CSE 332
Data Structures & Parallelism

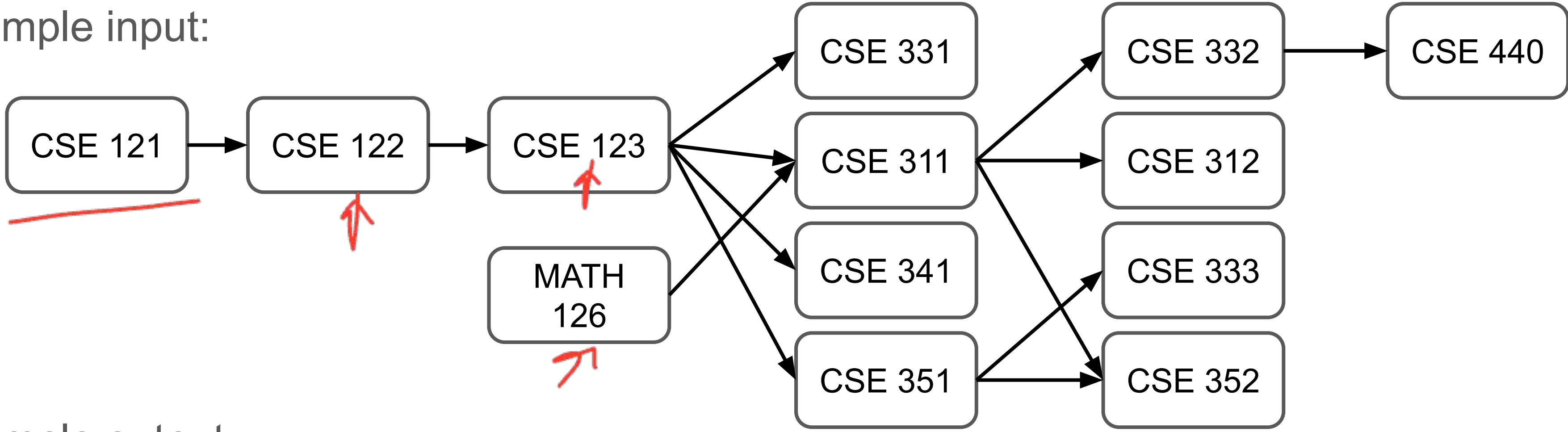
Topological Sort

Melissa Winstanley
Spring 2024

Topological Sort

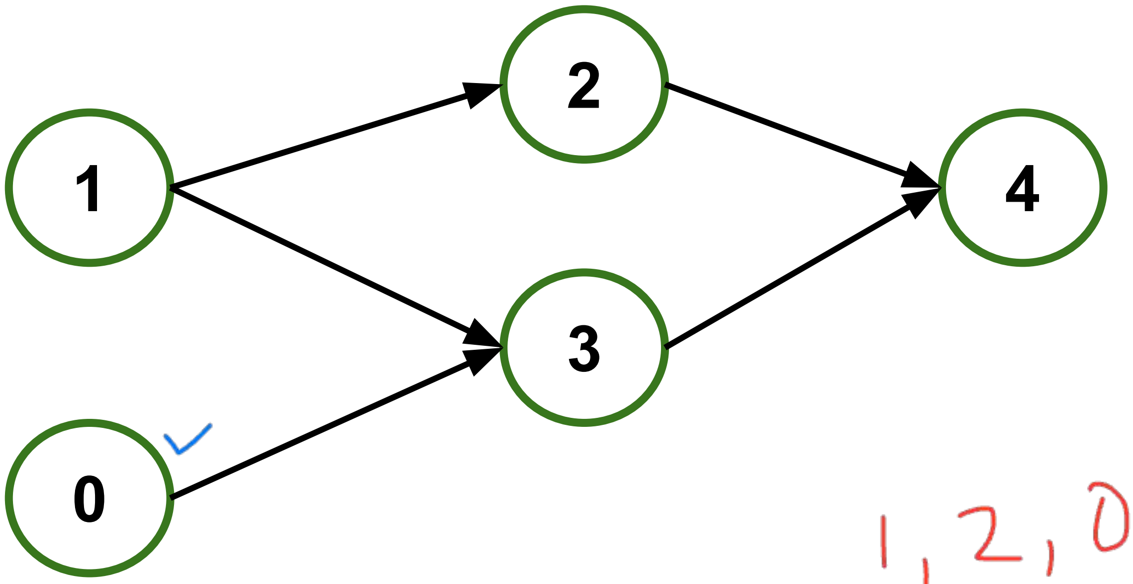
Problem: Given a DAG G=(V,E), output all the vertices in order such that if no vertex appears before any other vertex that has an edge to it

Example input:



Example output:

121, 126, 122, 123, 311, 331, 332, 312, 341, 351, 333, 440, 352



~~0, 3, 1, 2, 4~~

1, 2, 0, 3, 4

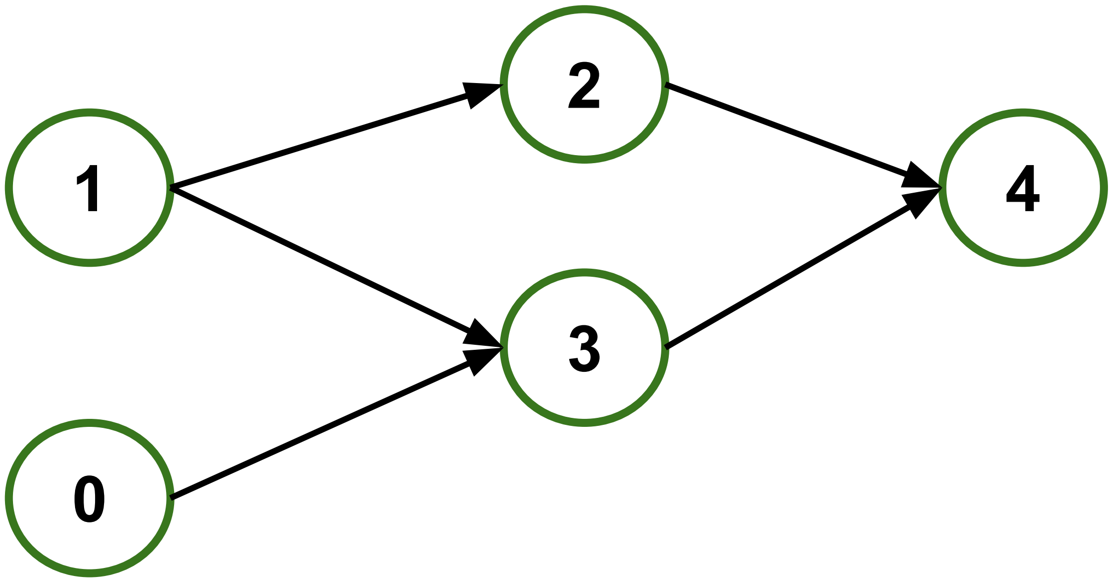
1, 0, 3, 2, 4

0, 1, 3, 2, 4

Valid Topological Sorts:

0, 1, 2, 3, 4

1, 0, 2, 3, 4



Valid Topological Sorts:

0, 1, 2, 3, 4

1, 0, 2, 3, 4

1, 2, 0, 3, 4

0, 1, 3, 2, 4

1, 0, 3, 2, 4

Questions and comments

- Why do we perform topological sorts only on DAGs?

  undir - need "first"
  cyclic - no "first"

- Is there always a unique answer?

  N

- What DAGs have exactly 1 answer?

  $\circ \to \circ \to \circ \to \circ \to \circ$

- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Topological Sort Uses

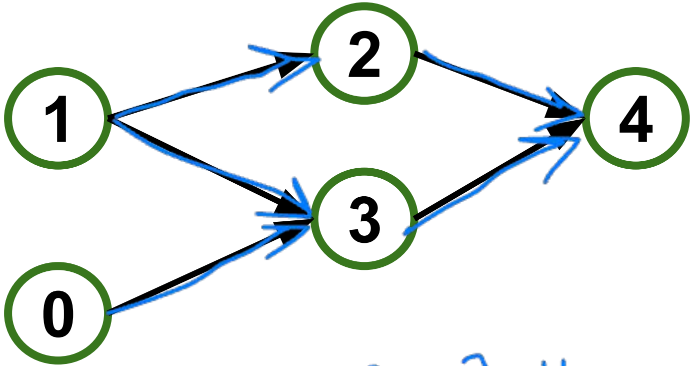
- Figuring out how to finish your degree
- Computing the order in which to recompute cells in a spreadsheet
- Determining the order to compile files using a Makefile
- In general, taking a dependency graph and coming up with an order of execution

A First Algorithm for Topological Sort

1. Label (“mark”) each vertex with its in-degree
   - Think “write in a field in the vertex”
   - Could also do this via a data structure (e.g., array) on the side

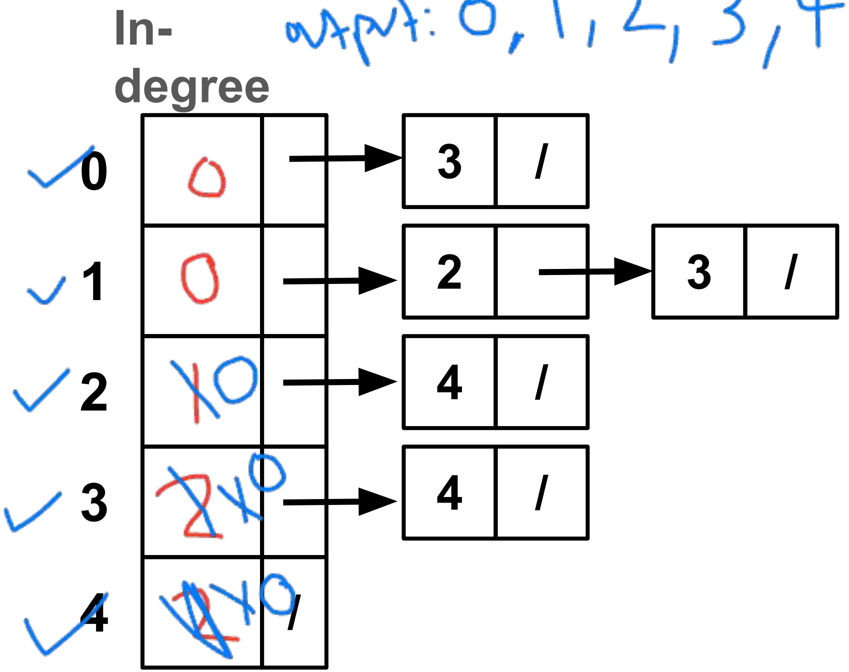
2. While there are vertices not yet output:
   a. Choose a vertex **v** labeled with in-degree of 0
   b. Output **v** and *conceptually* remove it from the graph
   c. For each vertex **w** adjacent to **v** (i.e. **w** such that **(v,w)** in **E**), decrement the in-degree of **w**

A First Algorithm for Topological Sort

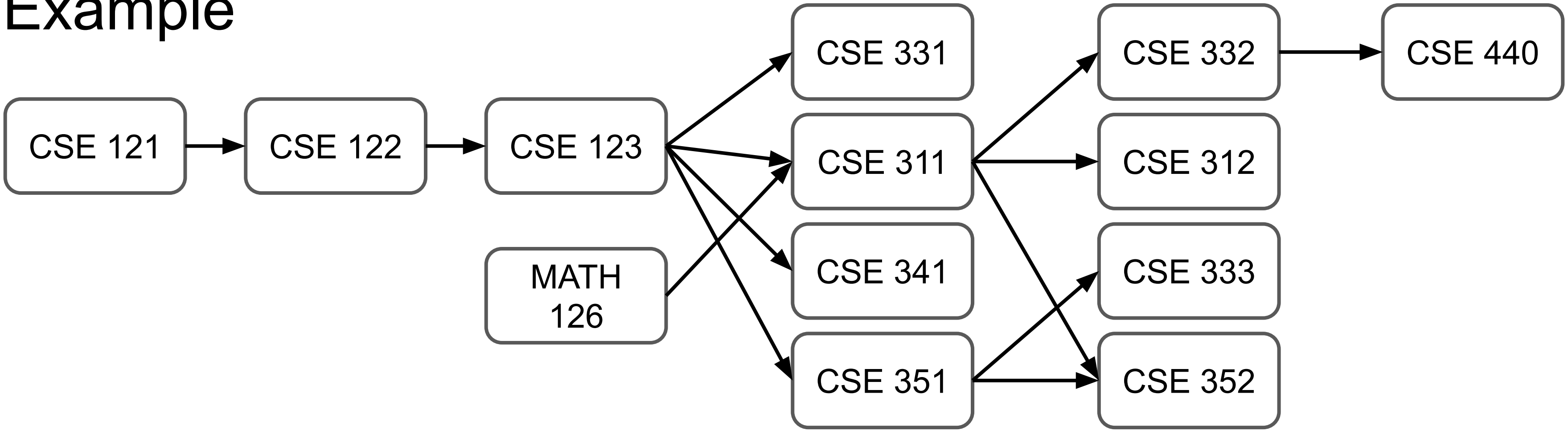


output: 0, 1, 2, 3, 4

1. Label ("mark") each vertex with its in-degree
   - Think "write in a field in the vertex"
   - Could also do this via a data structure (e.g., array) on the side
2. While there are vertices not yet output:
   a. Choose a vertex **v** labeled with in-degree of 0
   b. Output **v** and *conceptually* remove it from the graph
   c. For each vertex **w** adjacent to **v** (i.e. **w** such that **(v,w)** in **E**), decrement the in-degree of **w**

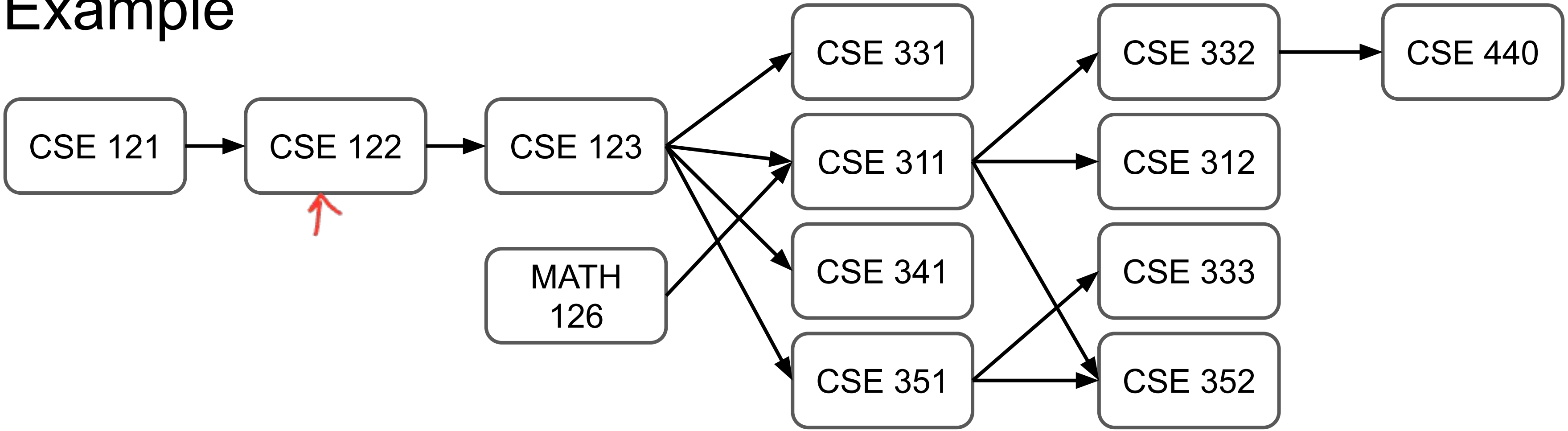


Example



Node:	121	122	123	126	311	312	331	332	333	341	351	352	440
Removed?													
In-degree:													
Output:													

Example



Node:	121	122	123	126	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x	x	x	x
In-degree:	0	1 0	1 0	0 0	2 1 0	1 0	1 0	1 0	1 0	1 0	1 0	2 1 0	1 0

Output: 121, 122, 123, 126, 331, 311, 341, 351, 332, 312, 333, 352, 440

A couple of things to note

- Needed a vertex with in-degree of 0 to start
  - No cycles
- Ties between vertices with in-degrees of 0 can be broken arbitrarily
  - Potentially many different correct orders

V E d

Topological Sort: Running time?

adj list

E+V

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

V times

V

O(1)

d

look at the array

$\cancel{V} + \cancel{E}$ $V^2 + \cancel{E}$

$(V+E) + V(V+d)$

Doing better

The trick is to avoid searching for a zero-degree node every time!

- Keep the “pending” zero-degree nodes in a list, stack, queue, box, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both O(1)

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
    a) **v** = dequeue()
    b) Output v and remove it from the graph
    c) For each vertex w adjacent to v (i.e. w such that (v,w) in E), decrement the in-degree of w, if new degree is 0, enqueue it

Topological Sort(optimized): Running time?

V+E

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
  for each w adjacent to v
    w.indegree--;
    if(w.indegree==0)
      enqueue(w);
}
```

√ O(1) d O(1)

V+E +V(d+1) → V+E