - BucketSort by each digit, one at a time, from least significant to most significant

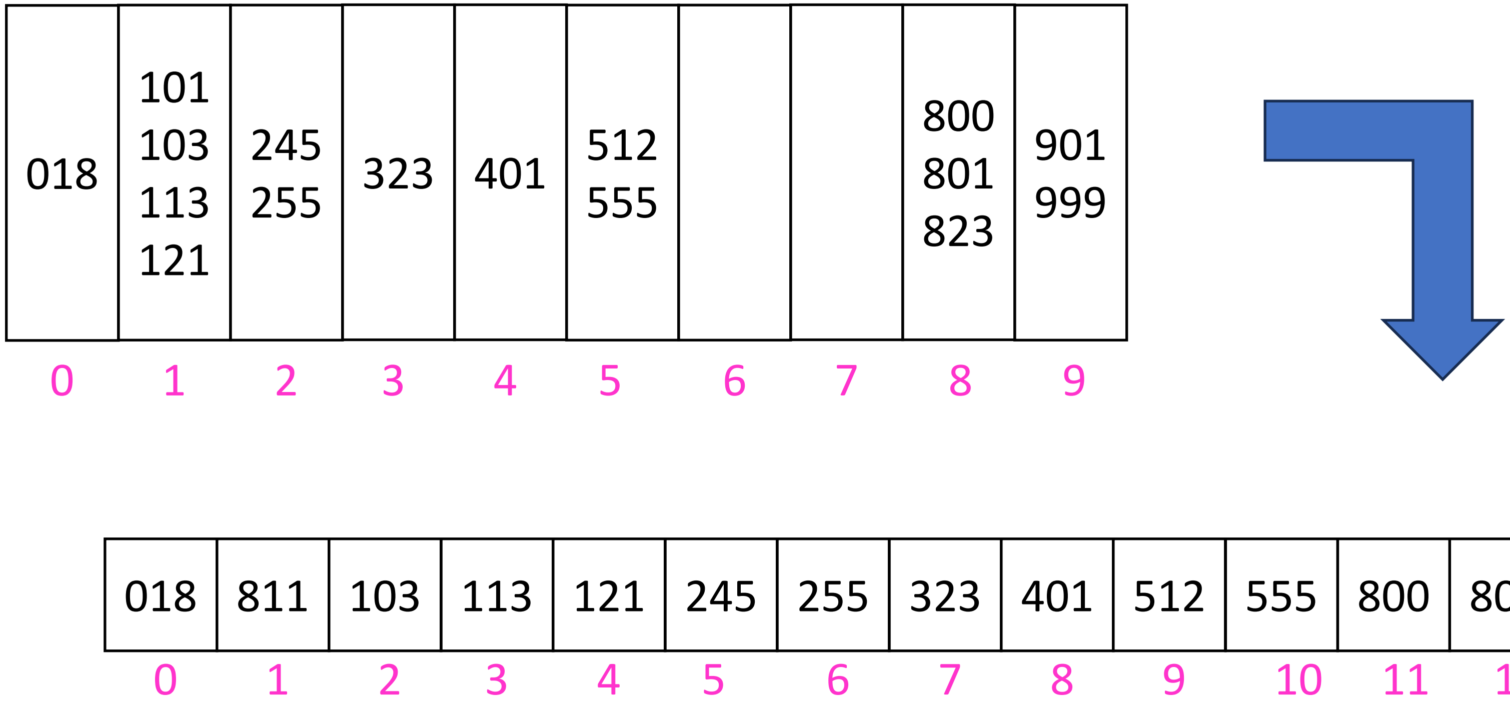
0	1	2	3	4	5	6	7	8	9
800 801 401 101 901 103	512 113 018	121 323 823		245	255 555				999

Place each element into a "bucket" according to its 100's place

0	1	2	3	4	5	6	7	8	9
018	101 103 113 121	245 255	323	401	512 555			800 801 823	901 999

RadixSort

- Radix: The base of a number system
  - We’ll use base 10, most implementations will use larger bases
- Idea:
  - BucketSort by each digit, one at a time, from least significant to most significant



018	101 103 113 121	245 255	323	401	512 555			800 801 823	901 999
0	1	2	3	4	5	6	7	8	9

Convert back into an array

018	811	103	113	121	245	255	323	401	512	555	800	801	823	901	999
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

RadixSort Running Time

- Suppose largest value is m
- Choose a radix (base of representation) b
- BucketSort all n things using b buckets
    - $\Theta(n+k)$
- Repeat once per each digit
    - $\log_b m$ iterations
- Overall:
    - $\Theta(n \log_b m + b \log_b m)$
- In practice, you can select the value of b to optimize running time
- When is this better than mergesort?