Exercise 1 Solutions

Question 5: Big-Oh Proofs

For a given $f(n)$ and $g(n)$ we must show that $f(n) \in O(g(n))$.

5.1 $f(n) = n^3$, $g(n) = n^3 - n$

Strategy: Select a value $c > 0$ and n_0 so that $\forall n > n_0$ we have that $n^3 \leq c \cdot (n^3 - n)$.

Proof: We select $c = 2$ and $n_0 = 3$. Next we demonstrate that the goal holds for this choice.

We will begin by applying algebra to rearrange the target inequality.

$$\begin{aligned} n^3 &\leq c \cdot (n^3 - n) \\ n^3 &\leq cn^3 - cn \\ n^2 &\leq cn^2 - c \\ n^2 + c &\leq cn^2 \\ c &\leq (c-1)n^2 \end{aligned}$$

This last statement is true for our choice of c and $\forall n > n_0$ because $(2-1)3^2 = 9$, and also the right-hand-side is increasing whereas the left-had side is constant. Therefore $f(n) \in O(g(n))$.

5.2 $f(n) = \log_2(n^4)$, $g(n) = \log_2(n^{1/4})$

Strategy: Select a value $c > 0$ and n_0 so that $\forall n > n_0$ we have that $\log_2(n^4) \leq c \log_2(n^{1/4})$.

Proof: We select $c = 16$ and $n_0 = 1$. Next we demonstrate that the goal holds for this choice.

We will begin by applying algebra to rearrange the target inequality.

$$\begin{aligned} \log_2(n^4) &\leq c \log_2(n^{1/4}) \\ 4 \log_2(n) &\leq \frac{c}{4} \log_2(n) \end{aligned}$$

This last statement is true for our choice of $c = 16$ and $\forall n > 1$ because $\frac{16}{4} \log_2(n) = 4 \log_2(n)$. Therefore $f(n) \in O(g(n))$.

5.3 $f(n) = \log_2(n)$, $g(n) = \log_{10}(n)$

Strategy: Select a value $c > 0$ and n_0 so that $\forall n > n_0$ we have that $\log_2(n) \leq c \log_{10}(n)$.

Proof: We select $c = \frac{1}{\log_{10}(2)}$ and $n_0 = 1$. Next we demonstrate that the goal holds for this choice.

We will begin by applying log rules to rearrange the target inequality.

$$\begin{aligned}\log_2(n) &\leq c\log_{10}(n)\\ \log_2(n) &\leq c\log_{10}(2^{\log_2(n)})\\ \log_2(n) &\leq c\log_2(n)\log_{10}(2)\end{aligned}$$

This last statement is true for our choice of $c = \frac{1}{\log_{10}(2)}$ and $\forall n > 1$ because $\frac{1}{\log_{10}(2)}\log_2(n)\log_{10}(2) = \log_2(n)$. Therefore $f(n) \in O(g(n))$.

5.4 $f(n) = n^{100}$, $g(n) = 2^n$

Strategy: Select a value $c > 0$ and n_0 so that $\forall n > n_0$ we have that $n^{100} \leq c2^n$.

Proof: We select $c = 1$ and $n_0 = 1000$. Next we demonstrate that the goal holds for this choice.

We will begin by applying taking the log of both sides of the inequality, and then performing log rules and algebra to simplify the expressions.

$$\begin{aligned}\log_2(n^{100}) &\leq \log_2(2^n)\\ 100\log_2(n) &\leq \log_2(2^n)\\ 100\log_2(n) &\leq n\end{aligned}$$

To summarize, we know now that $n^{100} \leq c2^n$ whenever $100\log_2(n) \leq n$. First, observe that this is true for $n = 1000$ because $\log_2(1000) < 10$. We will then show that $100\log_2(n) \leq n$ for all choices of $n > 1000$ via induction. Assume $100\log_2(n) \leq n$ for some choice of $n \geq n_0$, we will show that $100\log_2(n+1) \leq n+1$.

To begin, we will note that $n+1 \leq n(\frac{n_0+1}{n_0})$. This is because $n(\frac{n_0+1}{n_0}) = n\frac{n_0}{n_0} + \frac{n}{n_0}$ and $\frac{n_0}{n_0} = 1$ and $\frac{n}{n_0} \geq 1$. Therefore it's sufficient to show $100\log_2(n\frac{1001}{1000}) \leq n+1$. We next apply log rules to simplify:

$$\begin{aligned}100\log_2(n\frac{1001}{1000}) &\leq n+1\\ 100\log_2(n) + 100\log_2(\frac{1001}{1000}) &\leq n+1\end{aligned}$$

By our inductive hypothesis $100\log_2(n) \leq n$, and by inspection $100\log_2(\frac{1001}{1000}) < 1$, meaning the inequality holds, proving the inductive step. We can therefore conclude $100\log_2(n) \leq n$ for all choice of $n > 1000$ meaning $f(n) \in O(g(n))$.