CSE 332 Autumn 2024 Lecture 27: Complexity and Tractability

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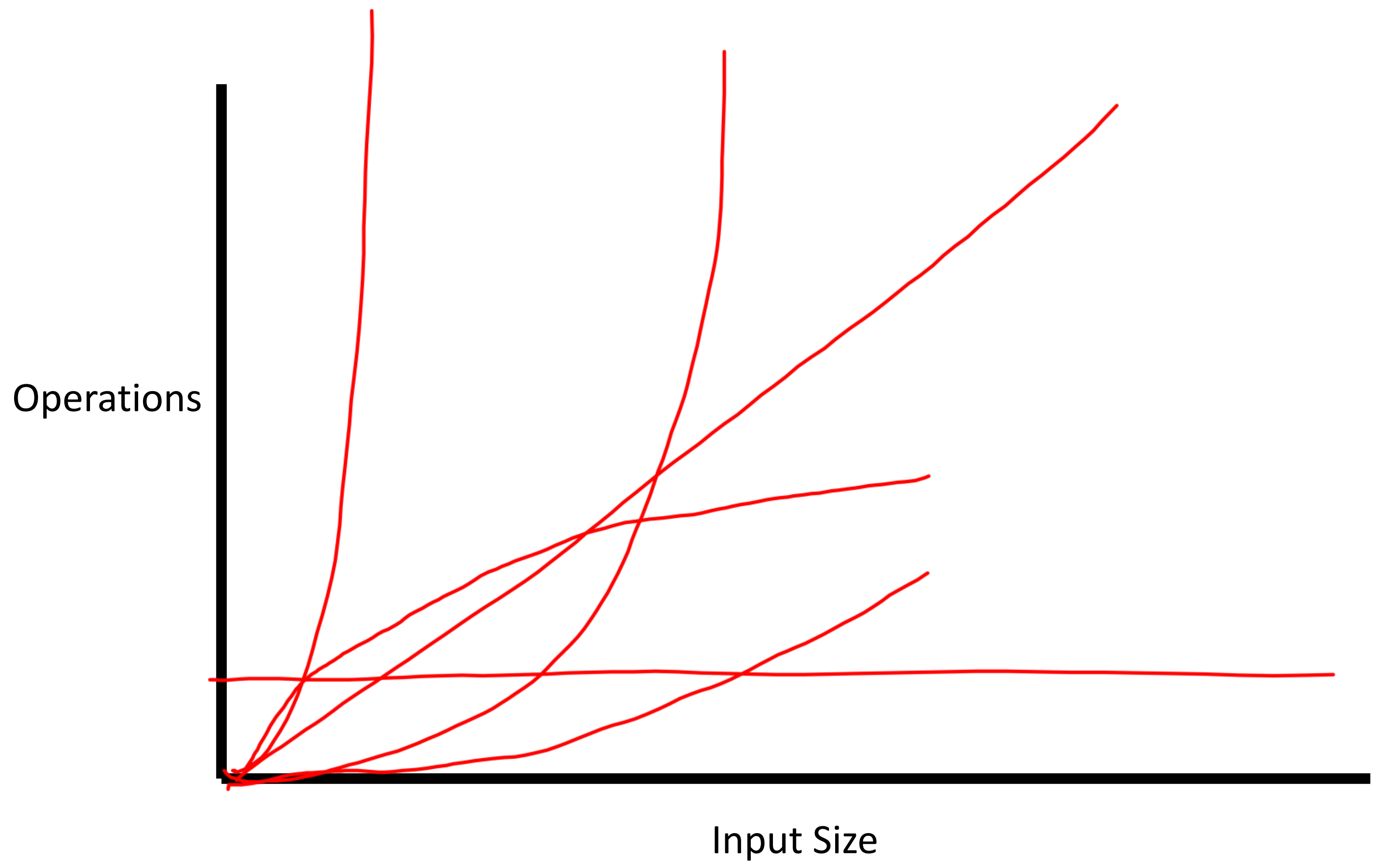
http://www.cs.uw.edu/332

Running Times



Running times we've seen:

- $\Theta(1)$
- $\Theta(\log n)$
- $\Theta(n)$
- $\Theta(n \log n)$
- $\Theta(n^2)$
- $\Theta(2^n)$

Running Times

n?

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

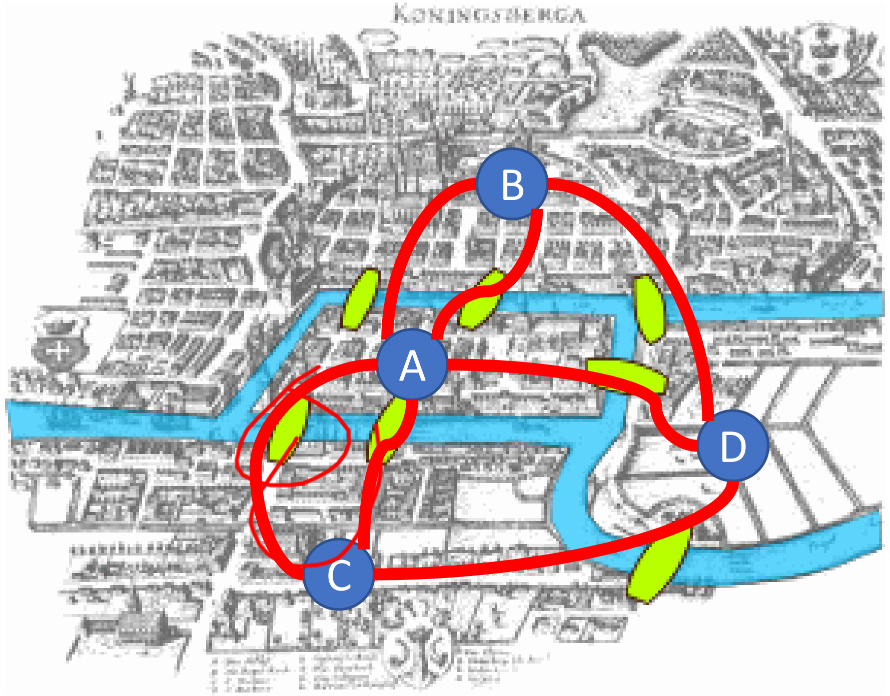
	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

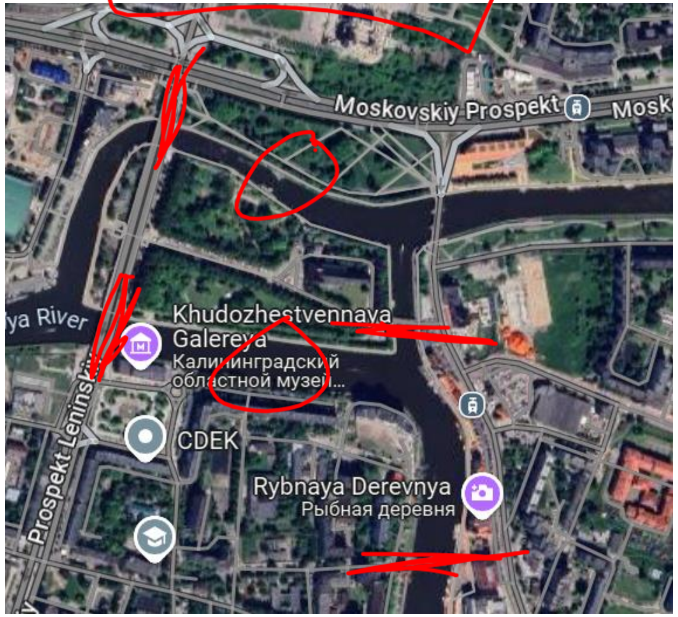
n?

Tractability

- Tractable:
  - Feasible to solve in the "real world"
- Intractable:
  - Infeasible to solve in the "real world"
- Whether a problem is considered "tractable" or "intractable" depends on the use case
  - For machine learning, big data, etc. tractable might mean $\mathrm{O}(n)$ or even $O(\log n)$
  - For most applications it's more like $O(n^3)$ or $O(n^2)$
- A strange pattern:
  - Most "natural" problems are either done in small-degree polynomial (e.g. n^2) or else exponential time (e.g. 2^n)
  - It's rare to have problems which require a running time of n^5, for example

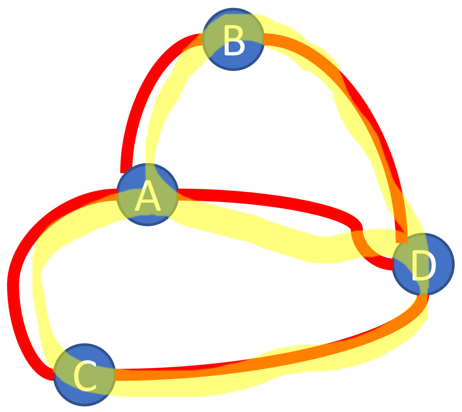
7 Bridges of Königsberg

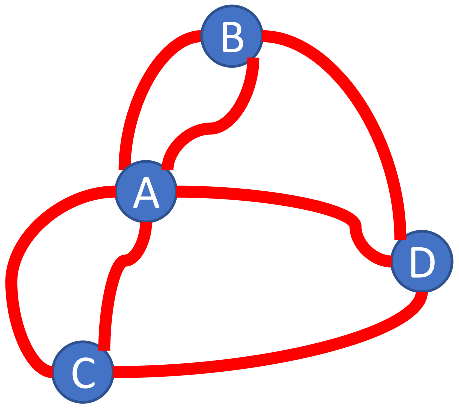




The Pregel River runs through the city of Koenigsberg, creating 2 islands. Among these 2 islands and the 2 sides of the river, there are 7 bridges. Is there any path starting at one landmass which crosses each bridge exactly once?

Euler Path Problem

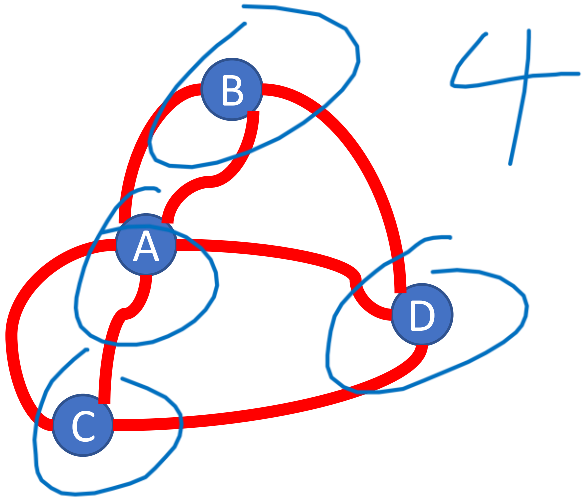




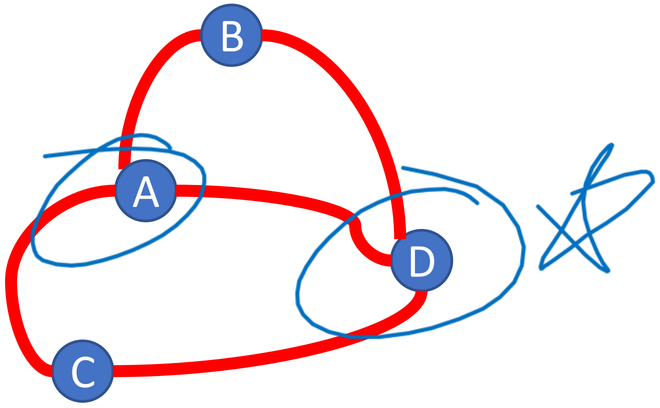
- Path:
  - A sequence of nodes $v_1, v_2, \ldots$ such that for every consecutive pair are connected by an edge (i.e. (v_i, v_{i+1}) is an edge for each i in the path)
- Euler Path:
  - A path such that every edge in the graph appears exactly once
    - If the graph is not simple then some pairs need to appear multiple times!
- Euler path problem:
  - Given an undirected graph $G = (V, E)$, does there exist an Euler path for G?

Examples

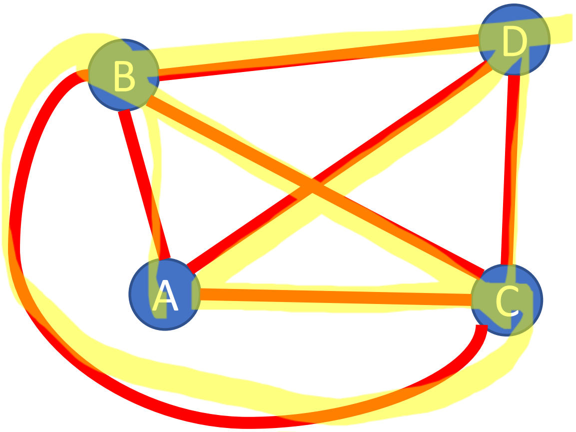
- Which of the graphs below have an Euler path?



No Euler path exists!



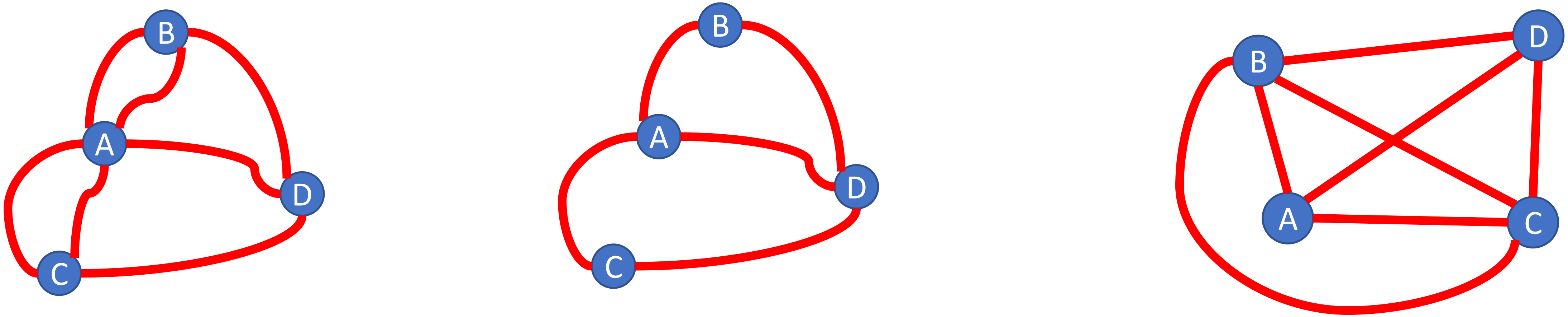
Euler path exists!
A, B, D, A, C, D



Euler path exists!
A, B, C, D, A, C, B, D

Euler's Theorem

- A graph has an Euler Path if and only if it is connected and has exactly 0 or 2 nodes with odd degree.



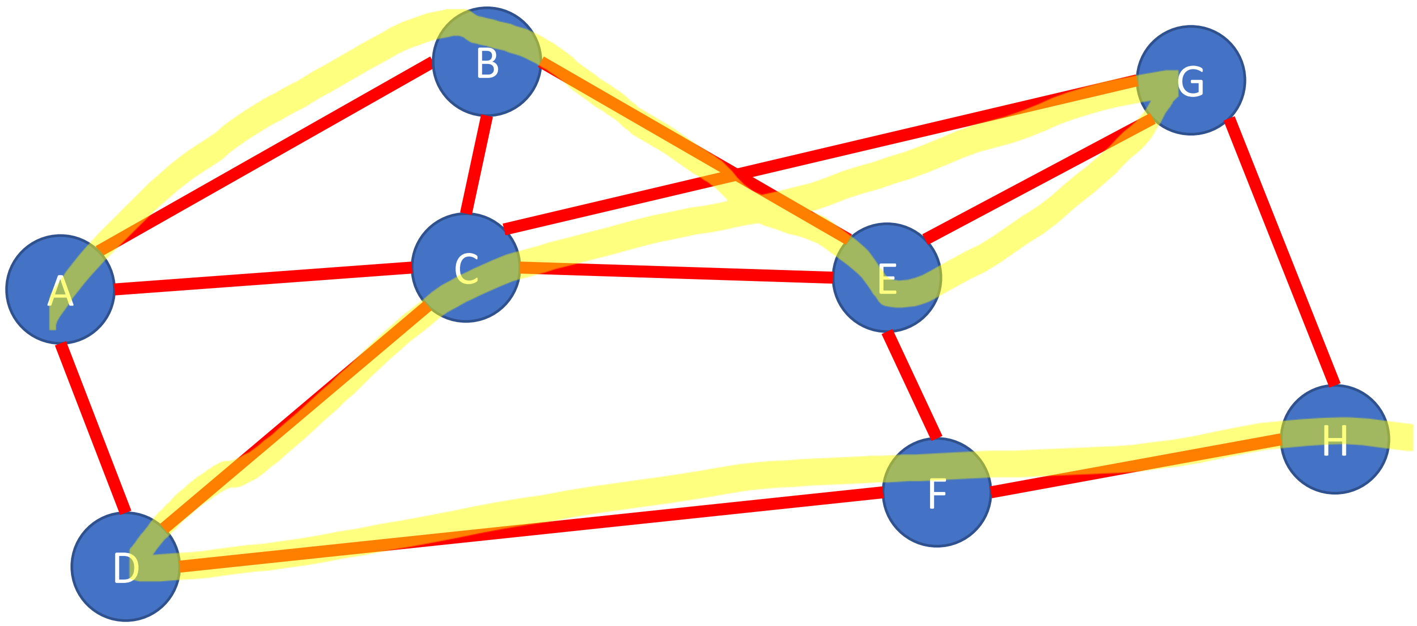
Algorithm for the Euler Path Problem

- Given an undirected graph $G = (V, E)$, does there exist an Euler path for G?
- Algorithm:
  - Check if the graph is connected BFS $V + E$
  - Check the degree of each node
  - If the number of nodes with odd degree is 0 or 2, return true
  - Otherwise return false
- Running time? $V + E$

A Seemingly Similar Problem

1.7^n 2^n

- Hamiltonian Path:
  - A path that includes every node in the graph exactly once
- Hamiltonian Path Problem:
  - Given a graph $G = (V, E)$, does that graph have a Hamiltonian Path?



True!
A, B, C, E, G, H, F, D

Algorithms for the Hamiltonian Path Problem

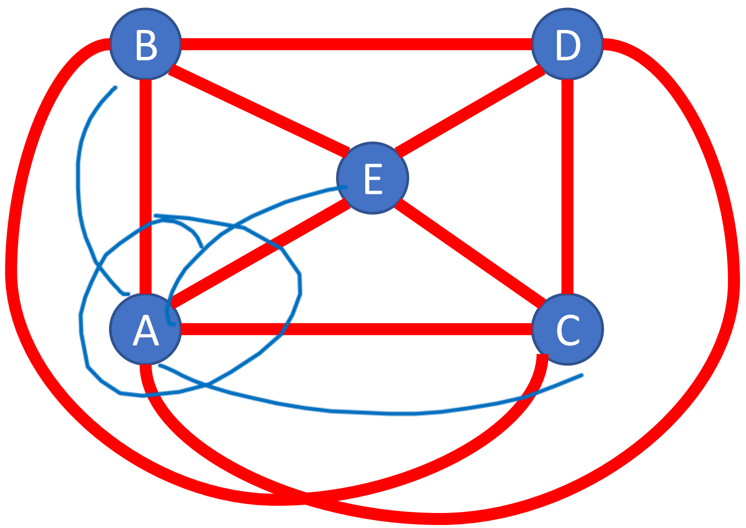
- Option 1:
  - Explore all possible simple paths through the graph
  - Check to see if any of those are length V
- Option 2:
  - Write down every sequence of nodes
  - Check to see if any of those are a path
- Both options are examples of an **Exhaustive Search ("Brute Force") algorithm**

Option 2: List all sequences, look for a path

- Running time:
  - $G = (V, E)$
  - Number of permutations of V is $|V|!$
    - $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$
  - How does $n!$ compare with 2^n?
    - $n! \in \Omega(2^n)$
  - Exponential running time!

Option 1: Explore all simple paths, check for one of length V

- Running time:
  - $G = (V, E)$
  - Number of paths
    - Pick a first node ($|V|$ choices)
    - Pick a neighbor (up to $|V| - 1$ choices)
    - Pick a neighbor (up to $|V| - 2$ choices)
    - …. Repeat $|V| - 1$ total times
    - Overall: $|V|!$ paths
  - Exponential running time



Complexity Classes

- A Complexity Class is a set of problems (e.g. sorting, Euler path, Hamiltonian path)
  - The problems included in a complexity class are those whose most efficient algorithm has a specific upper bound on its running time (or memory use, or…)
- Examples:
  - The set of all problems that can be solved by an algorithm with running time $O(n)$
    - Contains: Finding the minimum of a list, finding the maximum of a list, buildheap, summing a list, etc.
  - The set of all problems that can be solved by an algorithm with running time $O(n^2)$
    - Contains: everything above as well as sorting, Euler path
  - The set of all problems that can be solved by an algorithm with running time $O(n!)$
    - Contains: everything we've seen in this class so far

Complexity Classes and Tractability

- To explore what problems are and are not tractable, we give some complexity classes special names:
- Complexity Class P:
  - Stands for "Polynomial"
  - The set of problems which have an algorithm whose running time is $O(n^p)$ for some choice of $p \in \mathbb{R}$.
  - We say all problems belonging to P are "Tractable"
- Complexity Class EXP:
  - Stands for "Exponential"
  - The set of problems which have an algorithm whose running time is $O\left(2^{n^p}\right)$ for some choice of $p \in \mathbb{R}$
  - We say all problems belonging to $EXP - P$ are "Intractable"
    - Disclaimer: Really it's all problems outside of P, and there are problems which do not belong to EXP, but we're not going to worry about those in this class

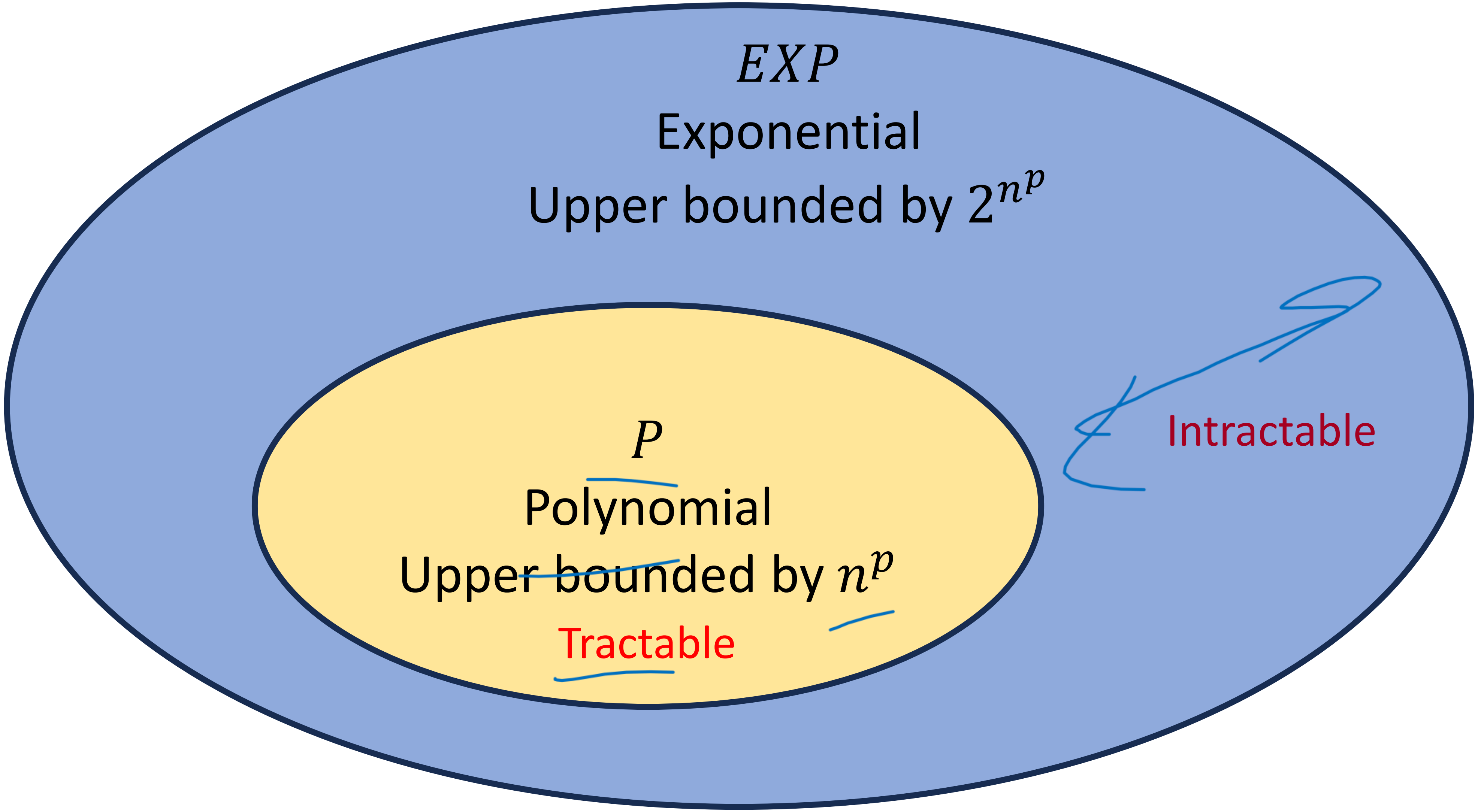
EXP and P

Important!

$P \subset EXP$

Every problem within P is also within EXP

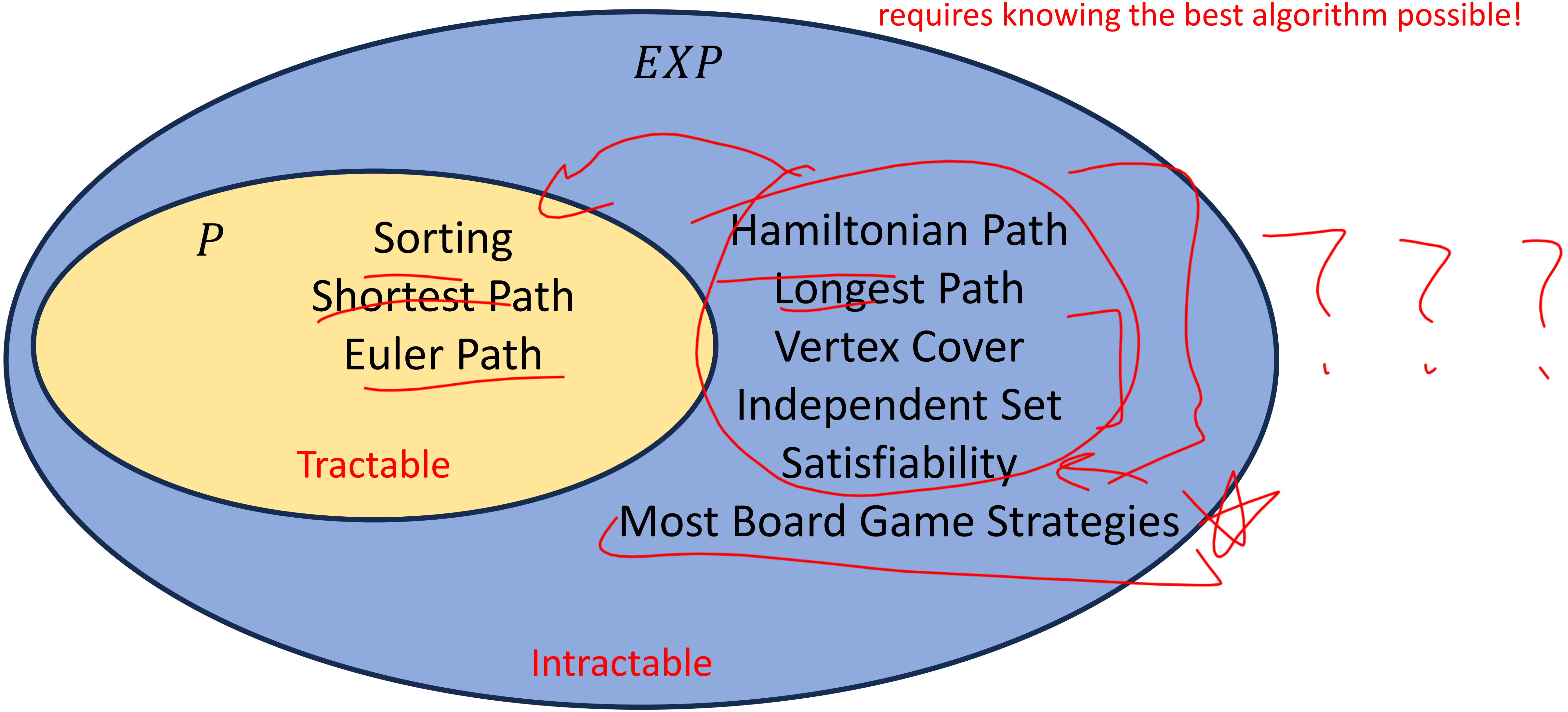
The intractable ones are the problems within EXP but NOT P



Members

<u>Important!</u>

Some of the problems we've listed in EXP could also be members of P. Since membership is determined by a problem's *most* efficient algorithm, knowing if a problem belongs to P requires knowing the best algorithm possible!



Studying Complexity and Tractability

- Organizing problems into complexity classes helps us to reason more carefully and flexibly about tractability
- The goal for each problem is to either
  - Find an efficient algorithm if it exists
    - i.e. show it belongs to P
  - Prove that no efficient algorithm exists
    - i.e. show it does not belong to P
- Complexity classes allow us to reason about sets of problems at a time, rather than each problem individually
  - If we can find more precise classes to organize problems into, we might be able to draw conclusions about the entire class
  - It may be easier to show a problem belongs to class C than to P, so it may help to show that $C \subseteq P$

Some problems in EXP seem “easier”

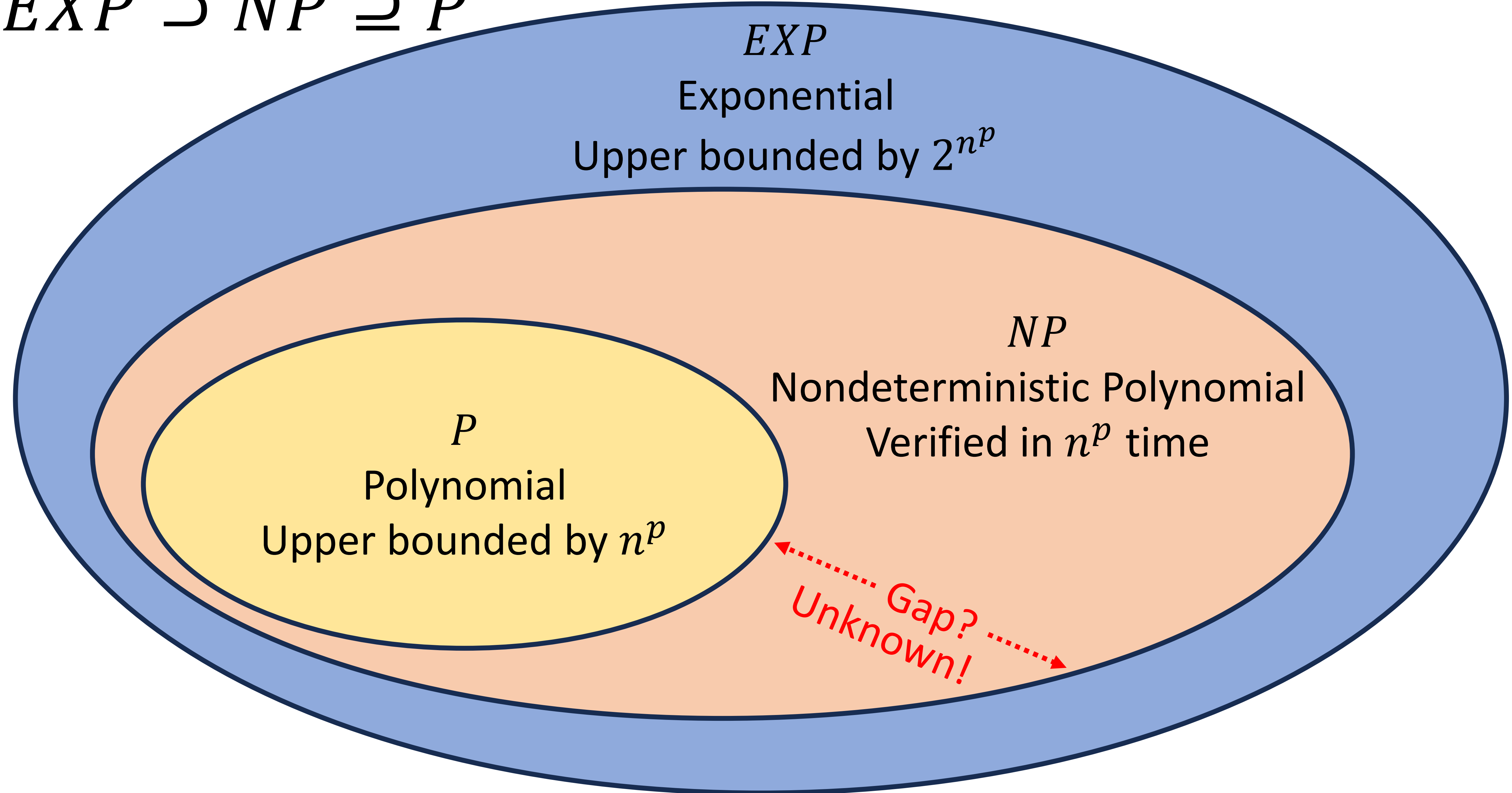
- There are some problems that we do not have polynomial time algorithms to solve, but provided answers are easy to check
- Hamiltonian Path:
  - It’s “hard” to look at a graph and determine whether it has a Hamiltonian Path
  - It’s “easy” to look at a graph and a candidate path together and determine whether THAT path is a Hamiltonian Path
    - It’s easy to **verify** whether a given path is a Hamiltonian path

Class NP

- NP
  - The set of problems for which a candidate solution can be verified in polynomial time
  - Stands for "Non-deterministic Polynomial"
    - Corresponds to algorithms that can guess a solution (if it exists), that solution is then verified to be correct in polynomial time
    - Can also think of as allowing a special operation that allows the algorithm to magically guess the right choice at each step of an exhaustive search
- $P \subseteq NP$
  - Why?

$\cancel{P} \overset{?}{\subsetneq} NP$

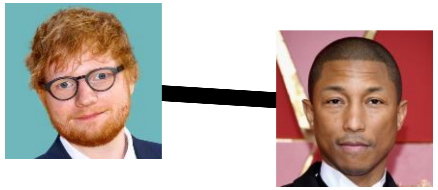
$EXP \supset NP \supseteq P$



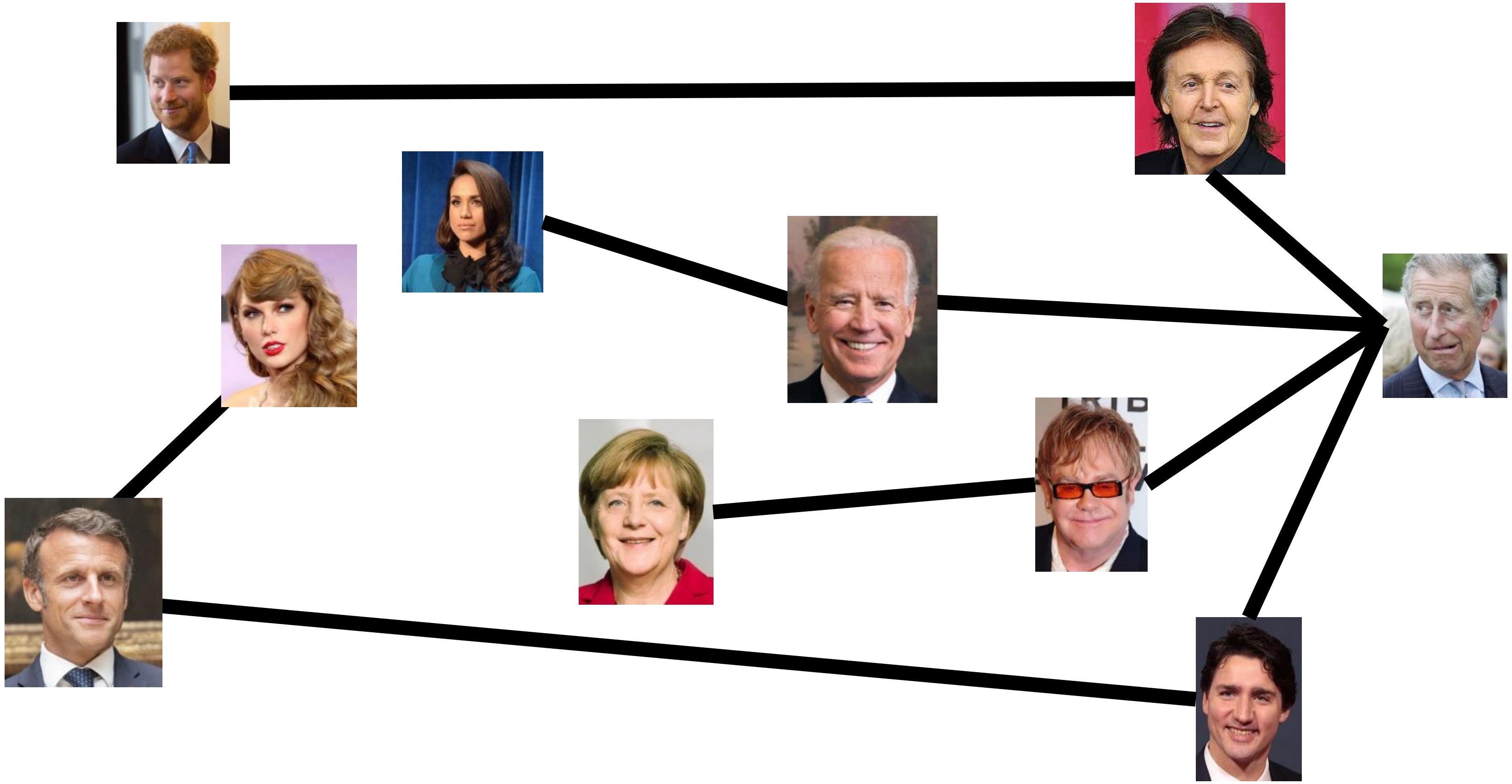
Solving and Verifying Hamiltonian Path

- Give an algorithm to solve Hamiltonian Path
  - Input: $G = (V, E)$
  - Output: True if G has a Hamiltonian Path
  - Algorithm: Check whether each permutation of V is a path.
    - Running time: $|V|!$, so does not show whether it belongs to P
- Give an algorithm to verify Hamiltonian Path
  - Input: $G = (V, E)$ and a sequence of nodes
  - Output: True if that sequence of nodes is a Hamiltonian Path
  - Algorithm:
    - Check that each node appears in the sequence exactly once
    - Check that the sequence is a path
    - Running time: $O(V \cdot E)$, so it belongs to NP

Party Problem



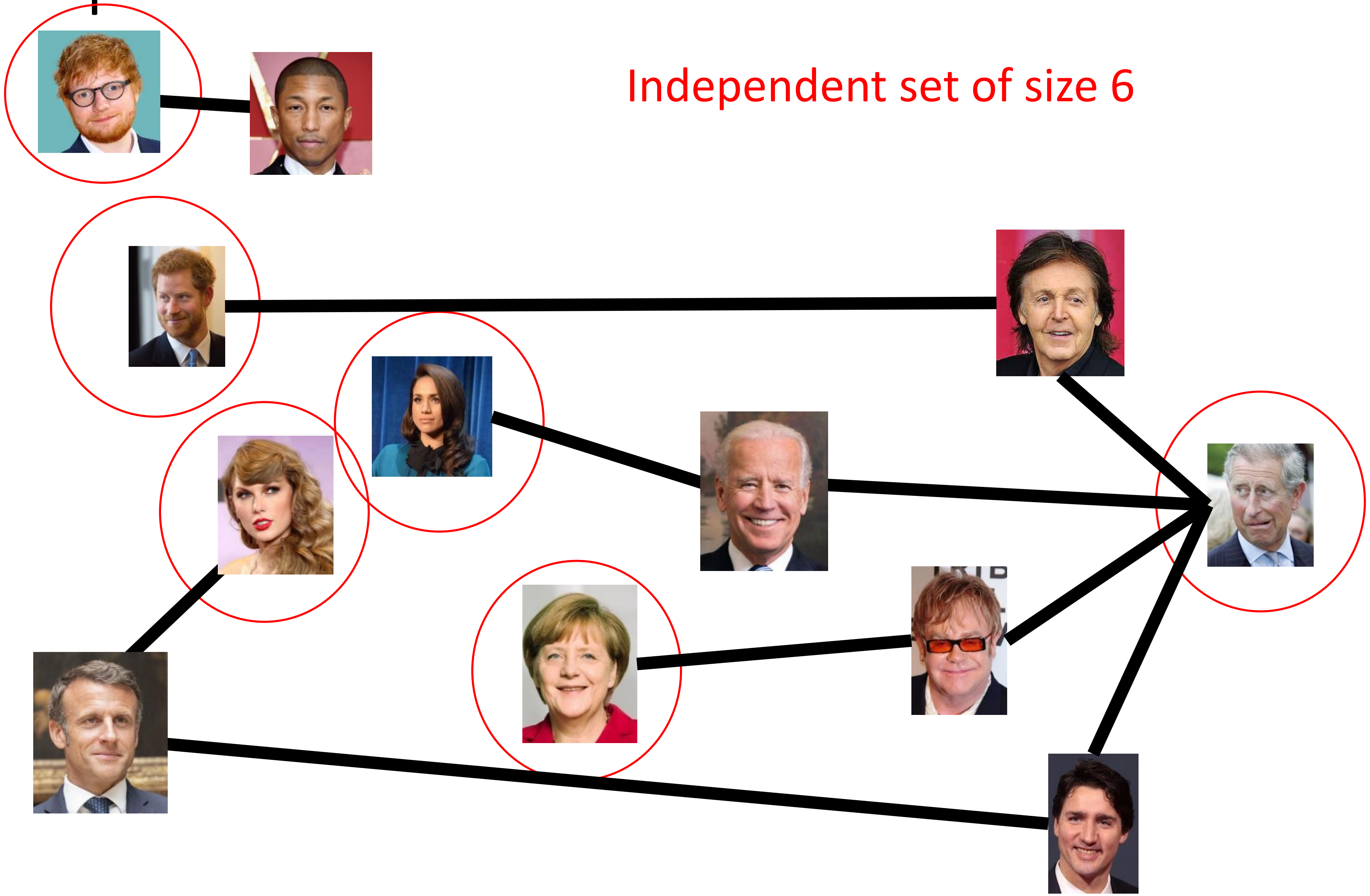
Draw Edges between people who don’t get along
How many people can I invite to a party if everyone must get along?



Independent Set

- Independent set:
  - $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Independent Set Problem:
  - Given a graph $G = (V, E)$ and a number k, determine whether there is an independent set S of size k

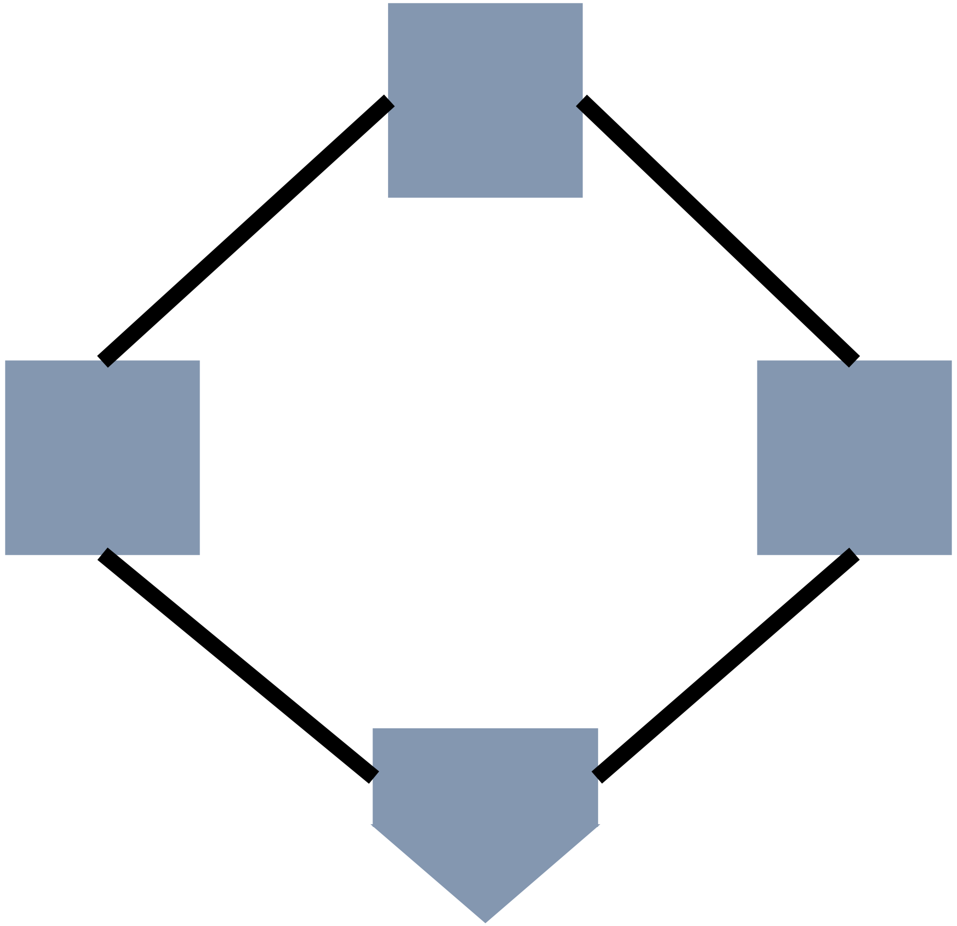
Example



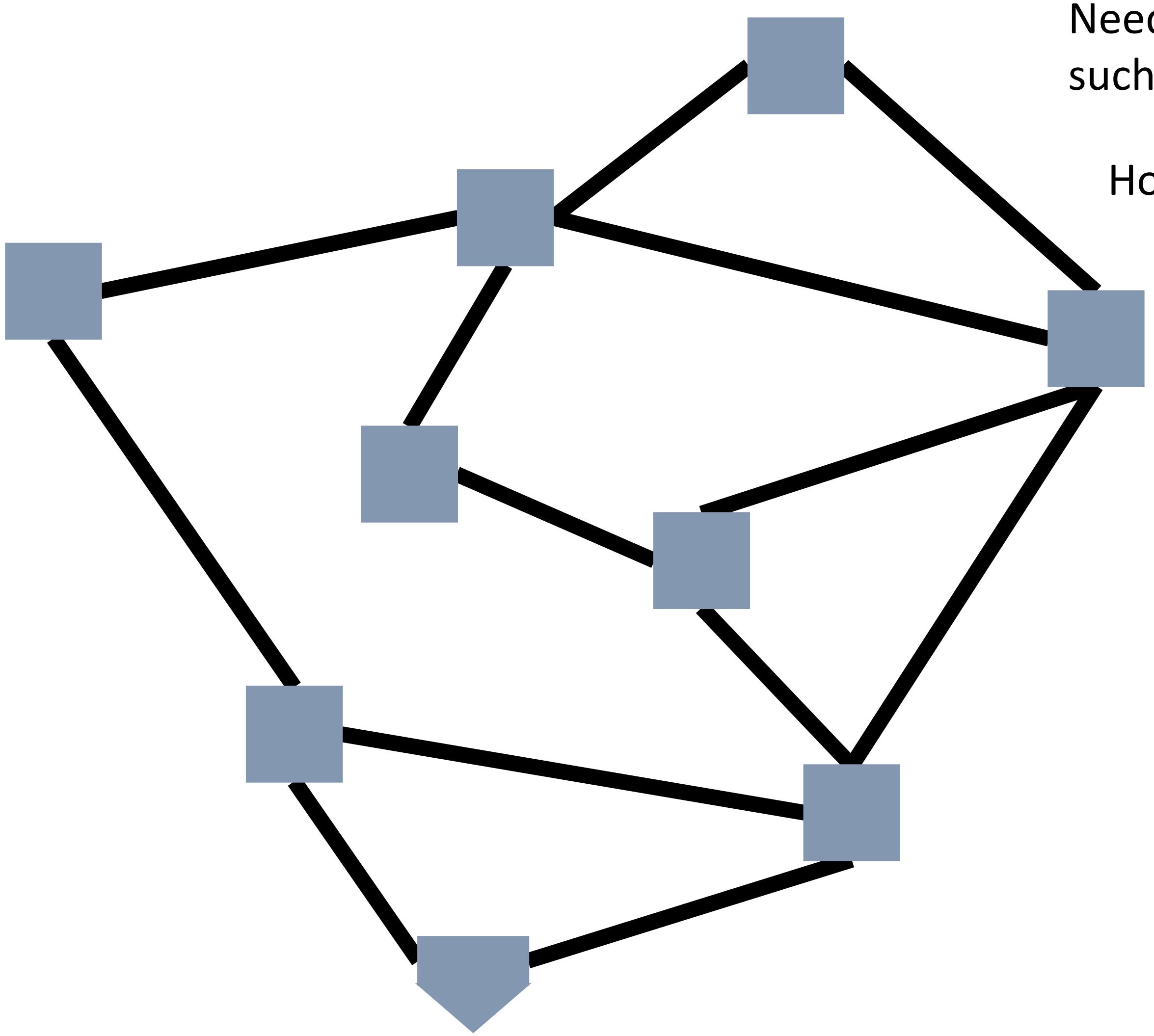
Solving and Verifying Independent Set

- Give an algorithm to **solve** independent set
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has an independent set of size k
- Give an algorithm to **verify** independent set
  - Input: $G = (V, E)$,a number k, and a set $S \subseteq V$
  - Output: True if S is an independent set of size k

Generalized Baseball



Generalized Baseball



Need to place defenders on bases such that every edge is defended

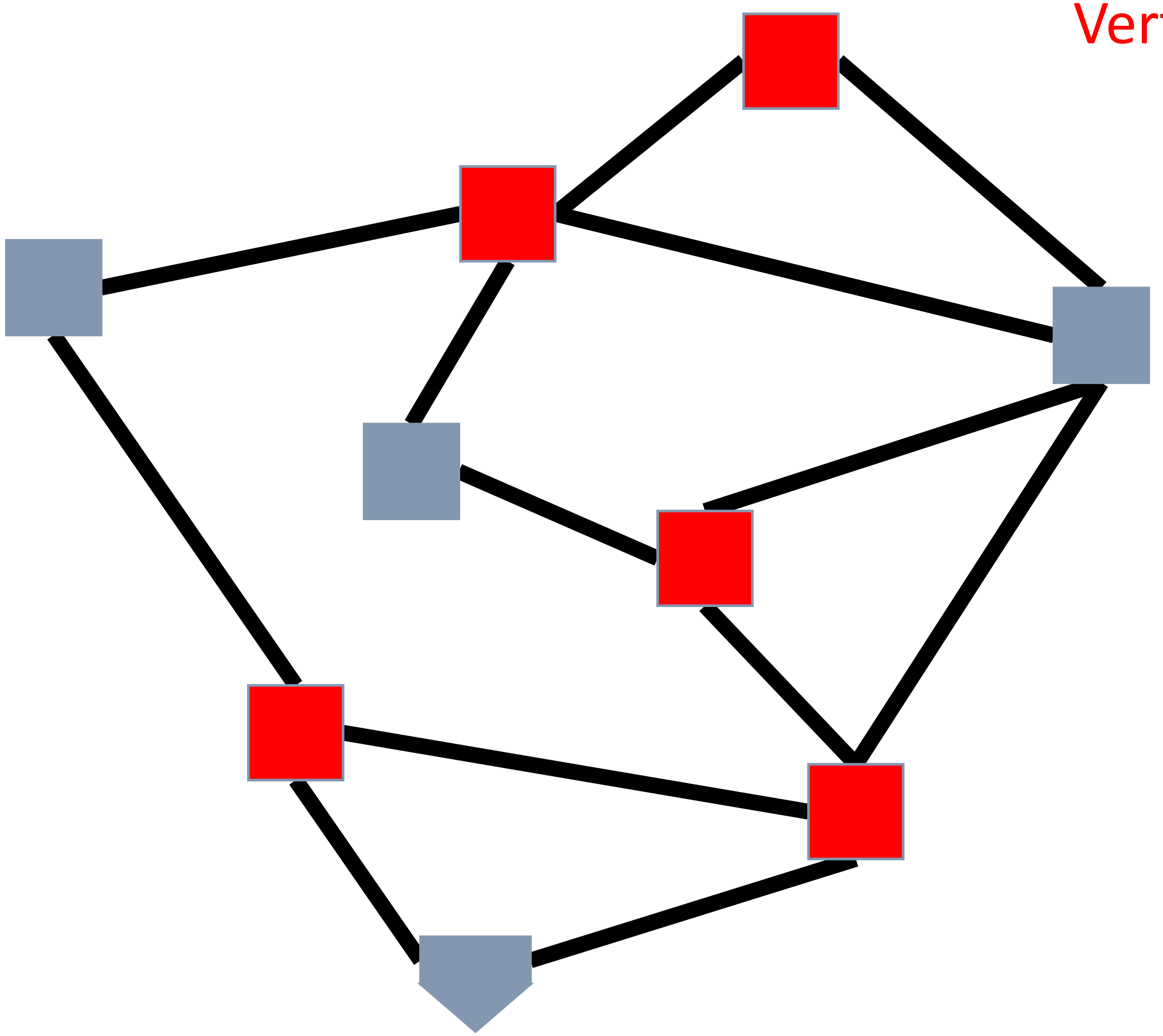
How many defenders would suffice?

Vertex Cover

- Vertex Cover:
  - $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Vertex Cover Problem:
  - Given a graph $G = (V, E)$ and a number k, determine if there is a vertex cover C of size k

Example

Vertex cover of size 5



Solving and Verifying Vertex Cover

- Give an algorithm to **solve** vertex cover
  - Input: $G = (V, E)$ and a number k
  - Output: True if G has a vertex cover of size k
- Give an algorithm to **verify** vertex cover
  - Input: $G = (V, E)$,a number k, and a set $S \subseteq E$
  - Output: True if S is a vertex cover of size k