CSE 332 Autumn 2024
Lecture 9: AVL Trees

Nathan Brunelle

http://www.cs.uw.edu/332

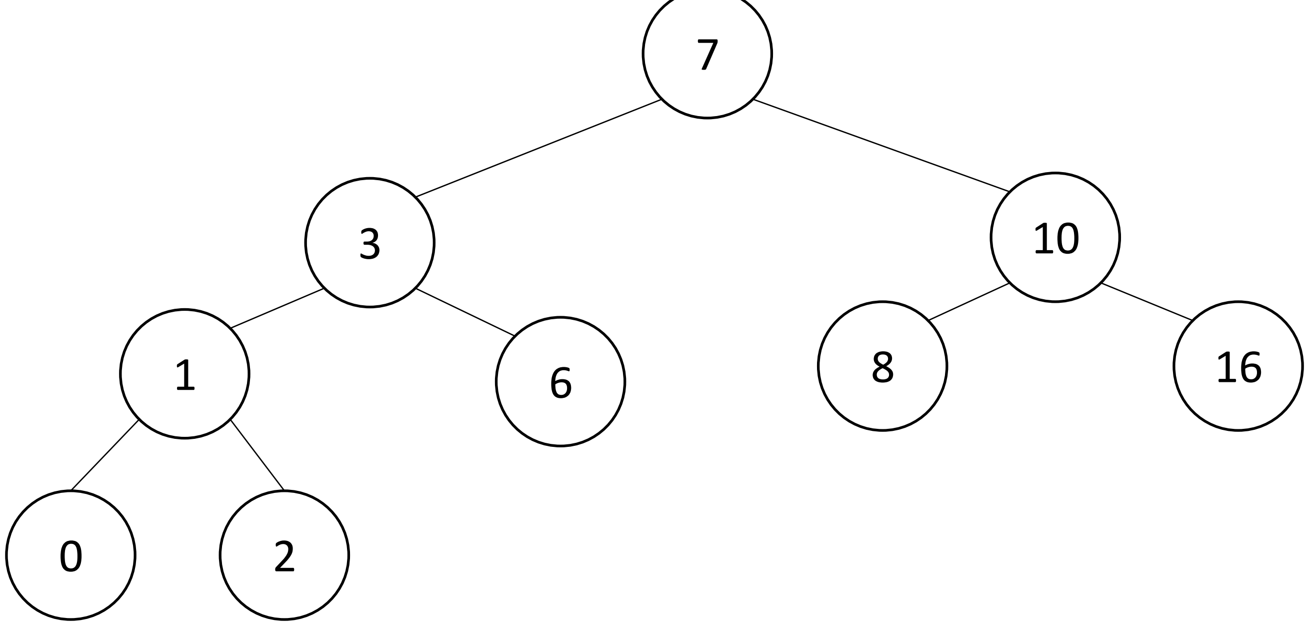
Dictionary (Map) ADT

- Contents:
  - Sets of key+value pairs
  - Keys must be comparable
- Operations:
  - insert(key, value)
    - Adds the (key,value) pair into the dictionary
    - If the key already has a value, overwrite the old value
      - Consequence: Keys cannot be repeated
  - find(key)
    - Returns the value associated with the given key
  - delete(key)
    - Remove the key (and its associated value)

Naïve attempts

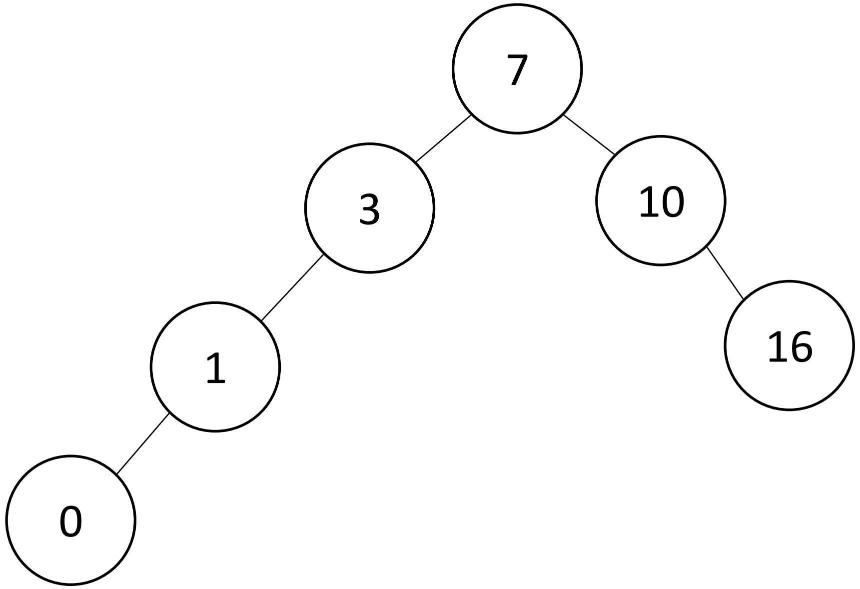
Data Structure	Time to insert	Time to find	Time to delete
Unsorted Array	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
Unsorted Linked List	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
Sorted Array	$\Theta(n)$	$\Theta(\log n)$	$\Theta(n)$
Sorted Linked List	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Heap	$\Theta(\log n)$	$\Theta(n)$	$\Theta(n)$
Binary Search Tree	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
AVL Tree	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$

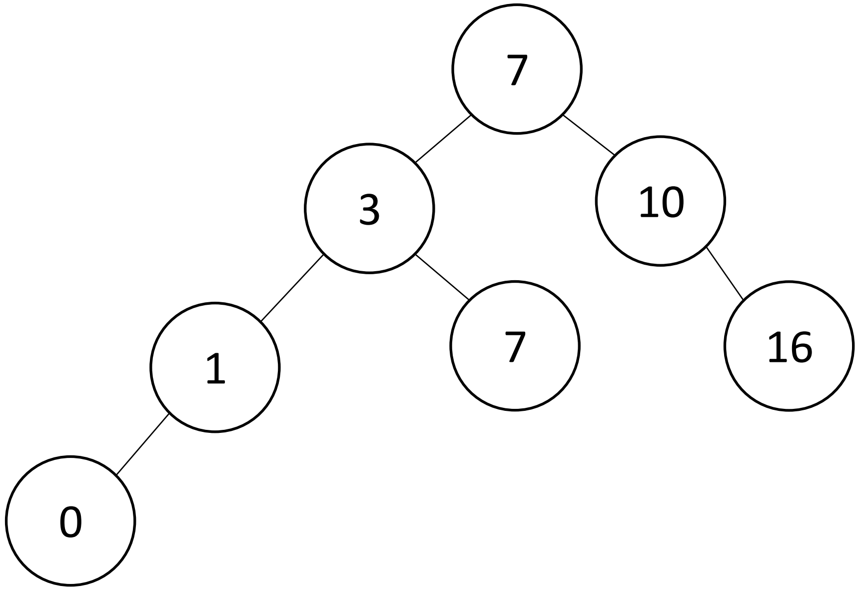
Binary Search Tree

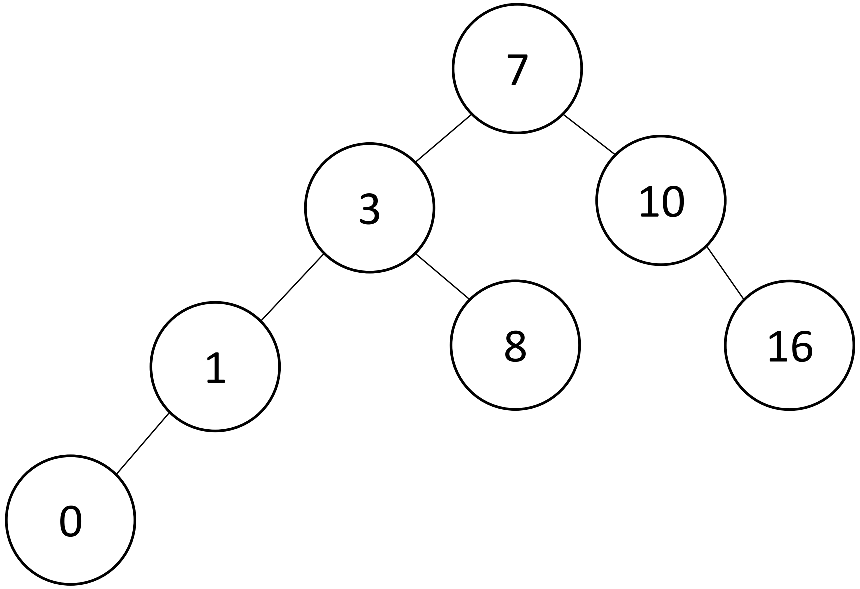


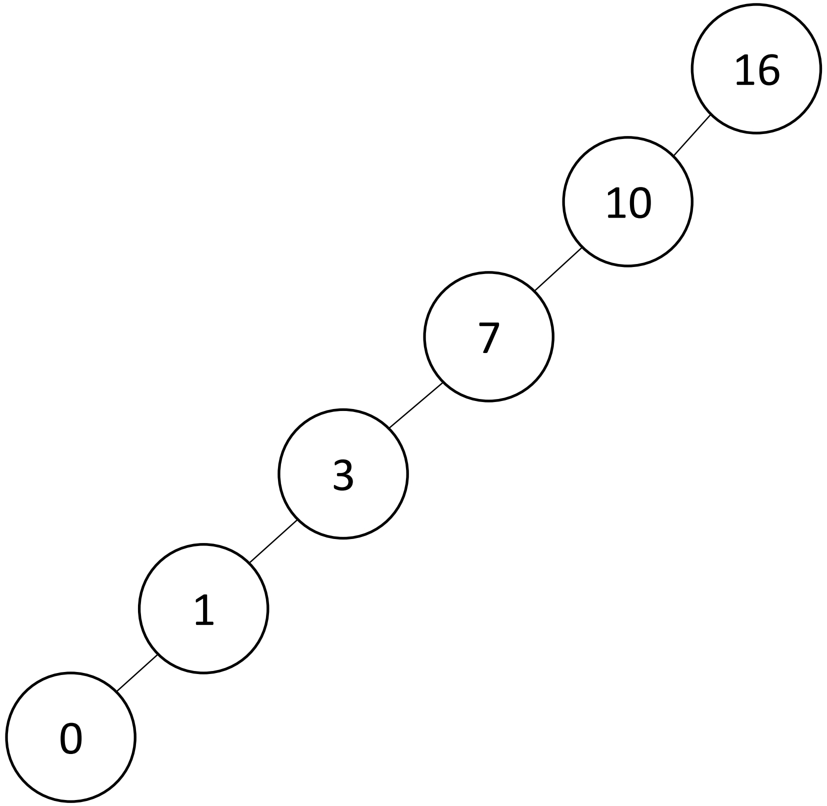
- Binary Tree
  - Definition:
    - Tree where each node has at most 2 children
- Order Property
  - All keys in the left subtree are smaller than the root
  - All keys in the right subtree are larger than the root
  - Consequence: cannot have repeated values

Are these BSTs?







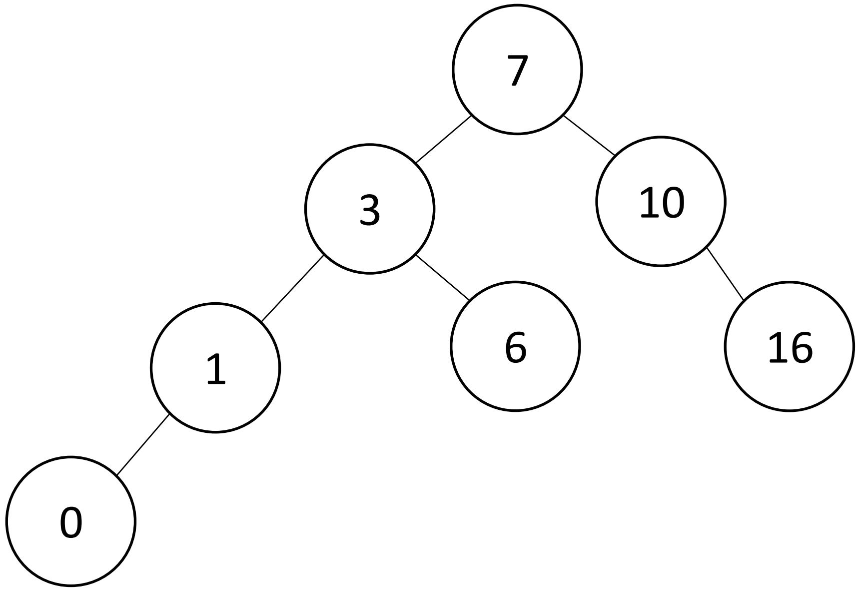


Aside: Why not use an array?

- We represented a heap using an array, finding children/parents by index
- We will represent BSTs with nodes and references. Why?
  - We might have “gaps” in our tree
  - Memory!
    - 2^n

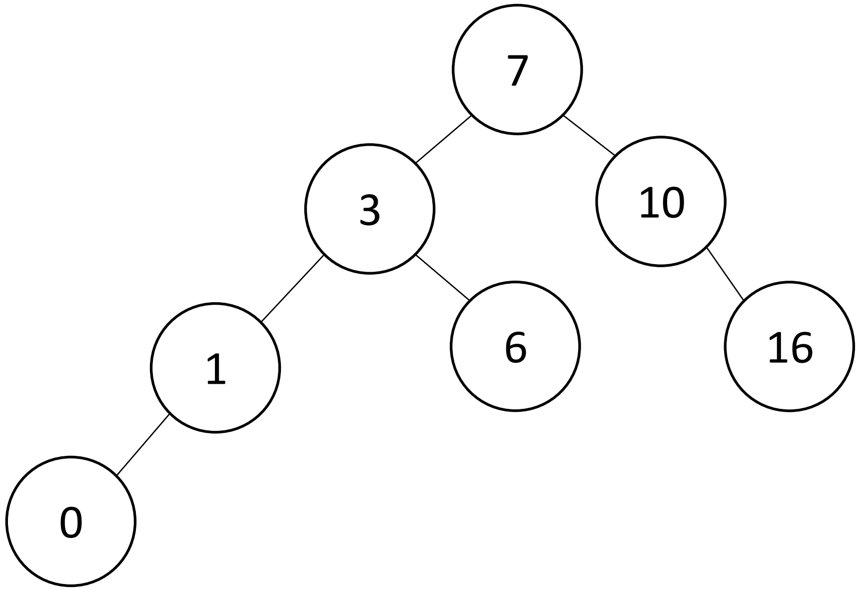
Find Operation (recursive)

```
find(key, root){
        if (root == Null){
                return Null;
        {
        if (key == root.key){
                return root.value;
        }
        if (key < root.key){
                return find(key, root.left);
        }
        if (key > root.key){
                return find(key, root.right);
        }
        return Null;
}
```

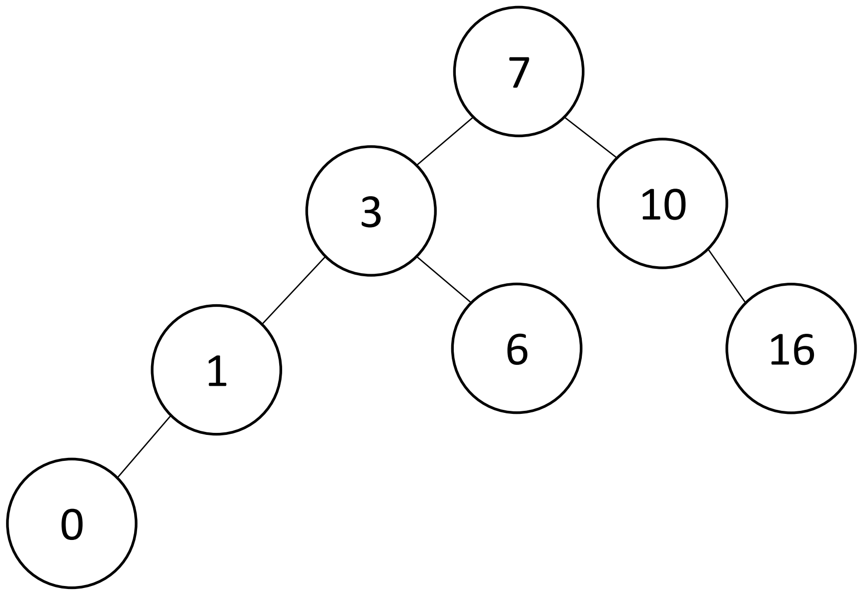


Find Operation (iterative)

```
find(key, root){
	while (root != Null && key != root.key){
		if (key < root.key){
			root = root.left;
		}
		else if (key > root.key){
			root = root.right;
		}
	}
	if (root == Null){
		return Null;
	}
	return root.value;
}
```



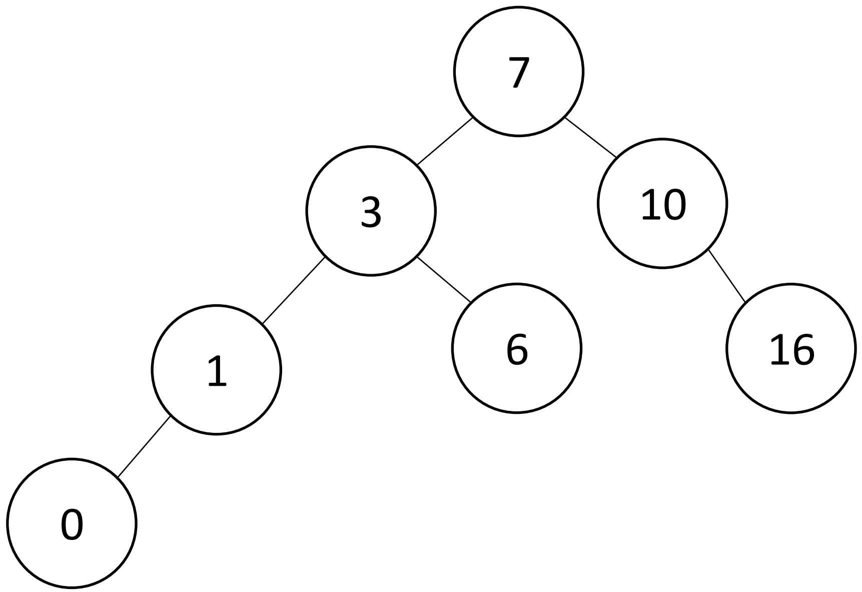
Insert Operation (recursive)




```
insert(key, value, root){
	root = insertHelper(key, value, root);
}
insertHelper(key, value, root){
	if(root == null)
		return new Node(key, value);
	if (root.key < key)
		root.right = insertHelper(key, value, root.right);
	else
		root.left = insertHelper(key, value, root.left);
	return root;
}
```

Note: Insert happens only at the leaves!

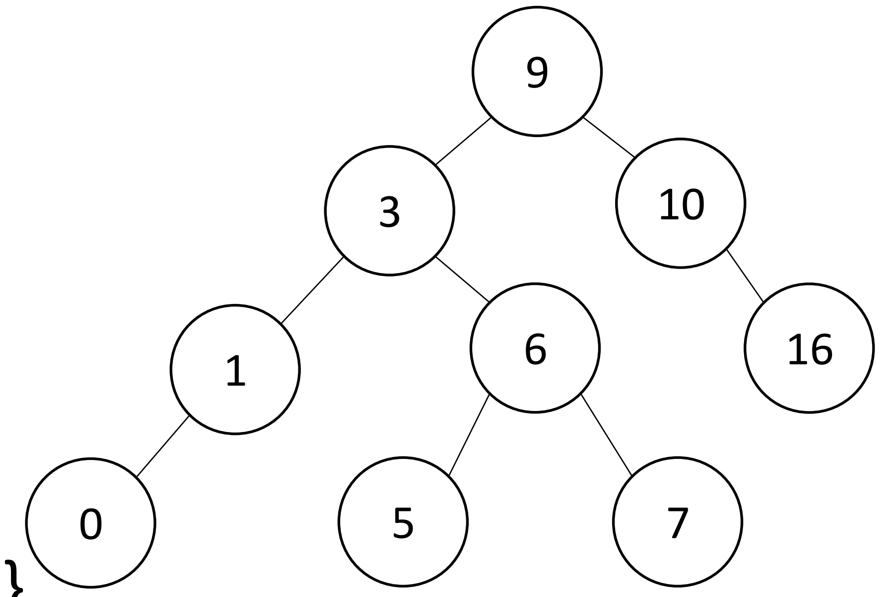
Insert Operation (iterative)




```
insert(key, value, root){
        if (root == Null){ this.root = new Node(key, value); }
        parent = Null;
        while (root != Null && key != root.key){
                parent = root;
                if (key < root.key){ root = root.left; }
                else if (key > root.key){ root = root.right; }
        }
        if (root != Null){ root.value = value; }
        else if (key < parent.key){ parent.left = new Node(key, value); }
        else{ parent.right = new Node (key, value); }
}
```

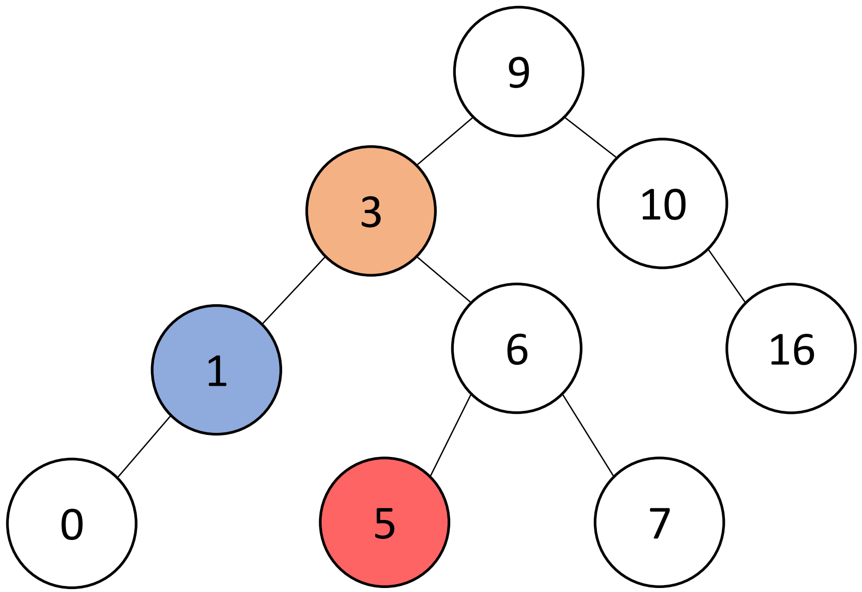
Note: Insert happens only at the leaves!

Delete Operation (iterative)



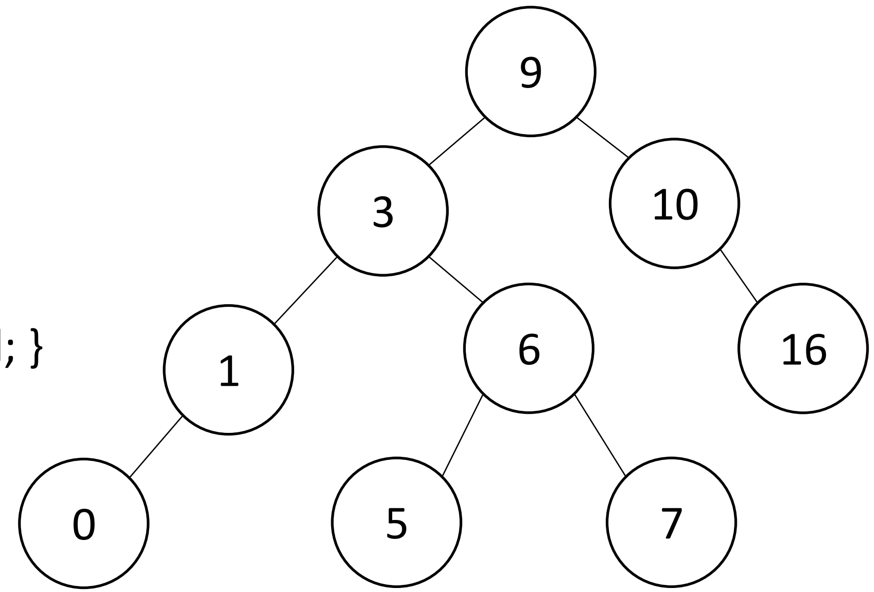

```
delete(key, root){
        while (root != Null && key != root.key){
                if (key < root.key){ root = root.left; }
                else if (key > root.key){ root = root.right; }
        }
        if (root == Null){ return; }
        // Now root is the node to delete, what happens next?
}
```

Delete – 3 Cases



- 0 Children (i.e. it's a leaf)
- 1 Child
  - Replace the deleted node with its child
- 2 Children
  - Replace the deleted with the largest node to its left or else the smallest node to its right

Finding the Max and Min



- Max of a BST:
  - Right-most Thing

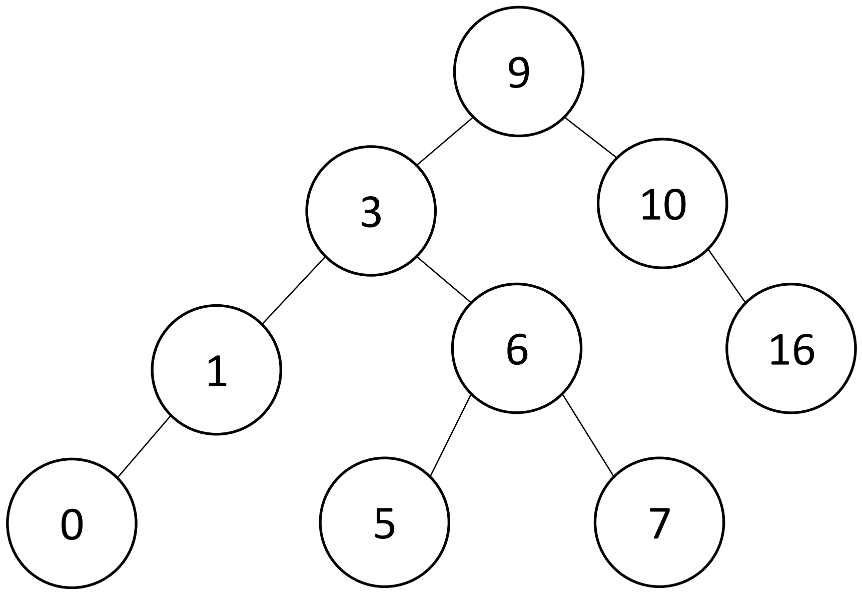
```
maxNode(root){
        if (root == Null){ return Null; }
        while (root.right != Null){
                root = root.right;
        }
        return root;
}
```

- Min of a BST:
  - Left-most Thing

```
minNode(root){
        if (root == Null){ return Null; }
        while (root.left != Null){
                root = root.left;
        }
        return root;
}
```

Delete Operation (iterative)

```
delete(key, root){
        while (root != Null && key != root.key){
                if (key < root.key){ root = root.left;  }
                else if (key > root.key){ root = root.right; }
        }
        if (root == Null){ return; }
        if (root has no children){
                make parent point to Null Instead;
        }
        if (root has one child){
                make parent point to that child instead;
        }
        if (root has two children){
                make parent point to either the max from the left or min from the right
        }
}
```



Worst Case Analysis

- For each of Find, insert, Delete:
  - Worst case running time matches height of the tree
- What is the maximum height of a BST with n nodes?
  - $\Theta(n)$

Improving the worst case

- How can we get a better worst case running time?
  - Add rules about the shape of our BST
- AVL Tree
  - A BST with some shape rules
    - Algorithms need to change to accommodate those

“Balanced” Binary Search Trees

- We get better running times by having “shorter” trees
- Trees get tall due to them being “sparse” (many one-child nodes)
- Idea: modify how we insert/delete to keep the tree more “full”

Idea 1: Both Subtrees of Root have same # Nodes

Idea 2: Both Subtrees of Root have same height

Idea 3: Both Subtrees of every Node have same # Nodes

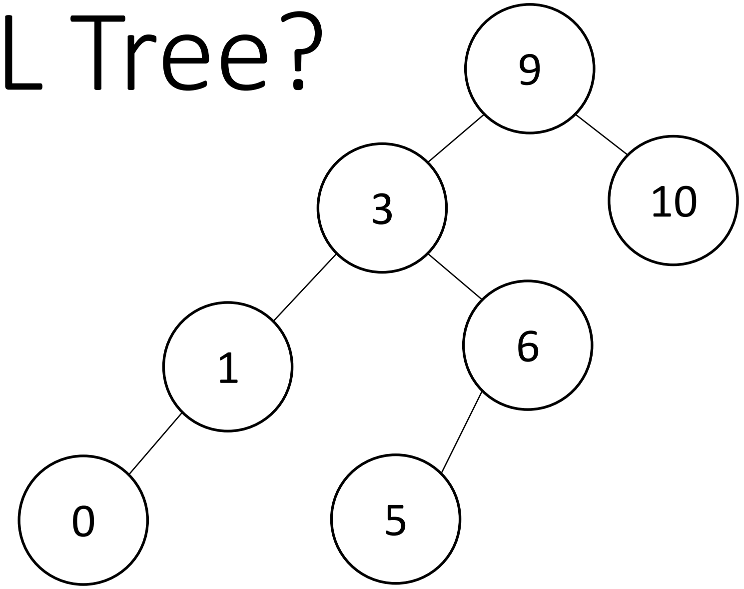
Idea 4: Both Subtrees of every Node have same height

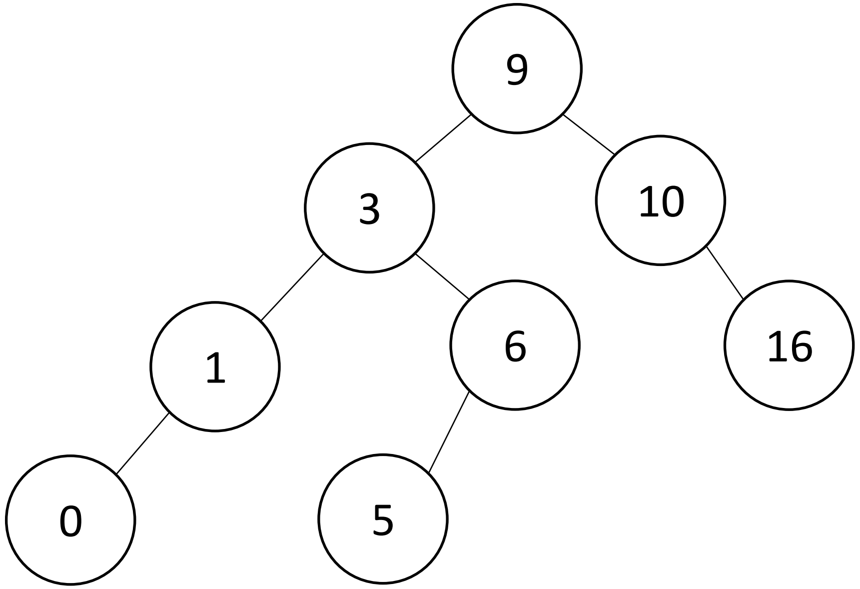
AVL Tree

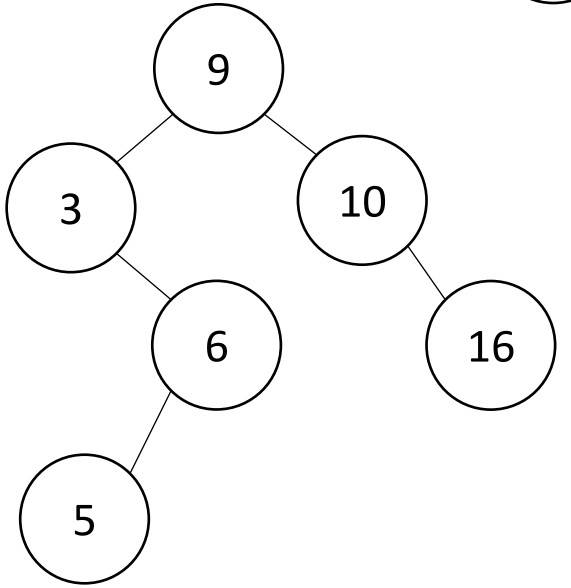
- A Binary Search tree that maintains that the left and right subtrees of every node have heights that differ by at most one.
  - height of left subtree and height of right subtree off by at most 1
  - Not too weak (ensures trees are short)
  - Not too strong (works for any number of nodes)

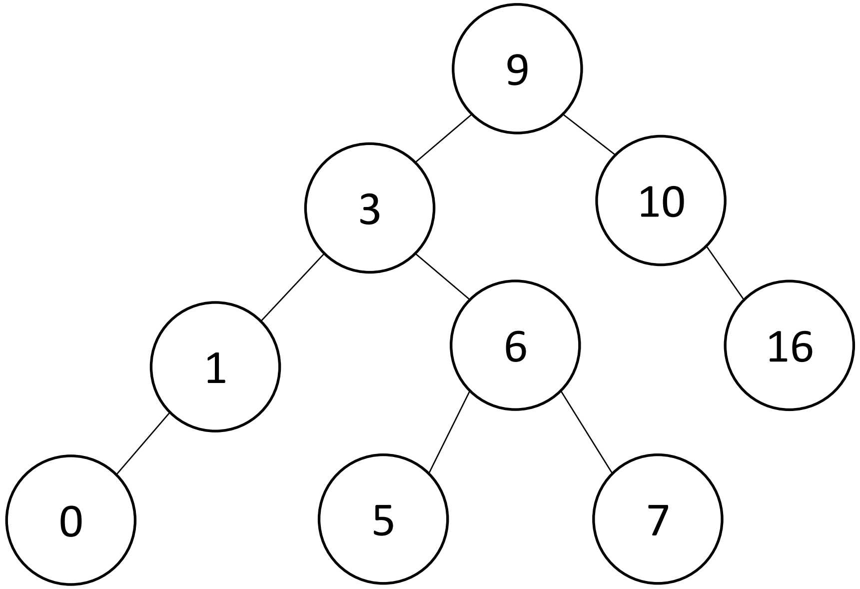
- Idea of AVL Tree:
  - When you insert/delete nodes, if tree is “out of balance” then modify the tree
  - Modification = “rotation”

Is it an AVL Tree?



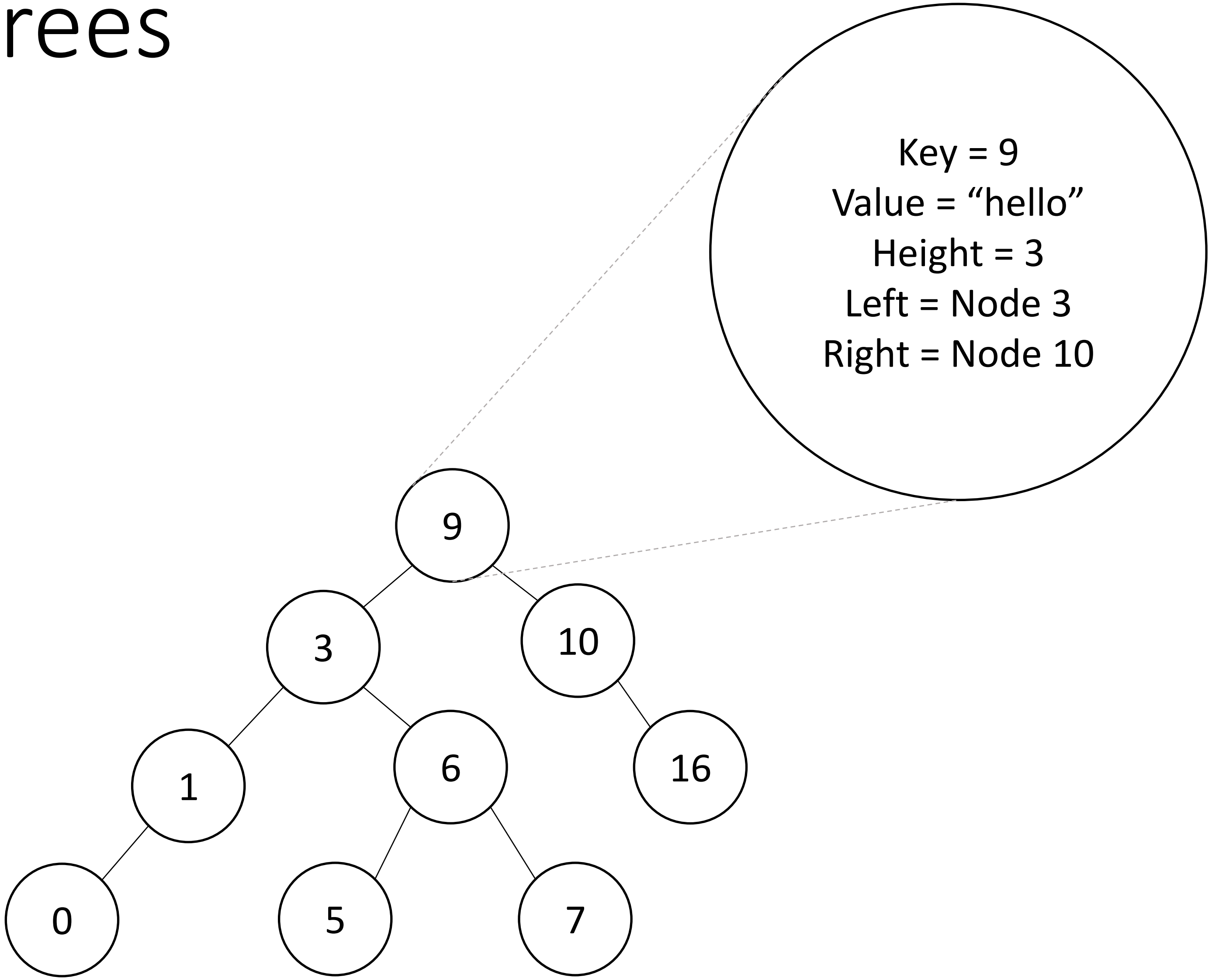






Using AVL Trees

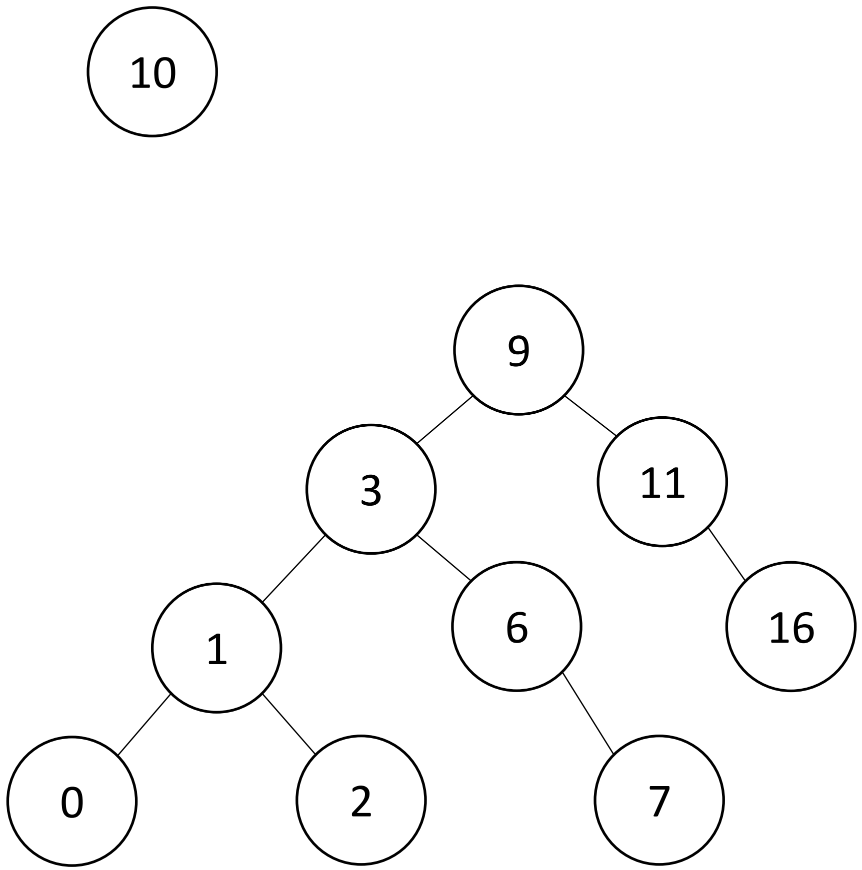
- Each node has:
  - Key
  - Value
  - Height
  - Left child
  - Right child



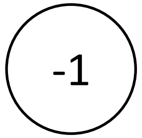
Inserting into an AVL Tree

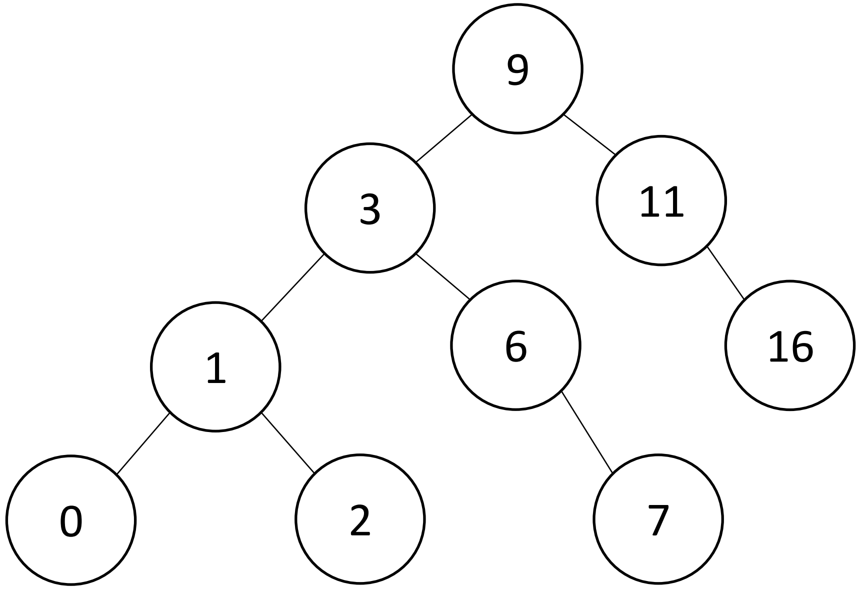
- Starts out the same way as BST:
  - “Find” where the new node should go
  - Put it in the right place (it will be a leaf)
- Next check the balance
  - If the tree is still balanced, you’re done!
  - Otherwise we need to do rotations

Insert Example

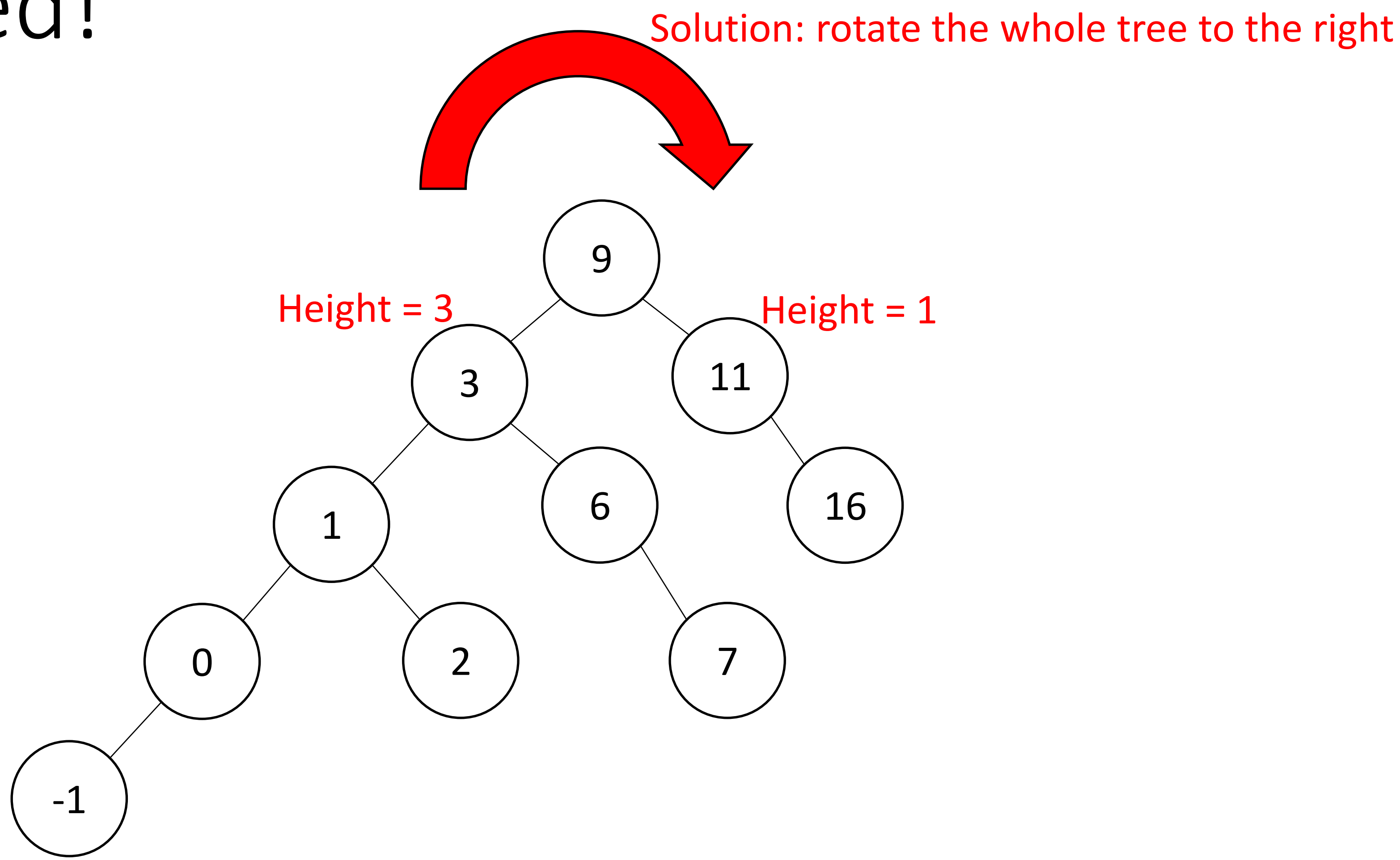


Insert Example

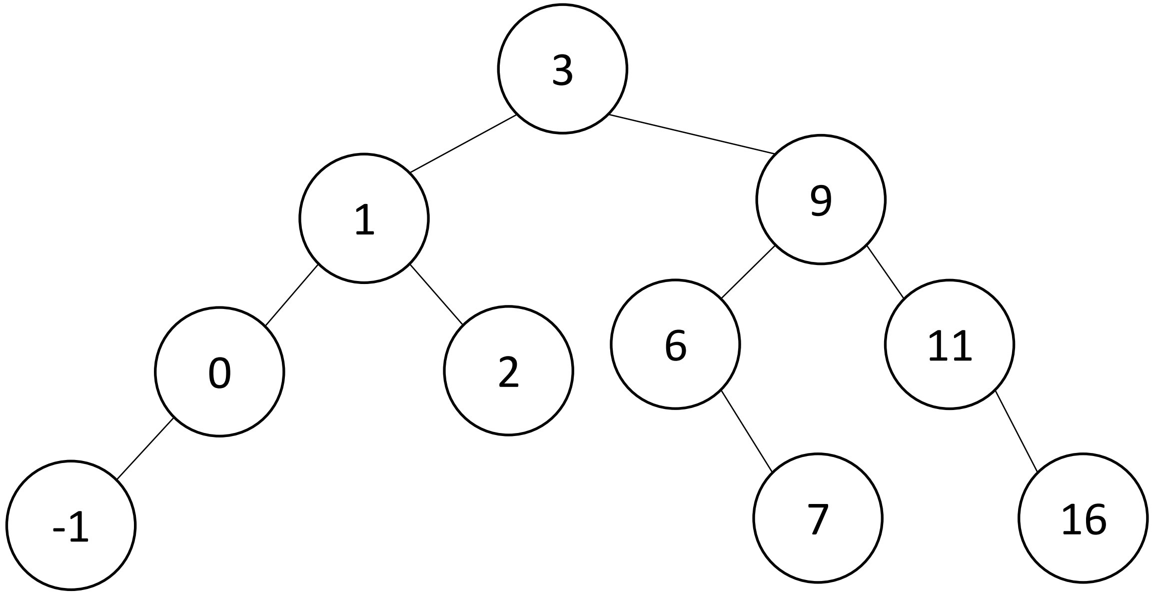




Not Balanced!

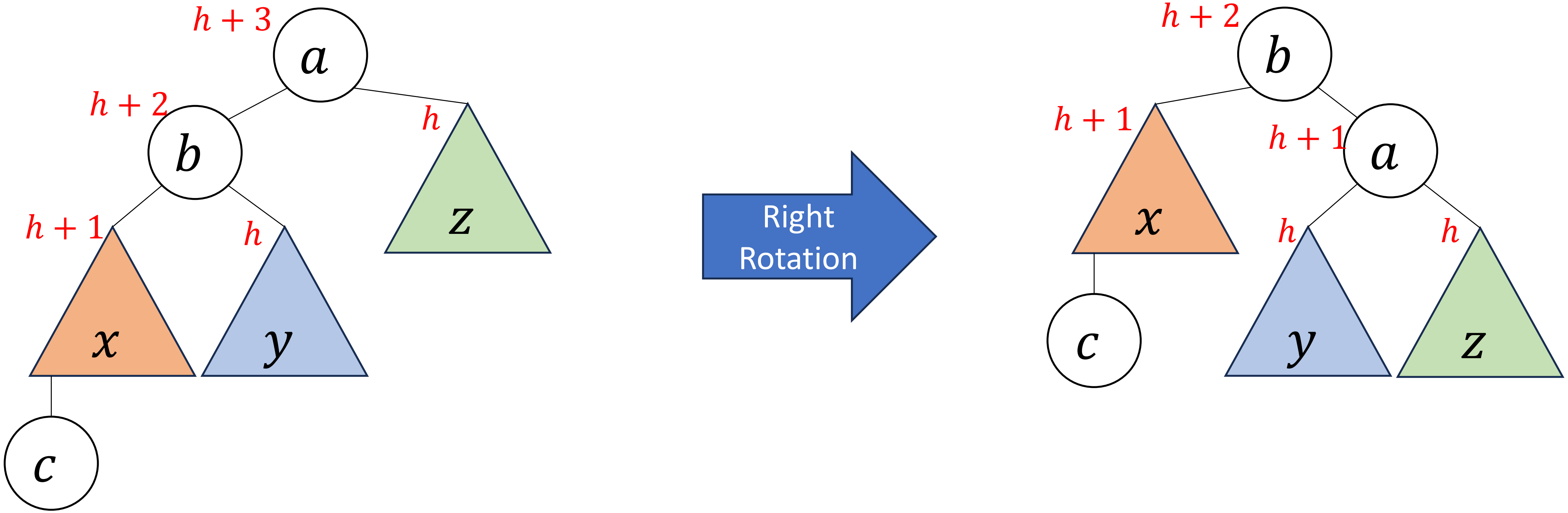


Balanced!

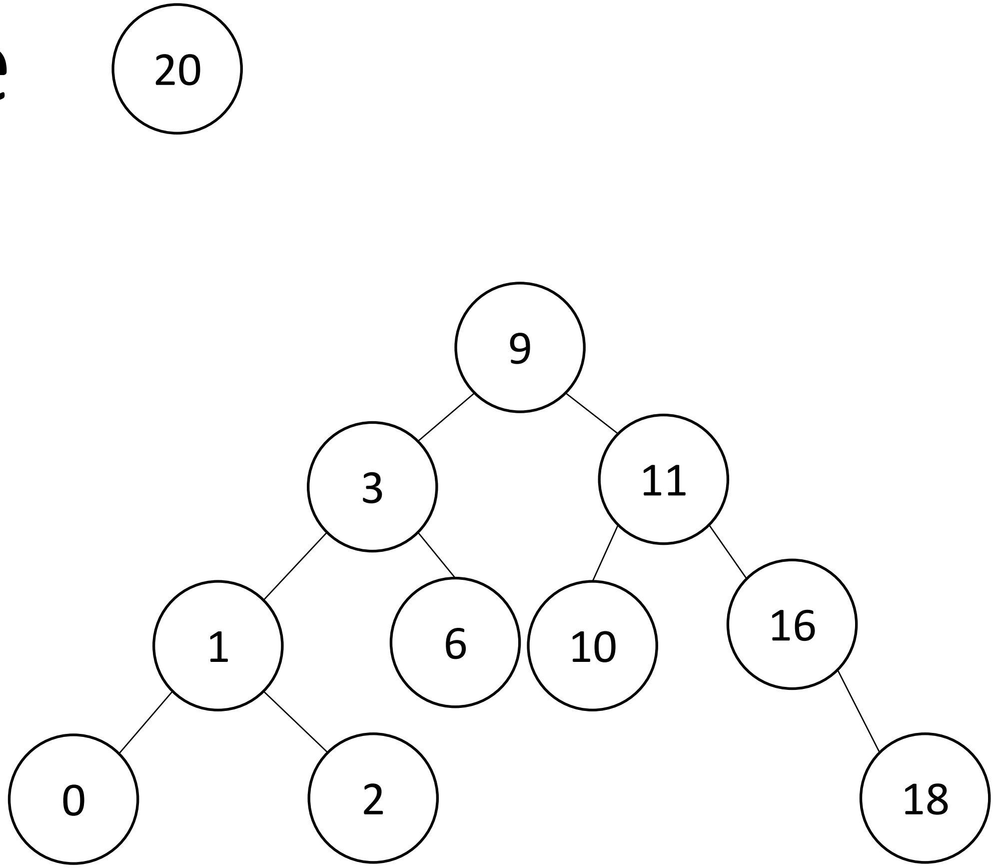


Right Rotation

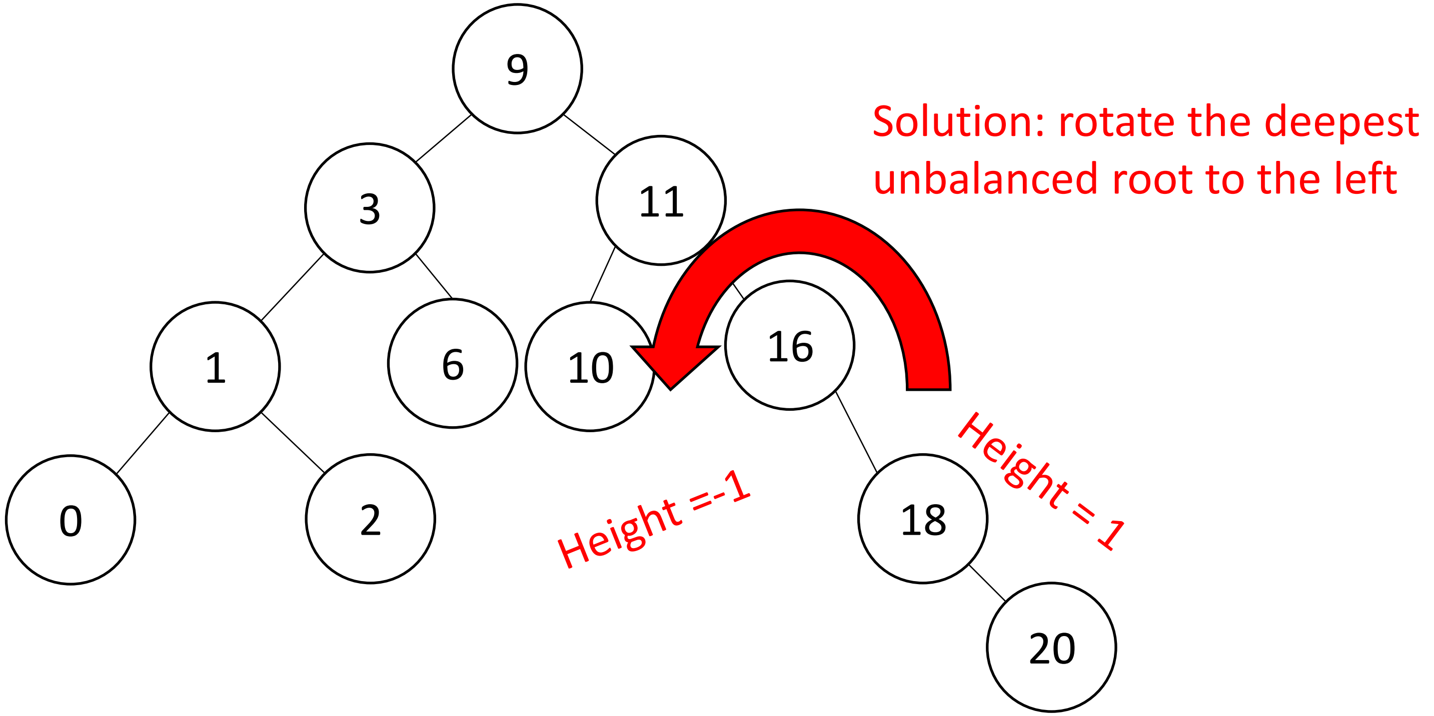
- Make the left child the new root
- Make the old root the right child of the new
- Make the new root's right subtree the old root's left subtree



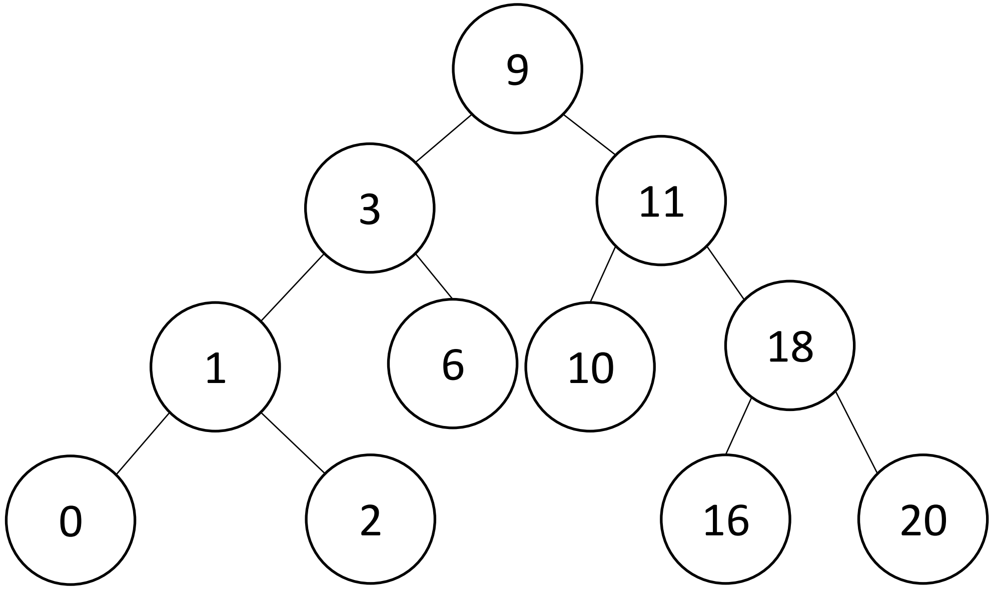
Insert Example



Not Balanced!

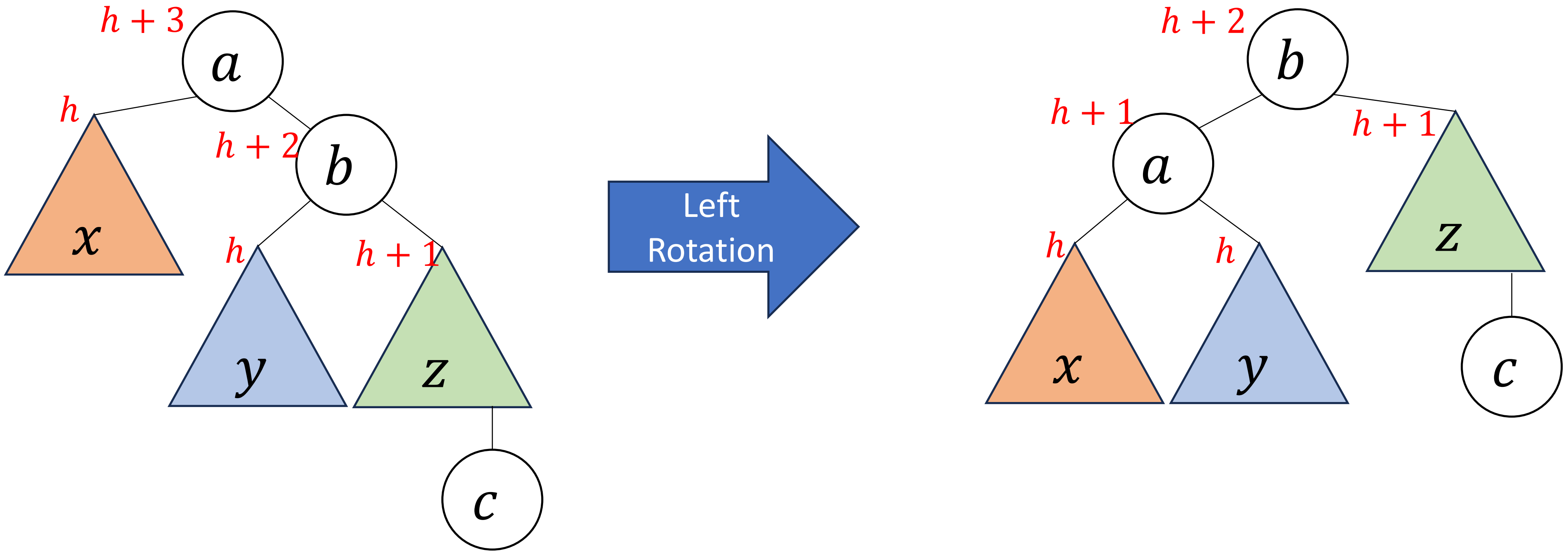


Balanced!



Left Rotation

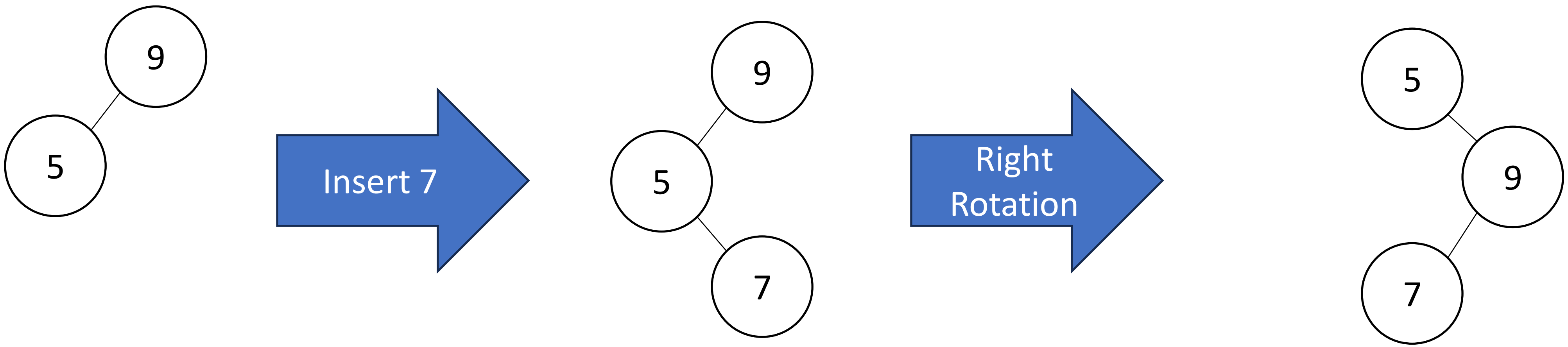
- Make the right child the new root
- Make the old root the left child of the new
- Make the new root's left subtree the old root's right subtree



Insertion Story So Far

- After insertion, update the heights of the node's ancestors
- Check for unbalance
- If unbalanced then at the deepest unbalanced root:
  - If the left subtree was deeper then rotate right
  - If the right subtree was deeper then rotate left

This is incomplete!
There are some cases
where this doesn't work!



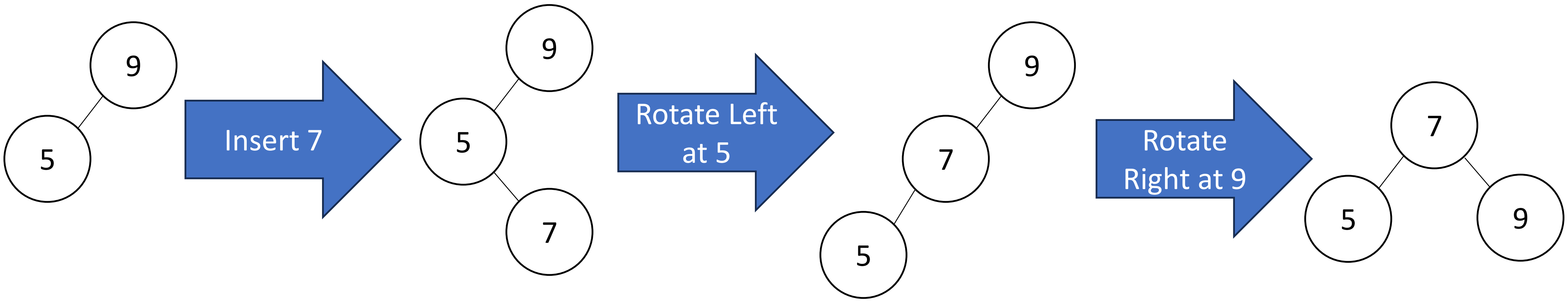
Insertion Story So Far

- After insertion, update the heights of the node's ancestors
- Check for unbalance
- If unbalanced then at the deepest unbalanced root:
  - Case LL: If we inserted in the **left** subtree of the **left** child then rotate right
  - Case RR: If we inserted in the **right** subtree of the **right** child then rotate left
  - Case LR: If we inserted into the **right** subtree of the **left** child then ???
  - Case RL: If we inserted into the **left** subtree of the **right** child then ???

Cases LR and RL require 2 rotations!

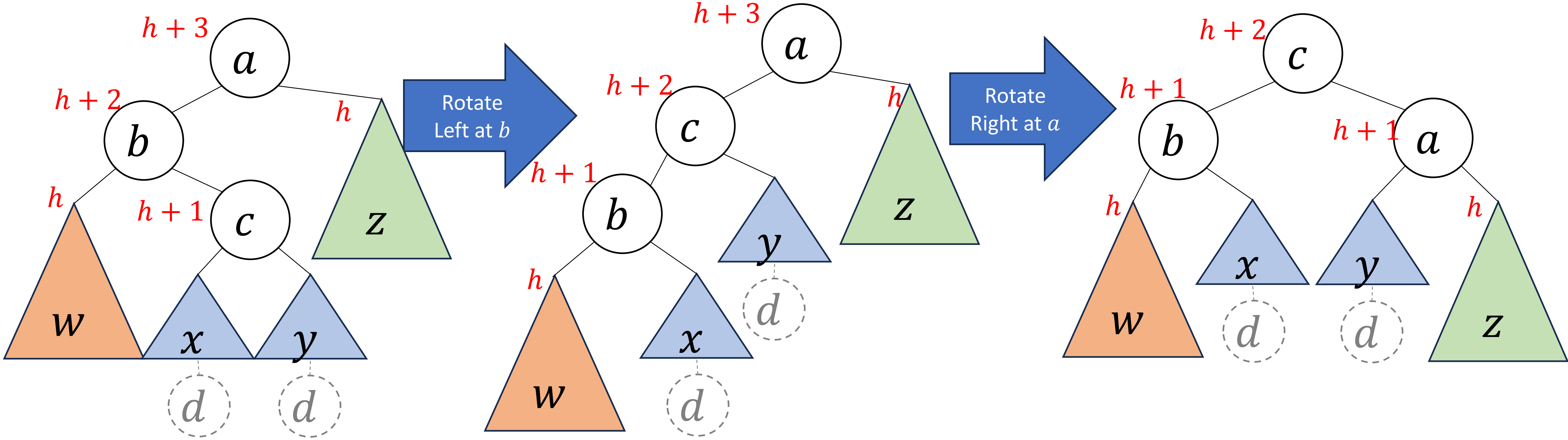
Case LR

- From deepest unbalanced root:
  - Rotate left at the left child
  - Rotate right at the root



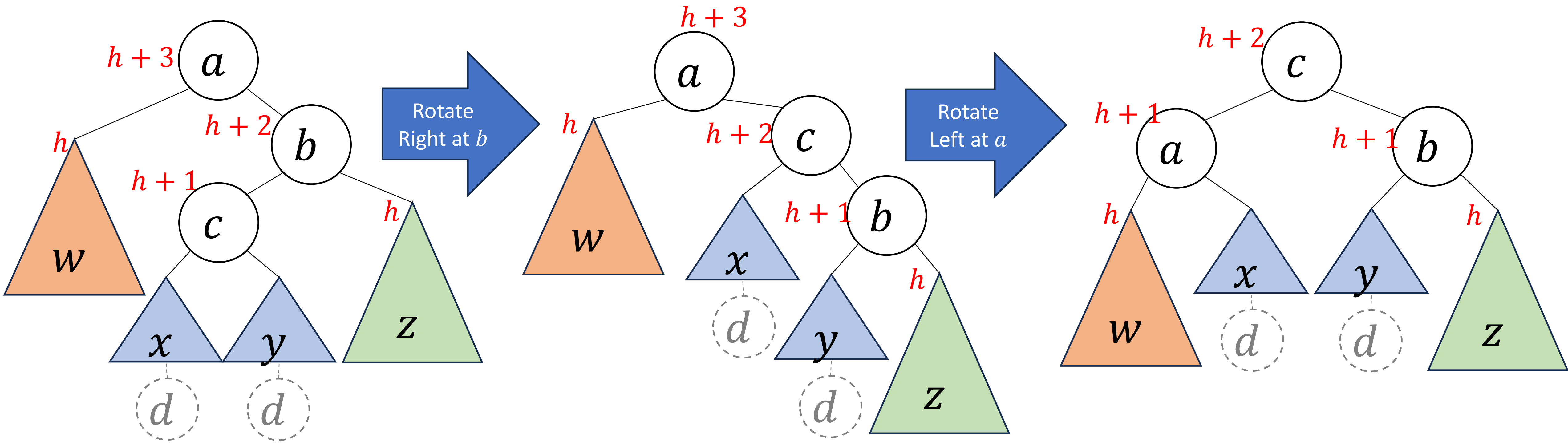
Case LR in General

- Imbalance caused by inserting in the left child's right subtree
- Rotate left at the left child
- Rotate right at the unbalanced node



Case RL in General

- Imbalance caused by inserting in the right child's left subtree
- Rotate right at the right child
- Rotate left at the unbalanced node



Insert Summary

- After a BST insertion, update the heights of the node's ancestors
- From leaf to root, check if each node is unbalanced
- If a node is unbalanced then at the deepest unbalanced node:
  - Case LL: If we inserted in the **left** subtree of the **left** child then: rotate right
  - Case RR: If we inserted in the **right** subtree of the **right** child then: rotate left
  - Case LR: If we inserted into the **right** subtree of the **left** child then: rotate left at the left child and then rotate right at the root
  - Case RL: If we inserted into the **left** subtree of the **right** child then: rotate right at the right child and then rotate left at the root
- Done after either reaching the root or applying **one** of the above cases

Delete Summary

- Tldr: same cases, reverse direction of rotation, may need to repeat with ancestors
- After a BST deletion, update the heights of the node's ancestors
- From leaf to root, check if each node is unbalanced
- If a node is unbalanced then at the deepest unbalanced node:
  - Case LL: If we deleted in the **left** subtree of the **left** child then: rotate left
  - Case RR: If we deleted in the **right** subtree of the **right** child then: rotate right
  - Case LR: If we deleted into the **right** subtree of the **left** child then: rotate right at the left child and then rotate left at the root
  - Case RL: If we deleted into the **left** subtree of the **right** child then: rotate left at the right child and then rotate right at the root
- Continue checking until reach the root