CSE 332 Autumn 2024
Lecture 26: Minimum Spanning Trees

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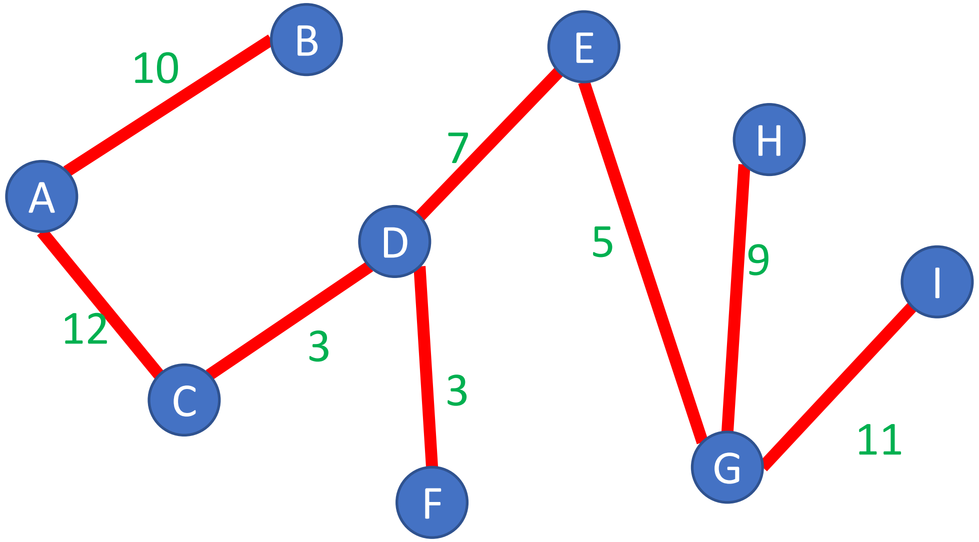
http://www.cs.uw.edu/332

Definition: Tree

A connected graph with no cycles



Note: A tree does not need a root, but they often do!

Definition: Tree

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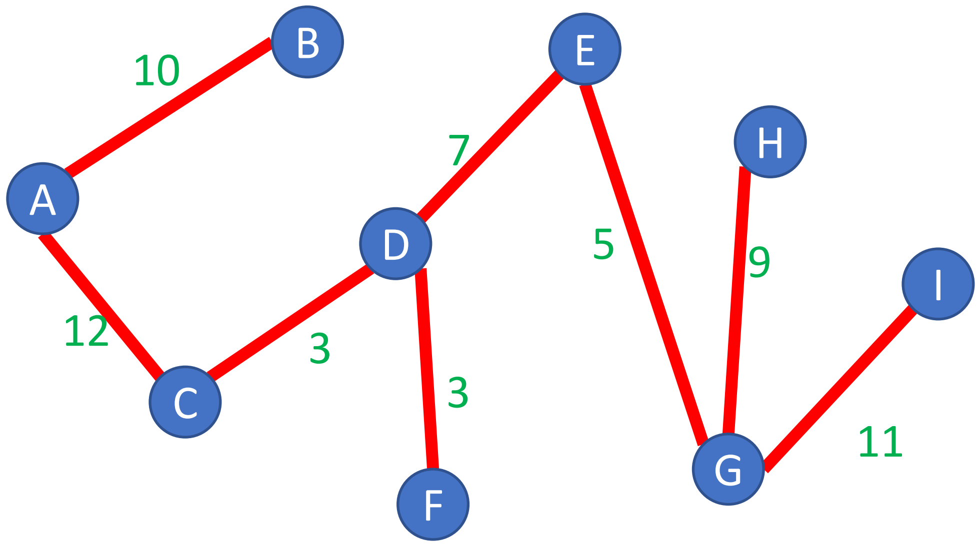
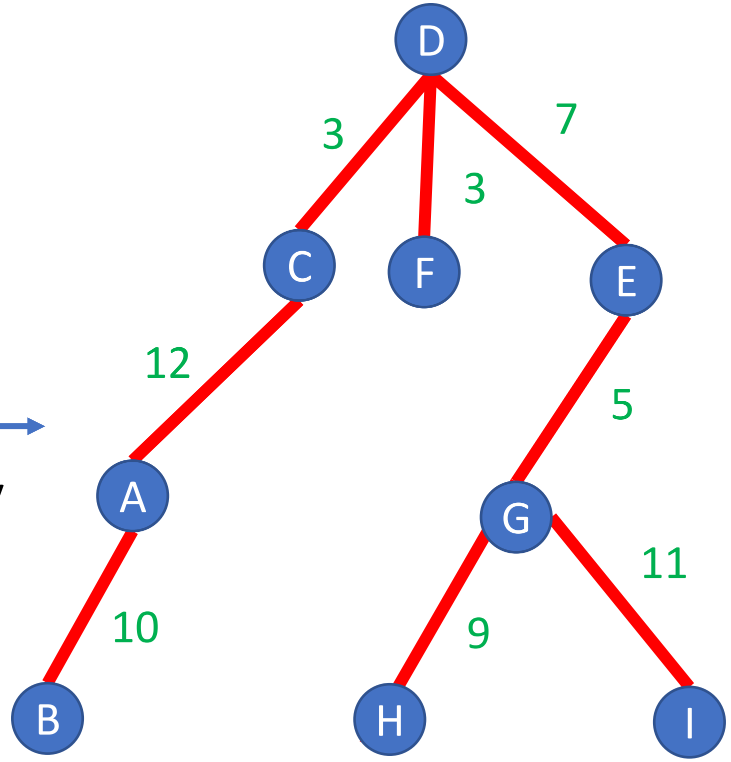
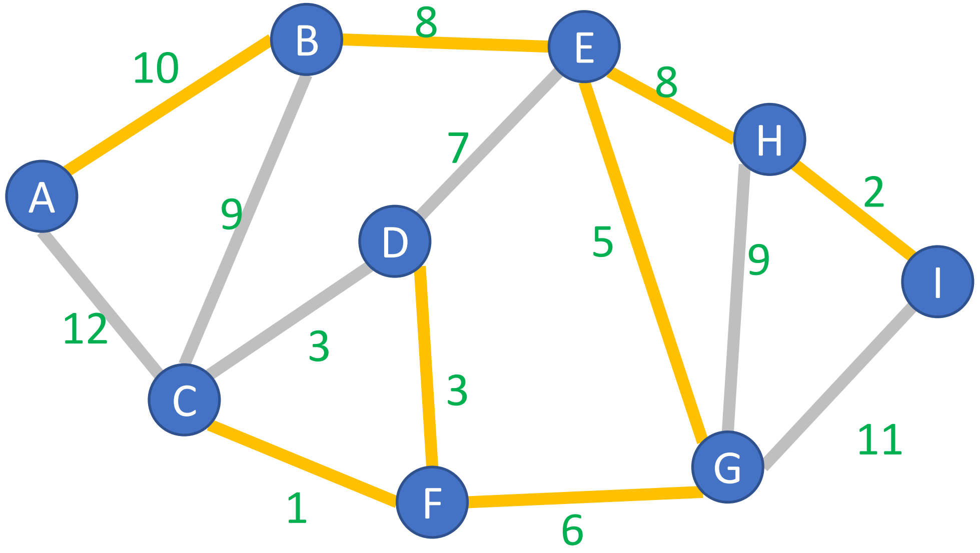


Pick some arbitrary root node and rearrange tree



Definition: Spanning Tree

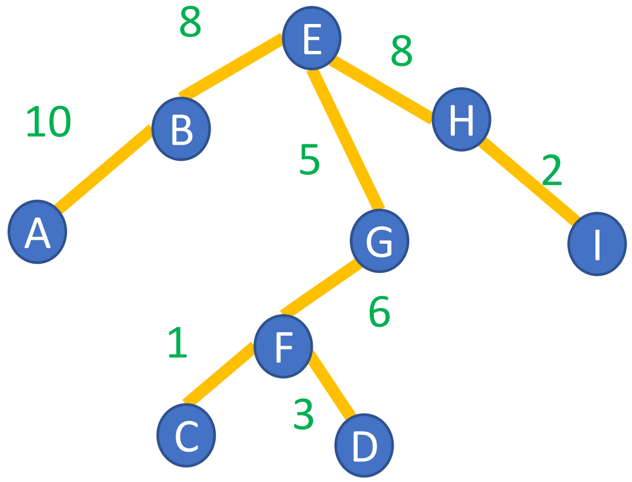
A Tree $\boldsymbol{T} = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$



How many edges does T have?

$V - 1$

Pick some arbitrary root node and rearrange tree

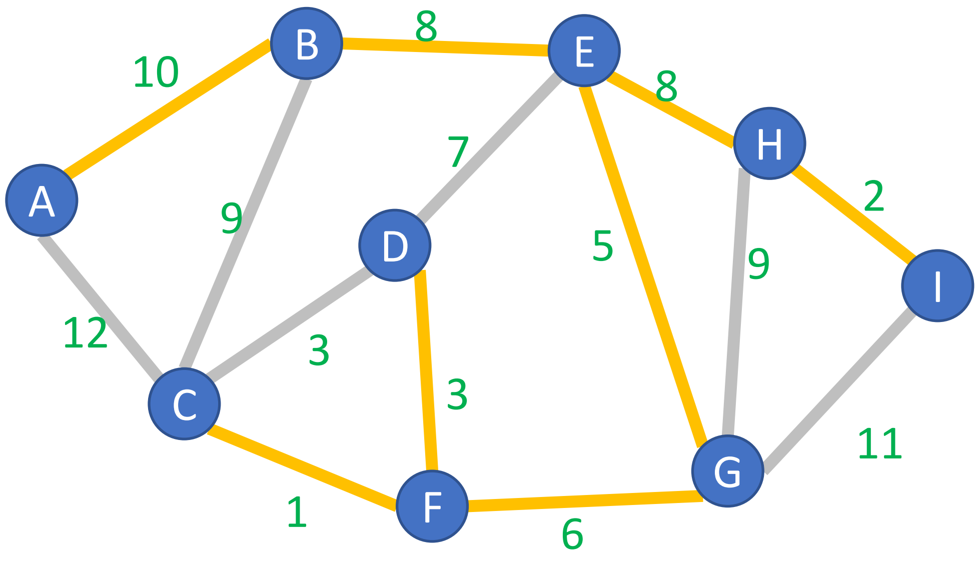


Any set of V-1 edges in the graph that doesn't have any cycles is guaranteed to be a spanning tree!

Any set of V-1 edges that connects all the nodes in the graph is guaranteed to be a spanning tree!

Definition: Minimum Spanning Tree

A Tree $\boldsymbol{T} = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$, that has minimal cost



$$Cost(T) = \sum_{e \in E_T} w(e)$$

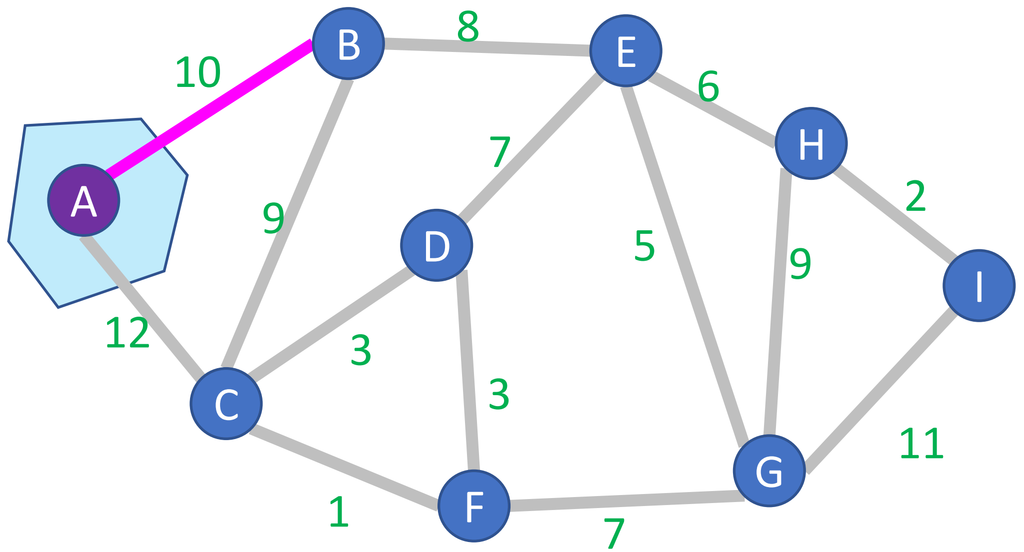
Prim's Algorithm

Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A



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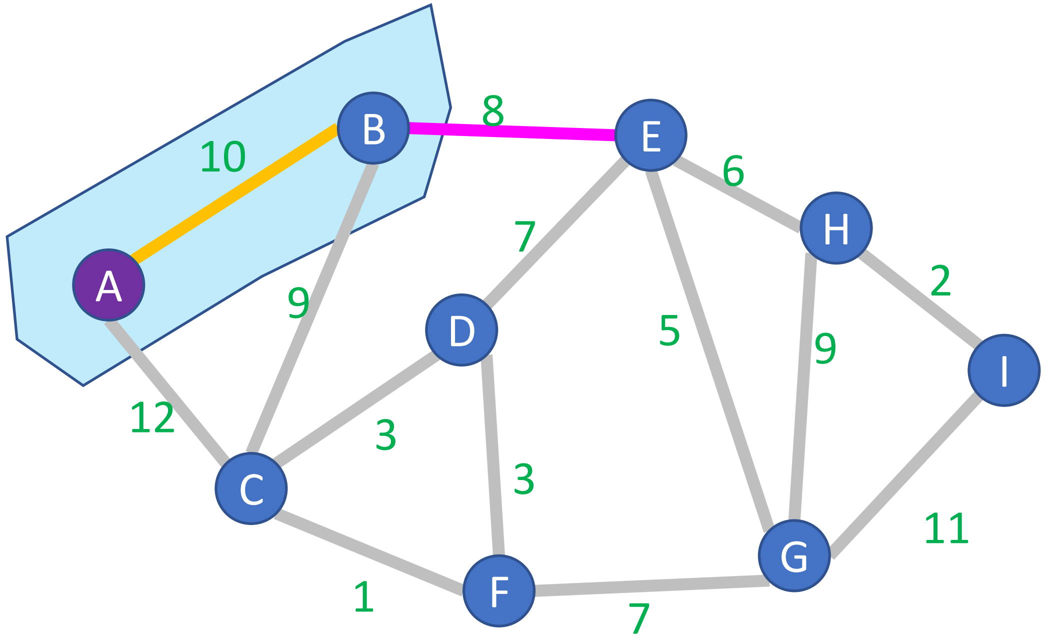

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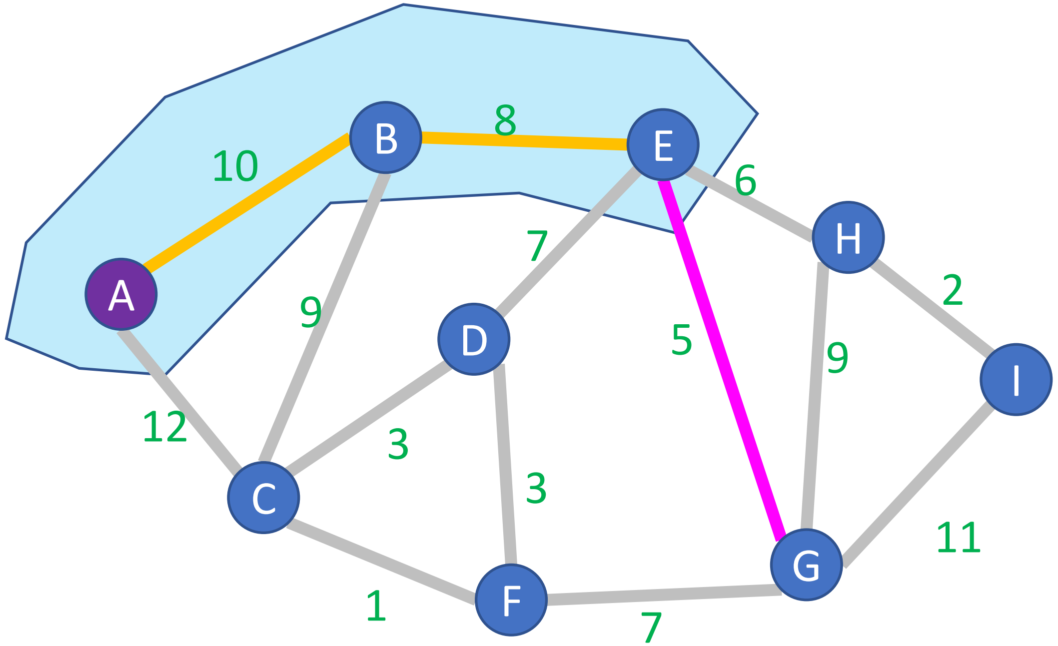

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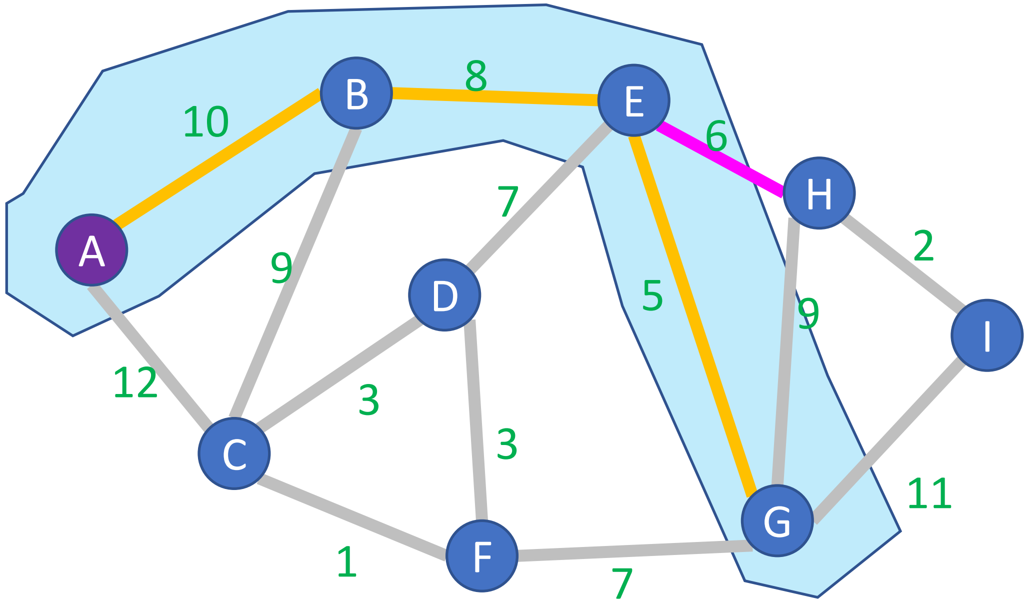


Prim's Algorithm

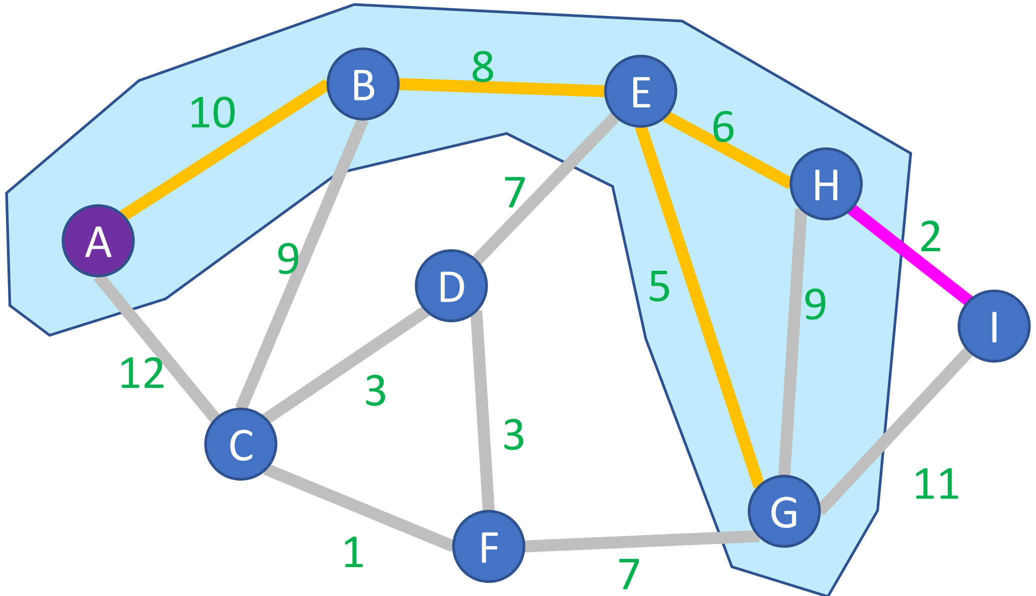
Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

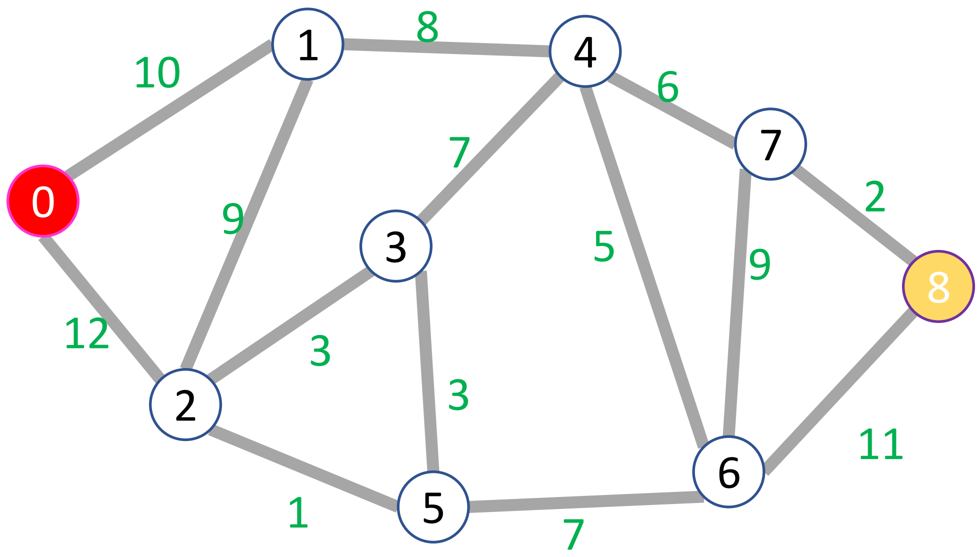
Add the min-weight edge which connects to node in A with a node not in A

Keep edges in a Heap
$O(E \log V)$

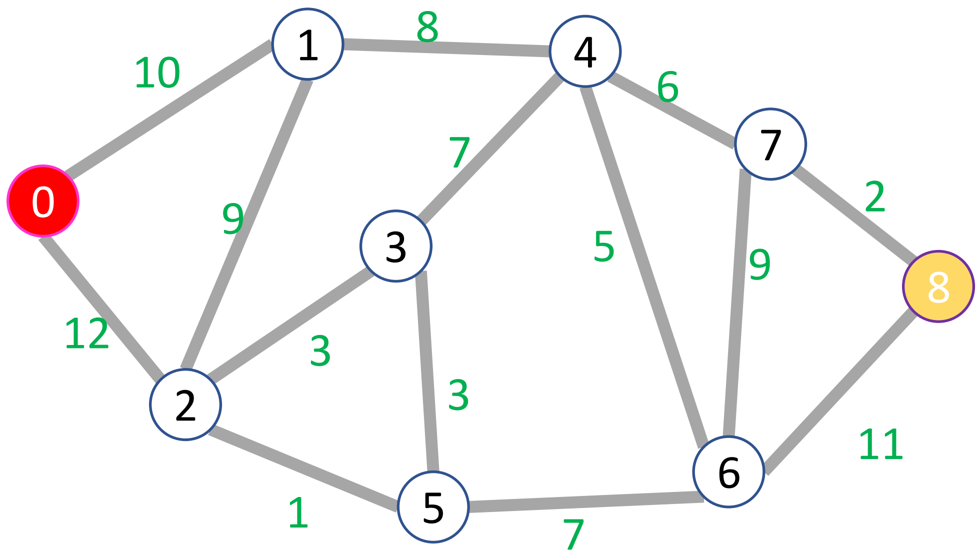


Dijkstra’s Algorithm

```
int dijkstras(graph, start, end){
        PQ = new minheap();
        PQ.insert(0, start);  // priority=0, value=start
        start.distance = 0;
        while (!PQ.isEmpty){
                current = PQ.extractmin();
                if (current.known){ continue;}
                current.known = true;
                for (neighbor : current.neighbors){
                        if (!neighbor.known){
                                new_dist = current.distance + weight(current,neighbor);
                                if(neighbor.dist != ∞){ PQ.insert(new_dist, neighbor);}
                                else if (new_dist < neighbor. distance){
                                        neighbor. distance = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return end.distance;
}
```



Prim's Algorithm

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int prims(graph, start, end){
        PQ = new minheap();
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                current = PQ.extractmin();
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                for (neighbor : current.neighbors){
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                                new_dist = weight(current,neighbor);
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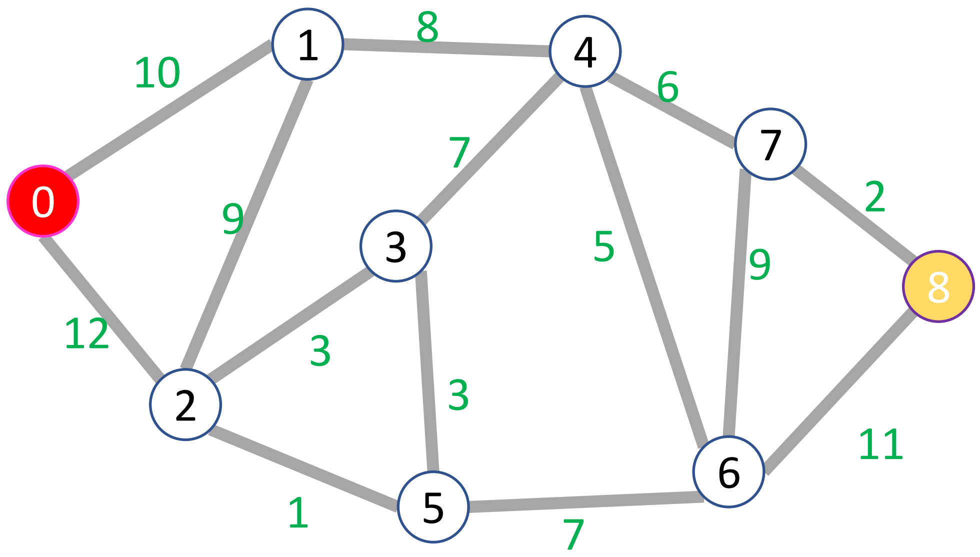
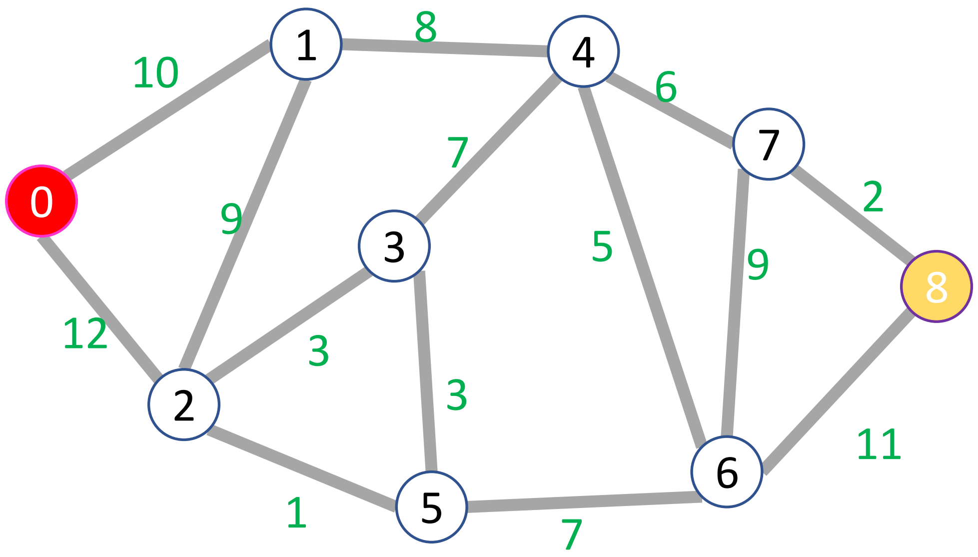


Prim’s Algorithm

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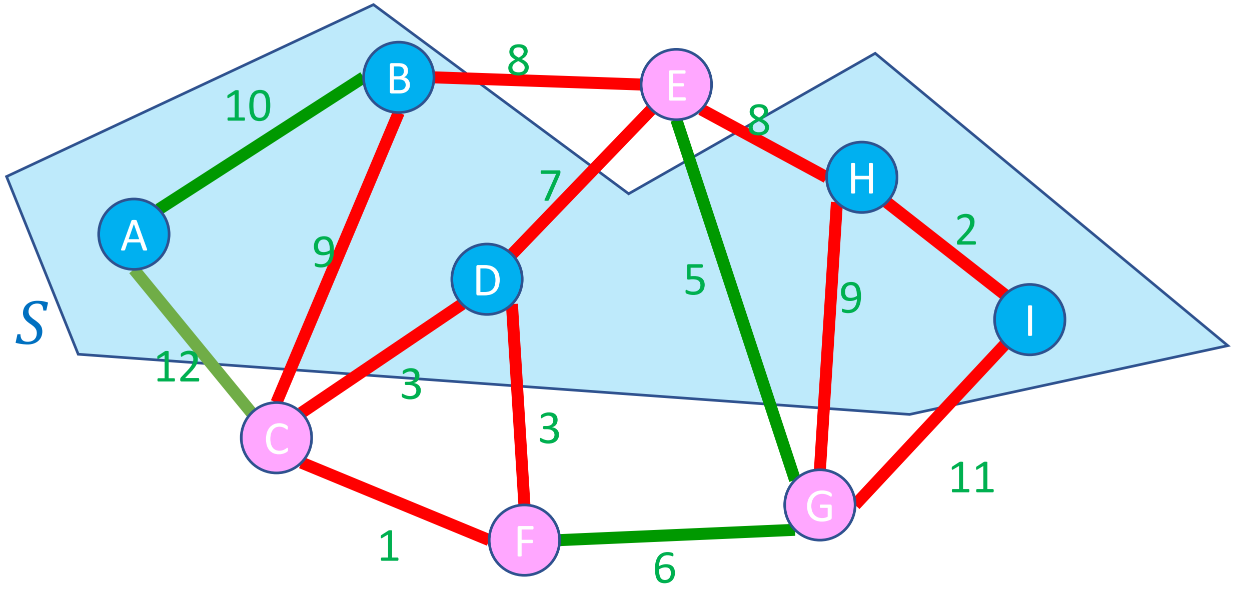
			}
		}
	}
	return end.distance;
}
```

Why does this work?

- To argue that Prim's produces a minimum spanning tree:
  - First we show that Prim's produces a spanning tree
    - Show two of:
      - Connected
      - Acyclic
      - $V - 1$ edges
  - Then we show that it is a minimum spanning tree
    - Show all edges chosen are MST edges
      - Using the "Cut Theorem"

Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, S and $V - S$



Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. (A, C)

A set of edges R Respects a cut if no edges cross the cut e.g. $R = \{(A, B), (E, G), (F, G)\}$

Cut Theorem

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.

Cut Theorem

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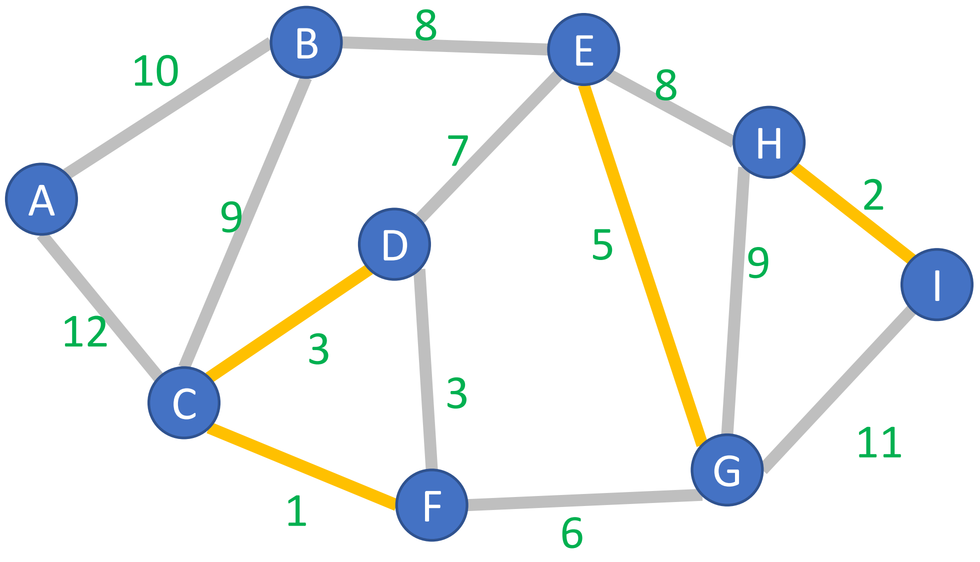

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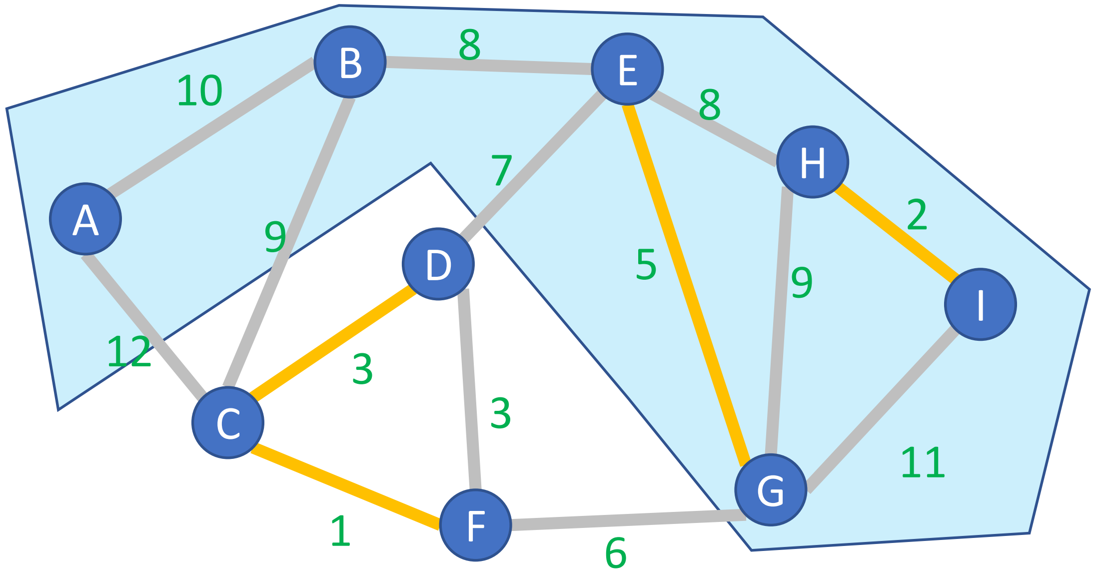

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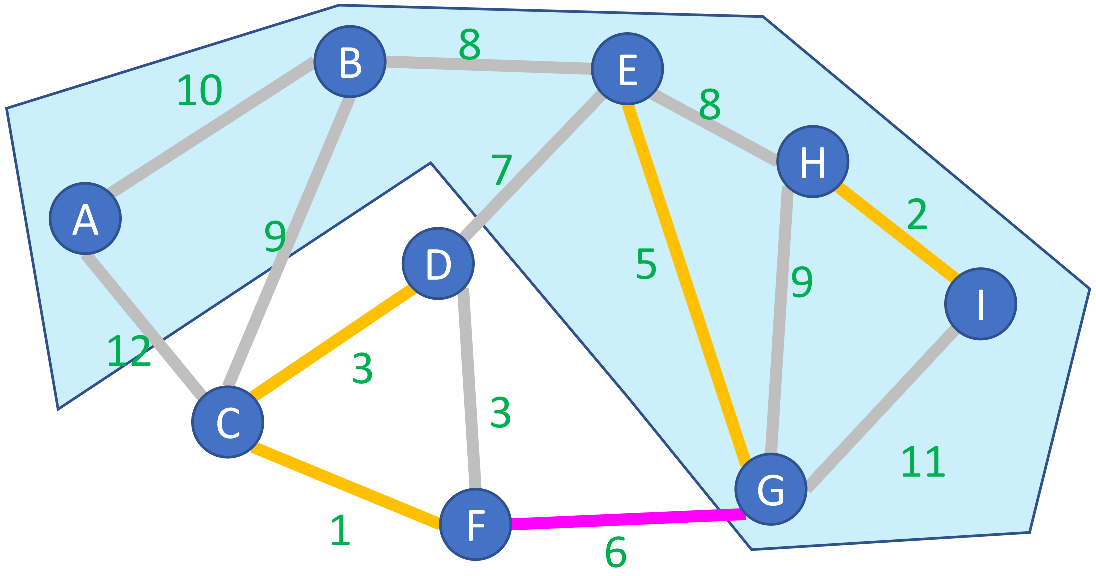

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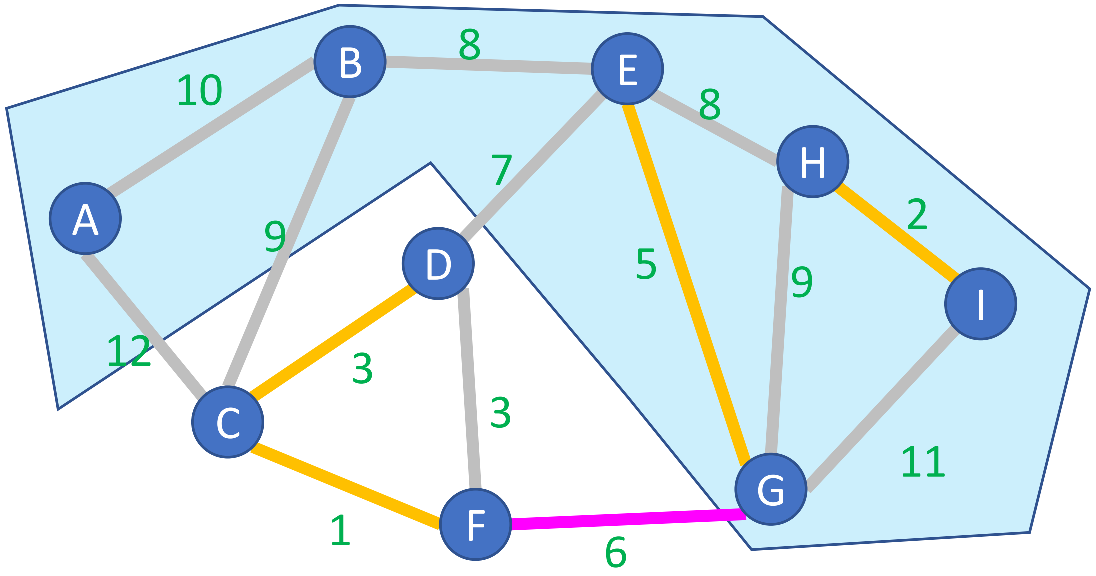
If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.

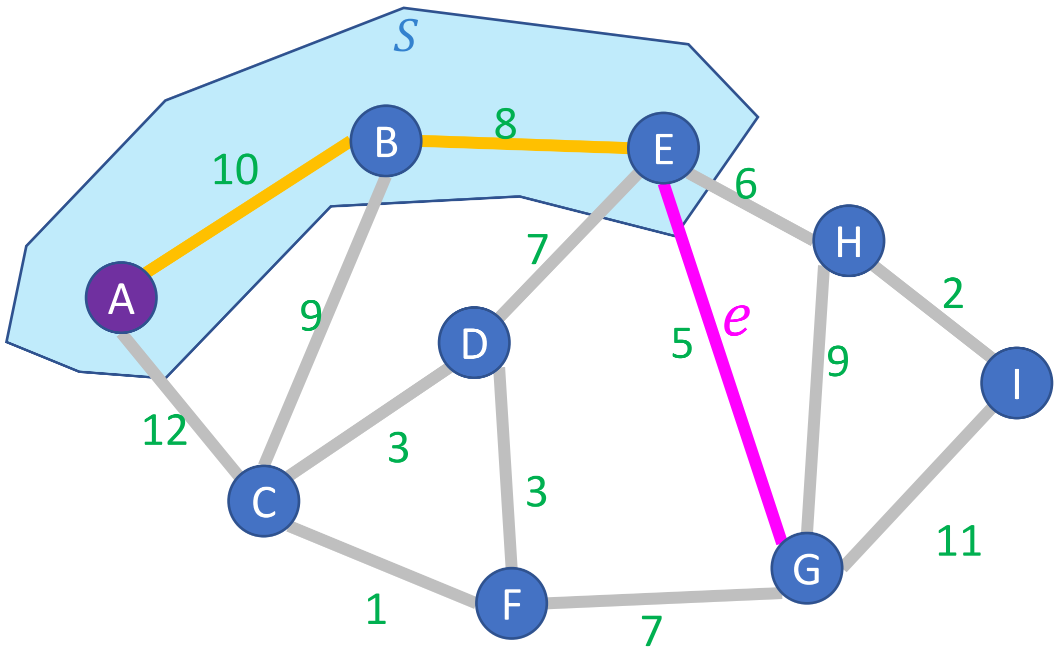


Proof of Prim's Algorithm

Start with an empty tree A

Repeat $V - 1$ times:

Add the min-weight edge that connects to a node not currently in the tree



Proof: By Structural Induction

Suppose we have some arbitrary set of edges A that Prims's has already selected to include in the MST. $e = (E, G)$ is the edge Prims's selects to add next

We know that there cannot exist a path from E to G using only edges in A because G has not been removed from the priority queue

We can cut the graph therefore into 2 disjoint sets:

- Nodes that have been removed from the priority queue
- All other nodes

e is the minimum cost edge that crosses this cut, so by the Cut Theorem, Prim's only selects MST edges!

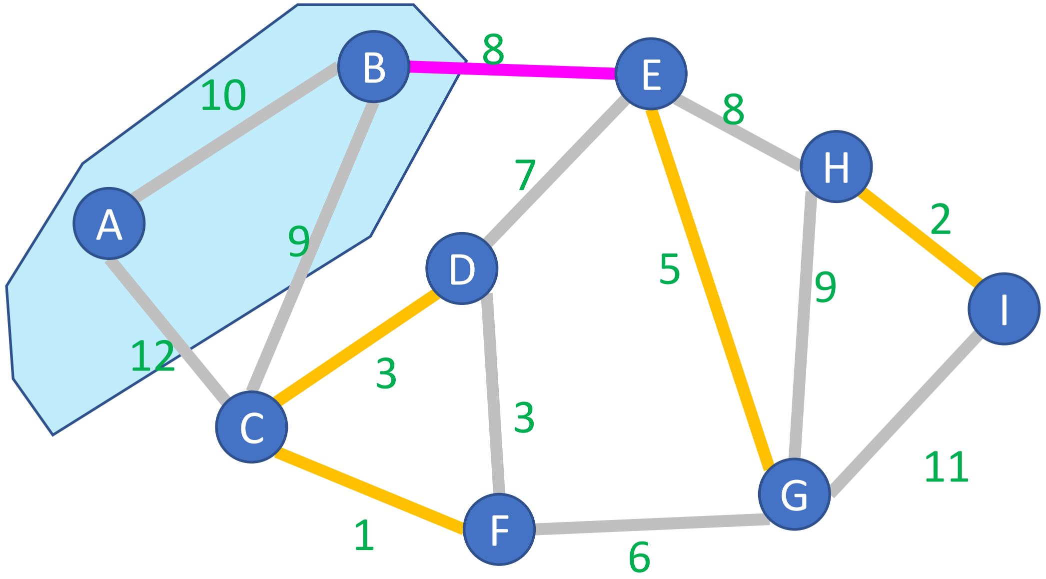
General MST Algorithm

Start with an empty tree A

Repeat $V-1$ times:

Pick a cut $(S, V-S)$ which A respects (typically implicitly)

Add the min-weight edge which crosses $(S, V-S)$



Prim's Algorithm

Start with an empty tree A

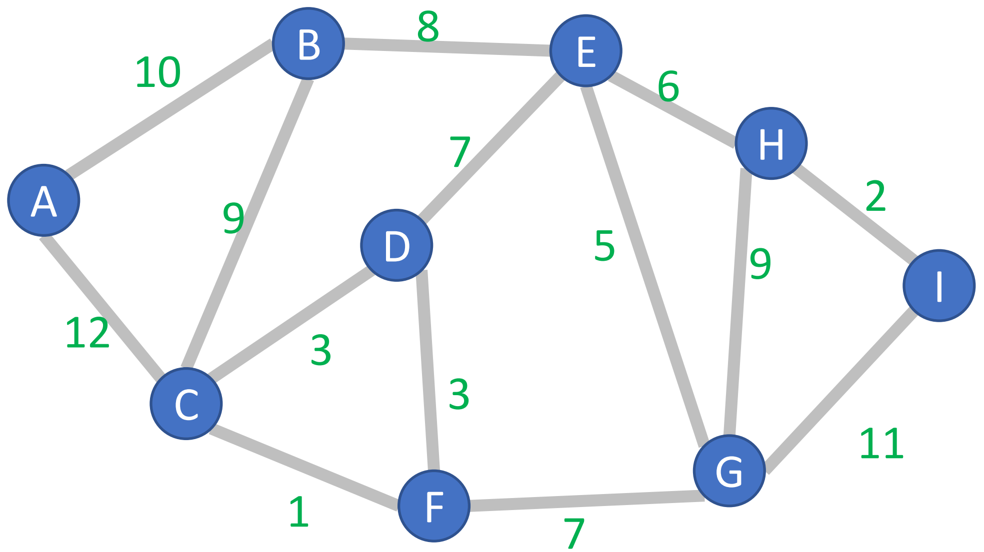
Repeat $V - 1$ times:

Pick a cut $(S, V - S)$ which A respects

Add the min-weight edge which crosses $(S, V - S)$

S is all endpoint of edges in A

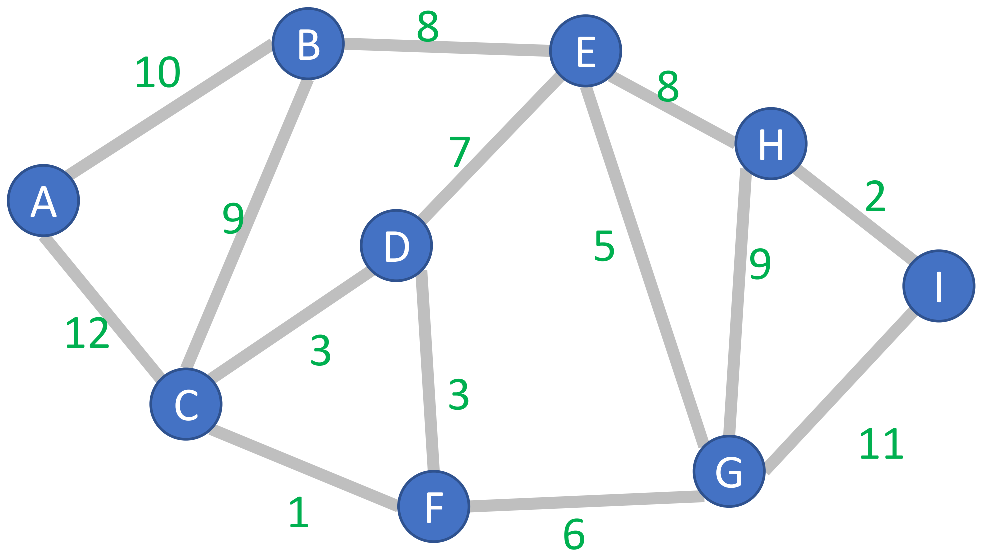
e is the min-weight edge that grows the tree



Kruskal's Algorithm

Start with an empty tree A

Add to A the lowest-weight edge that does not create a cycle



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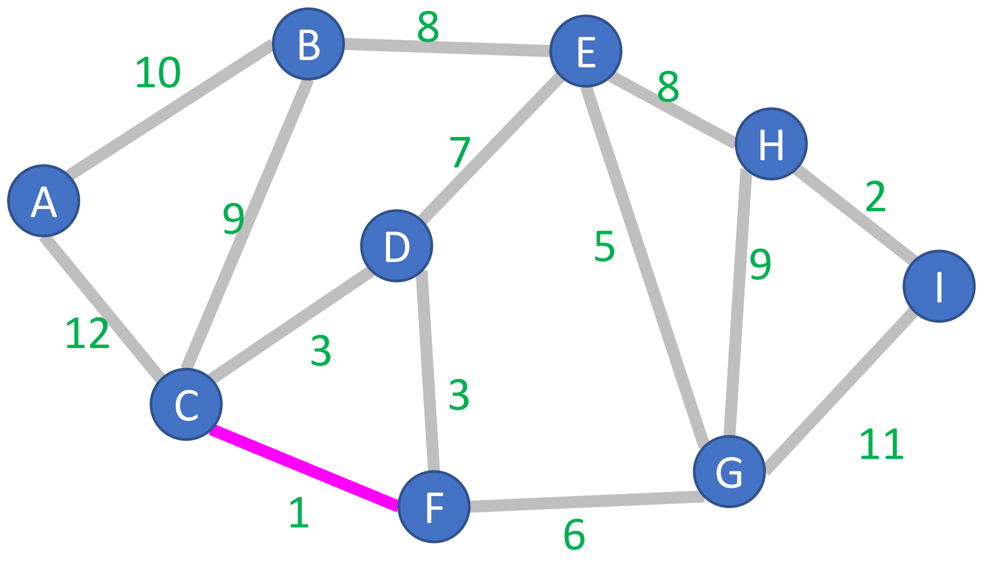

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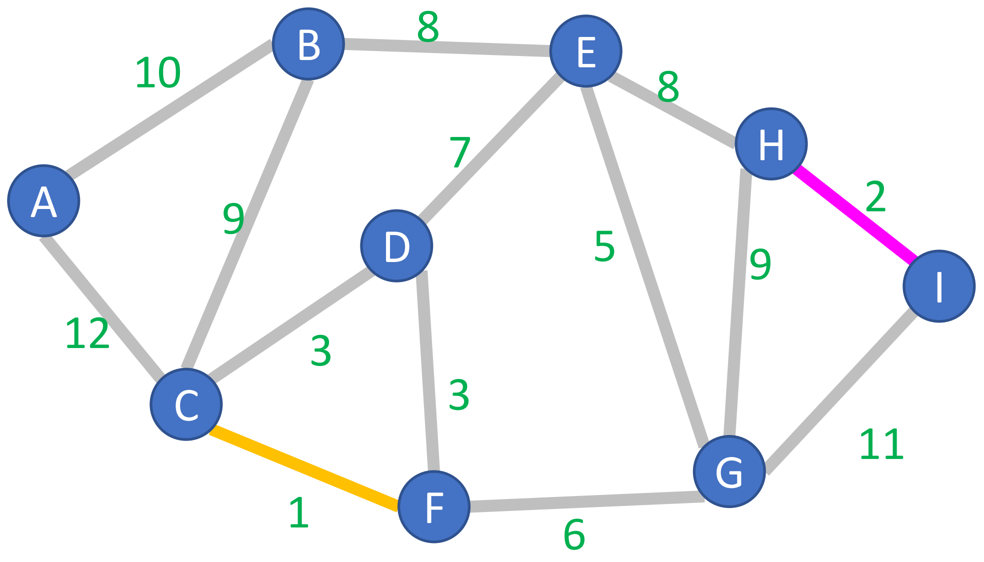

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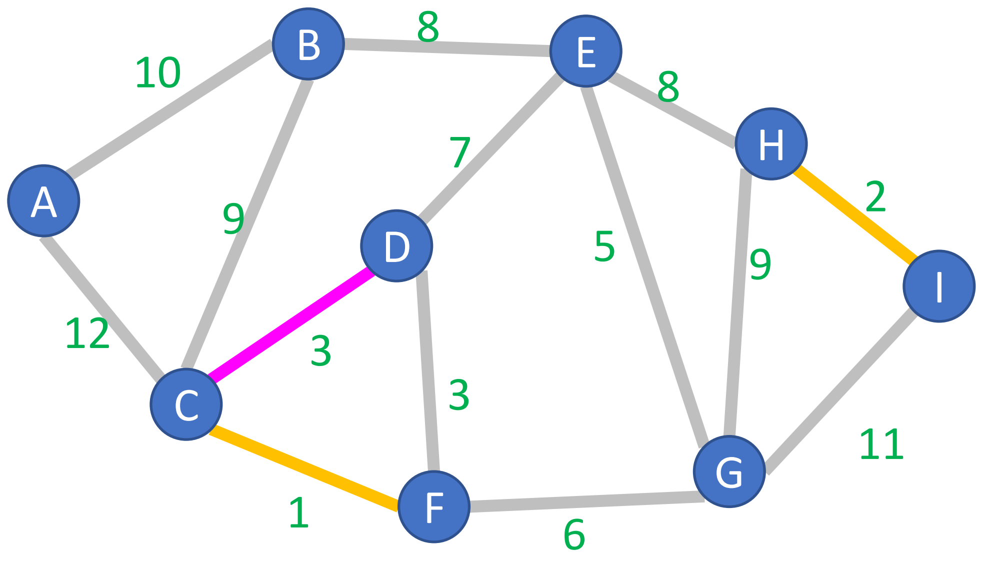

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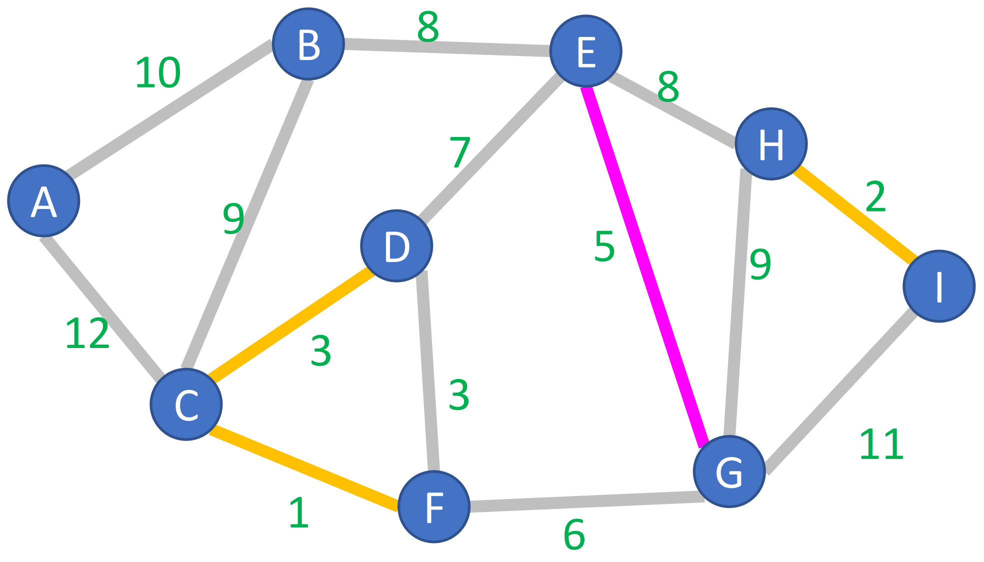

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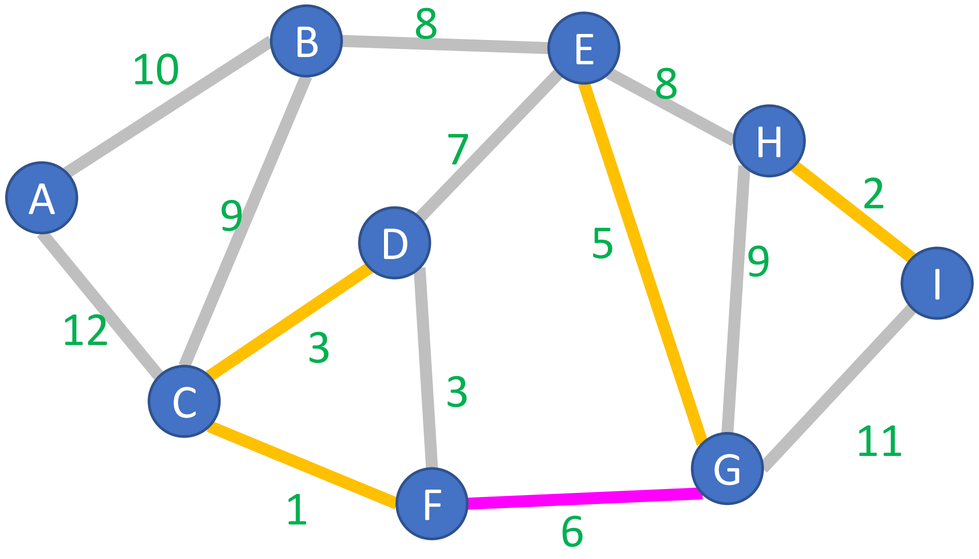
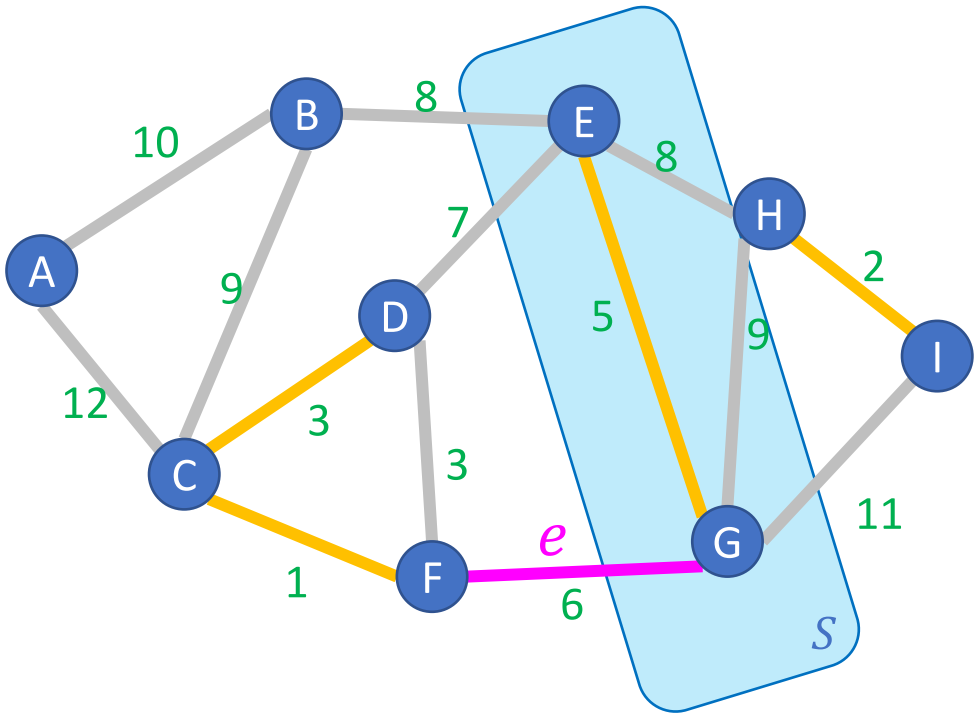


Correctness of Kruskal's Algorithm

- It's sufficient to just show that it follows the template of our "General MST Algorithm"
  - Show that for every edge chosen, it is the least-weight edge which crosses some cut that respects all already-chosen edges.

Proof of Kruskal's Algorithm

Start with an empty tree A
Repeat $V - 1$ times:
    Add the min-weight edge that doesn't cause a cycle



Proof: Suppose we have some arbitrary set of edges A that Kruskal's has already selected to include in the MST. $e = (F, G)$ is the edge Kruskal's selects to add next

We know that there cannot exist a path from F to G using only edges in A because e does not cause a cycle

We can cut the graph therefore into 2 disjoint sets:

- nodes reachable from G using edges in A
- All other nodes

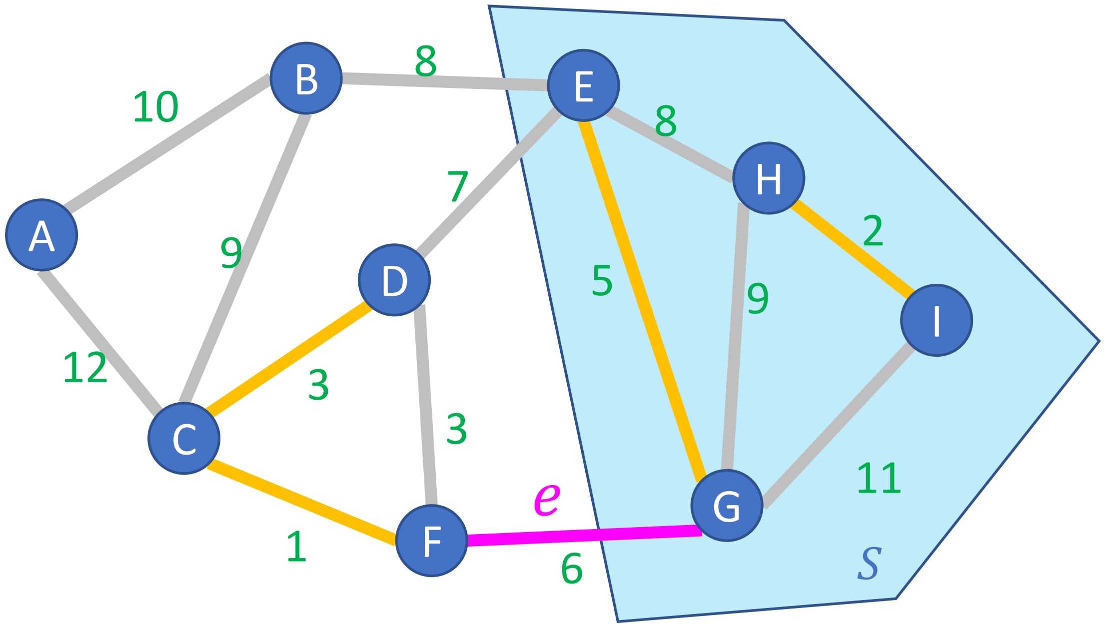
e is the minimum cost edge that crosses this cut, so by the Cut Theorem, Kruskal's is optimal!

Kruskal’s Algorithm Runtime

Start with an empty tree A

Repeat $V - 1$ times:

Add the min-weight edge that doesn’t cause a cycle



Keep edges in a Disjoint-set data structure (very fancy)

$O(E \log V)$