CSE 332 Autumn 2024
Lecture 18: Graphs 2

Nathan Brunelle

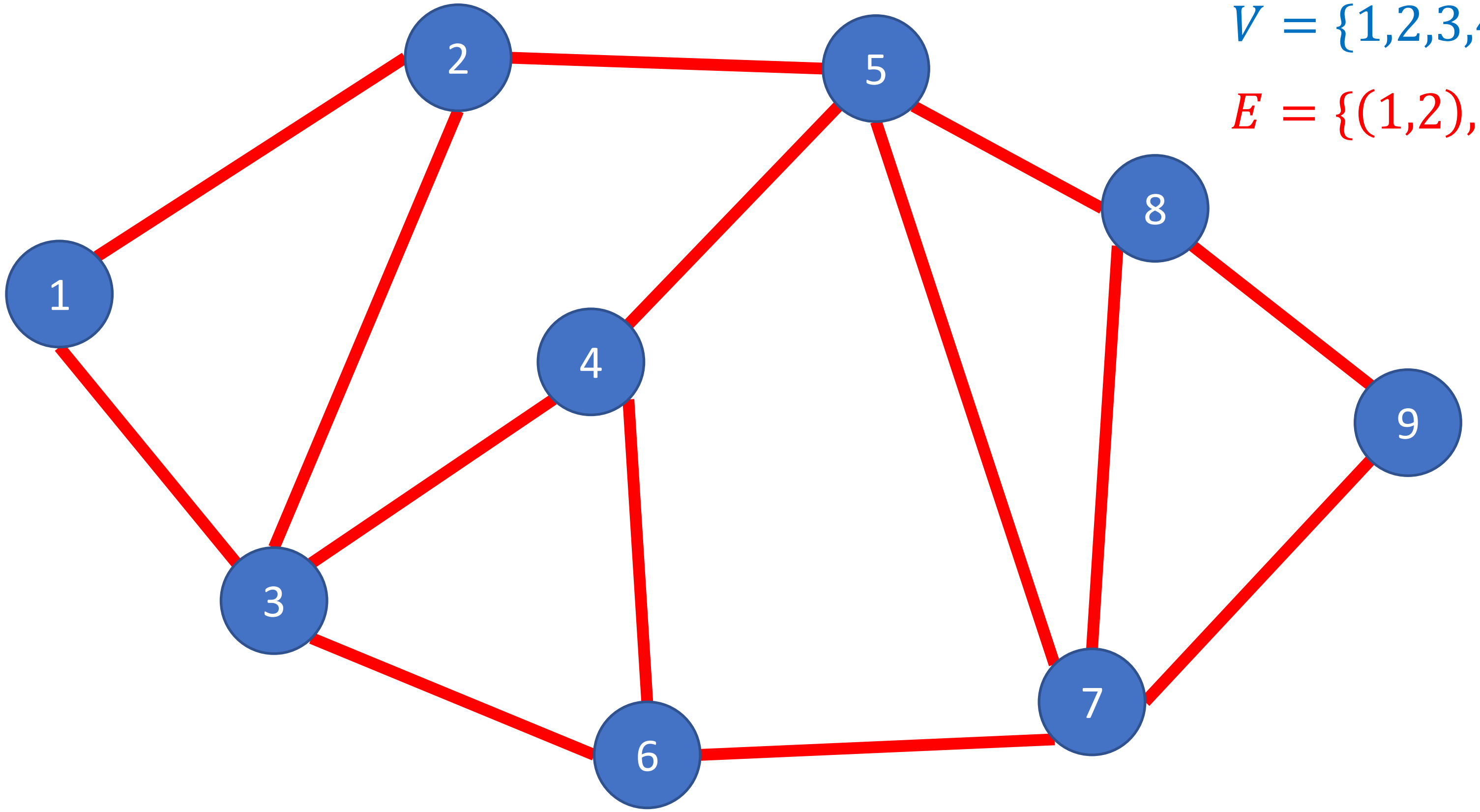
http://www.cs.uw.edu/332

Undirected Graphs

Vertices/Nodes

Definition: $G = (V, E)$

Edges



$V = \{1,2,3,4,5,6,7,8,9\}$

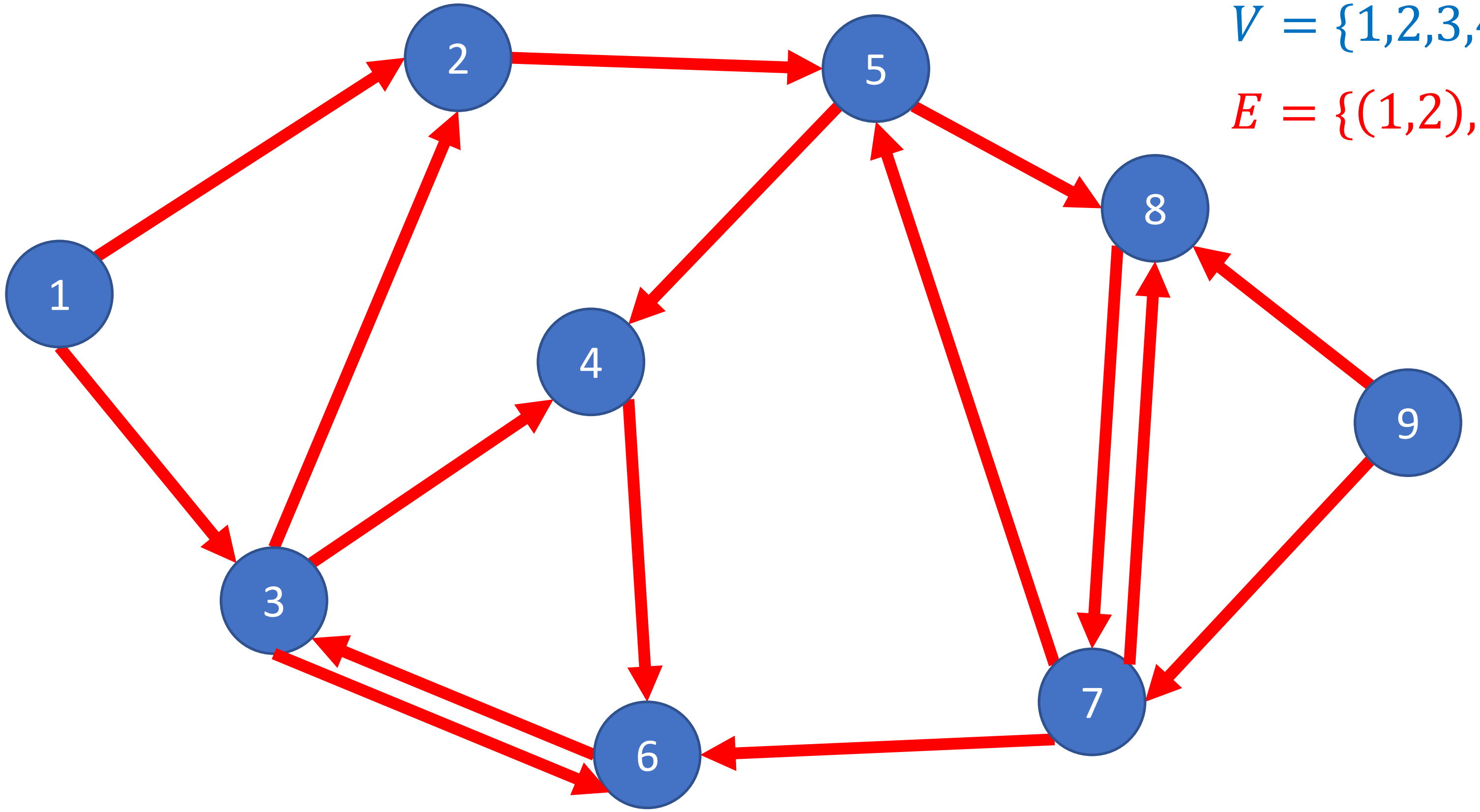
$E = \{(1,2), (2,3), (1,3), \dots\}$

Directed Graphs

Vertices/Nodes

Definition: $G = (V, E)$

Edges



$V = \{1,2,3,4,5,6,7,8,9\}$

$E = \{(1,2), (2,3), (1,3), \dots\}$

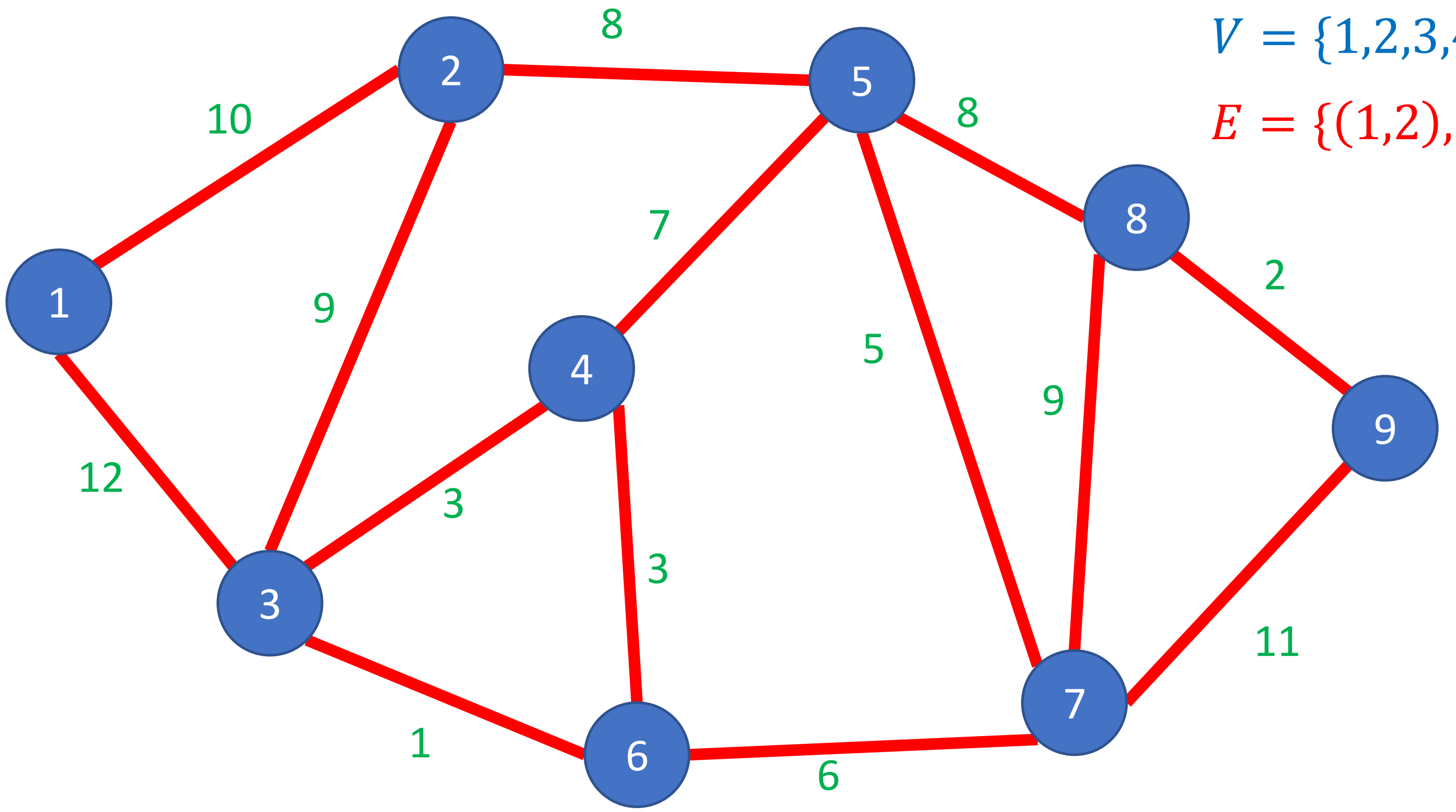
Weighted Graphs

Vertices/Nodes

Definition: $G = (V, E)$

Edges

$w(e)$ = weight of edge e

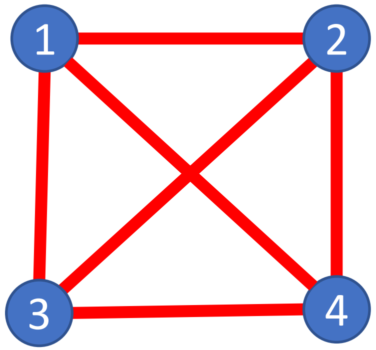


$V = \{1,2,3,4,5,6,7,8,9\}$

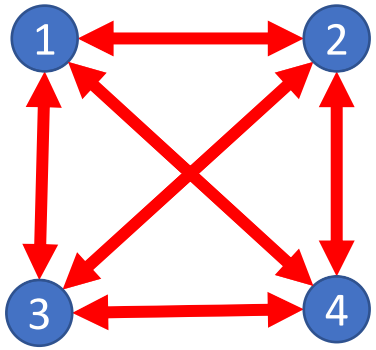
$E = \{(1,2), (2,3), (1,3), \dots\}$

Definition: Complete Graph

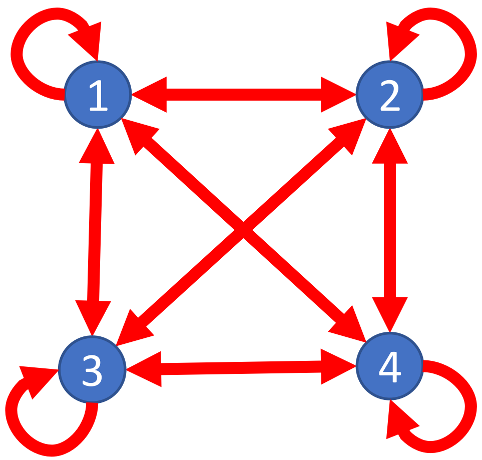
A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is an edge from v_1 to v_2



Complete
Undirected Graph



Complete
Directed Graph



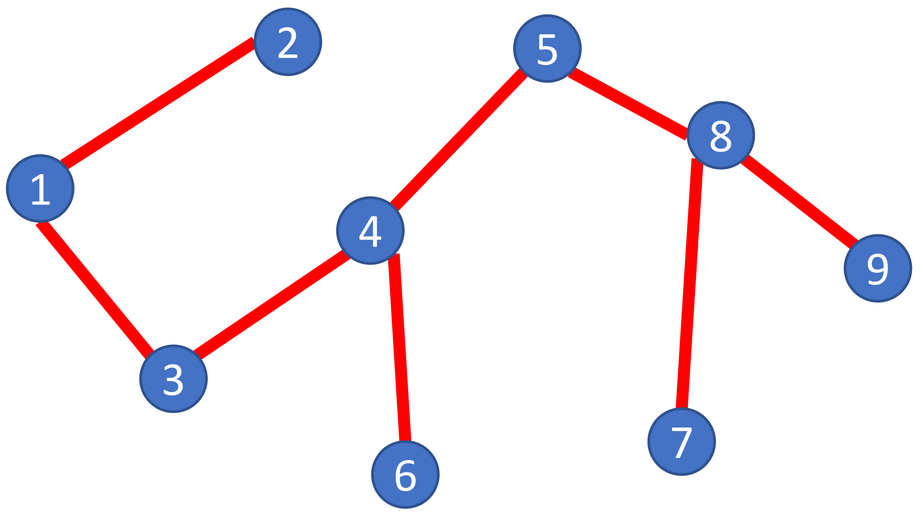
Complete Directed
Non-simple Graph

Graph Density, Data Structures, Efficiency

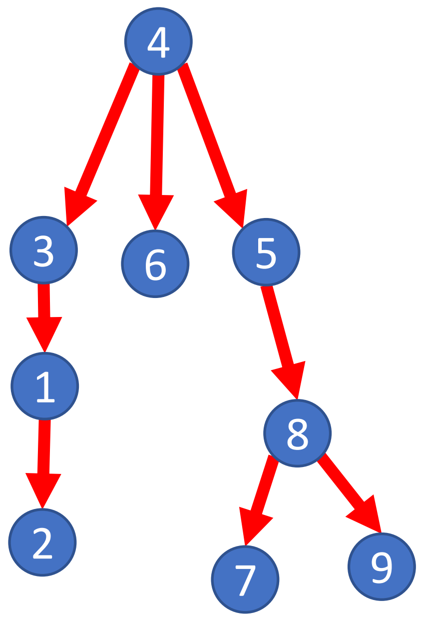
- The maximum number of edges in a graph is $\Theta(|V|^2)$:
  - Undirected and simple: $\frac{|V|(|V|-1)}{2}$
  - Directed and simple: $|V|(|V|-1)$
  - Direct and non-simple (but no duplicates): $|V|^2$
- If the graph is connected, the minimum number of edges is $|V|-1$
- If $|E| \in \Theta(|V|^2)$ we say the graph is **dense**
- If $|E| \in \Theta(|V|)$ we say the graph is **sparse**
- Because $|E|$ is not always near to $|V|^2$ we do not typically substitute $|V|^2$ for $|E|$ in running times, but leave it as a separate variable
  - However, $\log(|E|) \in \Theta(\log(|V|))$

Definition: Tree

A Graph $G = (V, E)$ is a tree if it is undirect, connected, and has no cycles (i.e. is acyclic). Often one node is identified as the "root"



A Tree

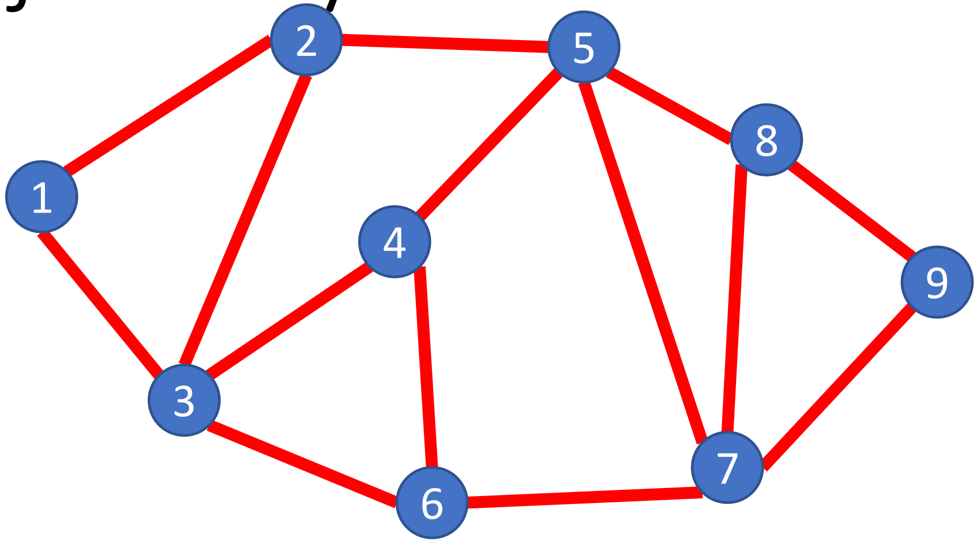


A Rooted Tree

Graph Operations

- To represent a Graph (i.e. build a data structure) we need:
  - Add Edge
  - Remove Edge
  - Check if Edge Exists
  - Get Neighbors (incoming)
  - Get Neighbors (outgoing)

Adjacency List



1	2	3		
2	1	3	5	
3	1	2	4	6
4	3	5	6	
5	2	4	7	8
6	3	4	7	
7	5	6	8	9
8	5	7	9	
9	7	8		

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n+m)$

Add Edge (v,w): $\Theta(\deg(v))$

Remove Edge (v,w): $\Theta(\deg(v))$

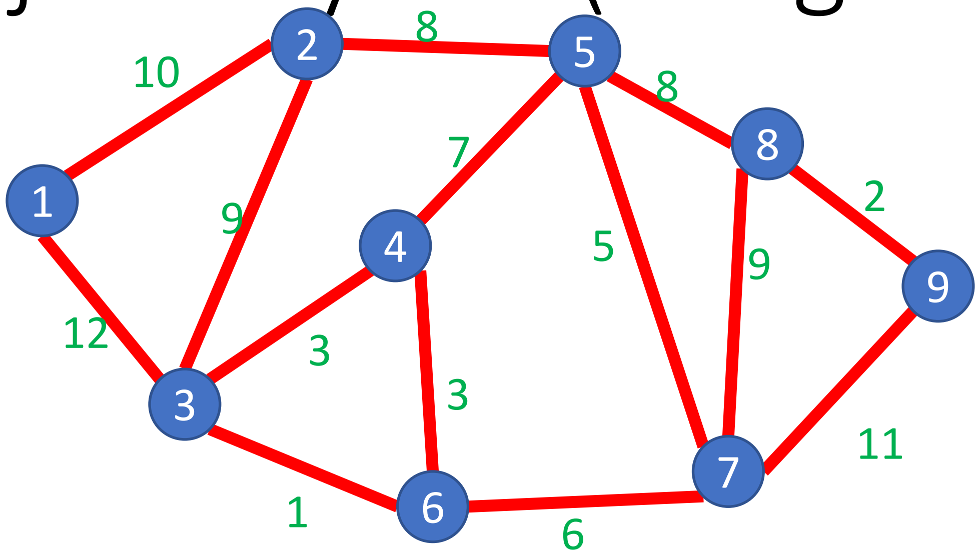
Check if Edge (v,w) Exists: $\Theta(\deg(v))$

Get Neighbors (incoming): $\Theta(n+m)$

Get Neighbors (outgoing): $\Theta(\deg(v))$

$$\boxed{\begin{aligned}|V| &= n\\ |E| &= m\end{aligned}}$$

Adjacency List (Weighted)



1	2 (10)	3 (12)		
2	1 (10)	3 (9)	5 (8)	
3	1 (12)	2 (9)	4 (3)	6 (1)
4	3 (3)	5 (7)	6 (3)	
5	2 (8)	4 (7)	7 (5)	8 (8)
6	3 (1)	4 (3)	7 (6)	
7	5 (5)	6 (6)	8 (9)	9 (11)
8	5 (8)	7 (9)	9 (2)	
9	7 (11)	8 (2)		

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n+m)$

Add Edge (v,w): $\Theta(\deg(v))$

Remove Edge (v,w): $\Theta(\deg(v))$

Check if Edge (v,w) Exists: $\Theta(\deg(v))$

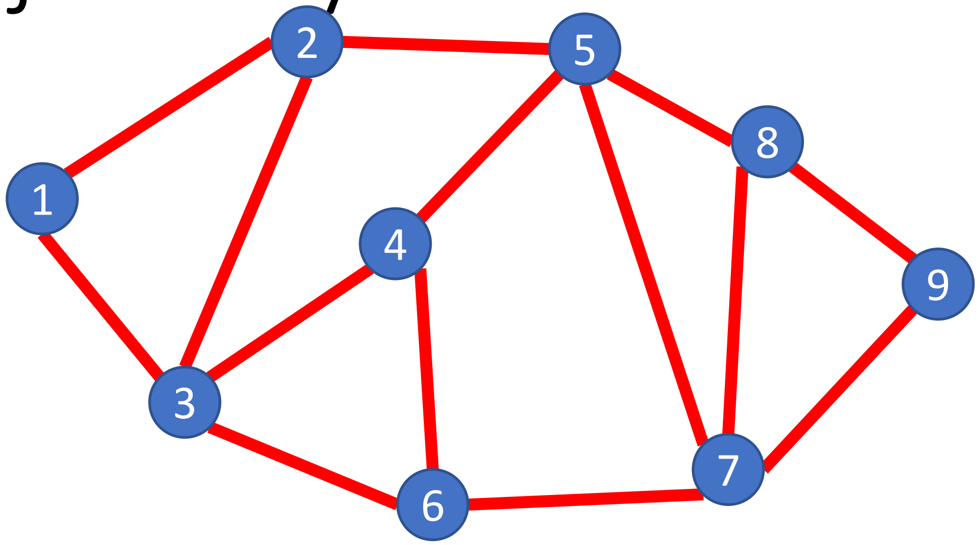
Get Neighbors (incoming): $\Theta(n+m)$

Get Neighbors (outgoing): $\Theta(\deg(v))$

$|V| = n$

$|E| = m$

Adjacency Matrix



	1	2	3	4	5	6	7	8	9
1		1	1						
2	1		1		1				
3	1	1		1		1			
4			1		1	1			
5		1		1			1	1	
6			1	1			1		
7					1	1		1	1
8					1		1		1
9							1	1	

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(?)$

Add Edge (v, w): $\Theta(?)$

Remove Edge (v, w): $\Theta(?)$

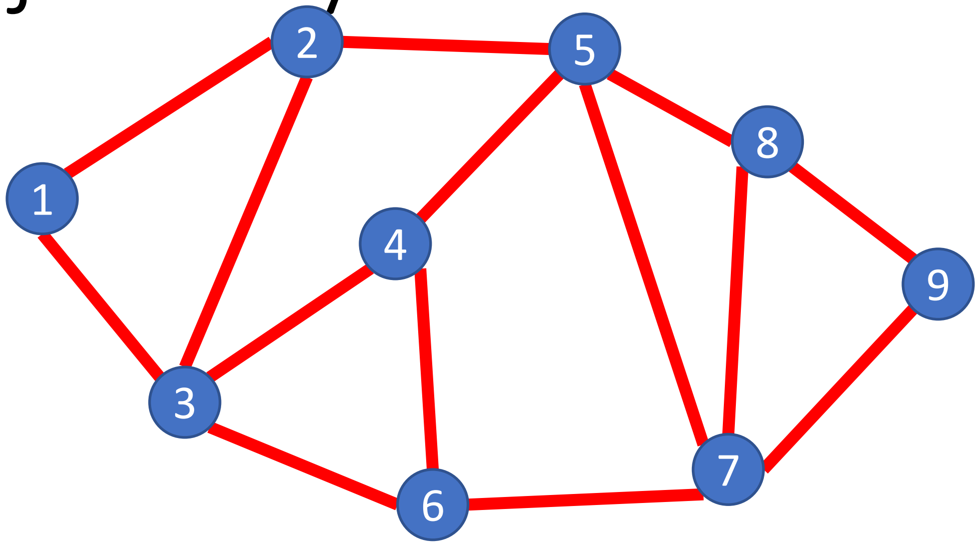
Check if Edge (v, w) Exists: $\Theta(?)$

Get Neighbors (incoming): $\Theta(?)$

Get Neighbors (outgoing): $\Theta(?)$

$|V| = n$
$|E| = m$

Adjacency Matrix



	1	2	3	4	5	6	7	8	9
1		1	1						
2	1		1		1				
3	1	1		1		1			
4			1		1	1			
5		1		1			1	1	
6			1	1			1		
7					1	1		1	1
8					1		1		1
9							1	1	

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n^2)$

Add Edge (v, w): $\Theta(1)$

Remove Edge (v, w): $\Theta(1)$

Check if Edge (v, w) Exists: $\Theta(1)$

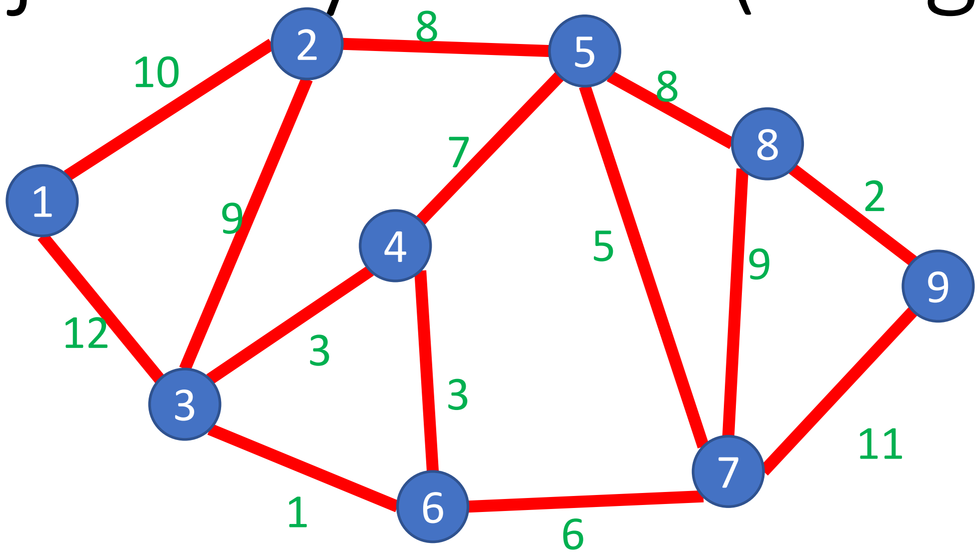
Get Neighbors (incoming): $\Theta(n)$

Get Neighbors (outgoing): $\Theta(n)$

$|V| = n$

$|E| = m$

Adjacency Matrix (weighted)



<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n^2)$

Add Edge (v, w): $\Theta(1)$

Remove Edge (v, w): $\Theta(1)$

Check if Edge (v, w) Exists: $\Theta(1)$

Get Neighbors (incoming): $\Theta(n)$

Get Neighbors (outgoing): $\Theta(n)$

$|V| = n$
$|E| = m$

	1	2	3	4	5	6	7	8	9
1		10	12						
2	10		9		8				
3	12	9		3		1			
4			3		7	3			
5		8		7			5	8	
6			1	3			1		
7					5	1		9	11
8					8		9		2
9							11	2	

Comparison

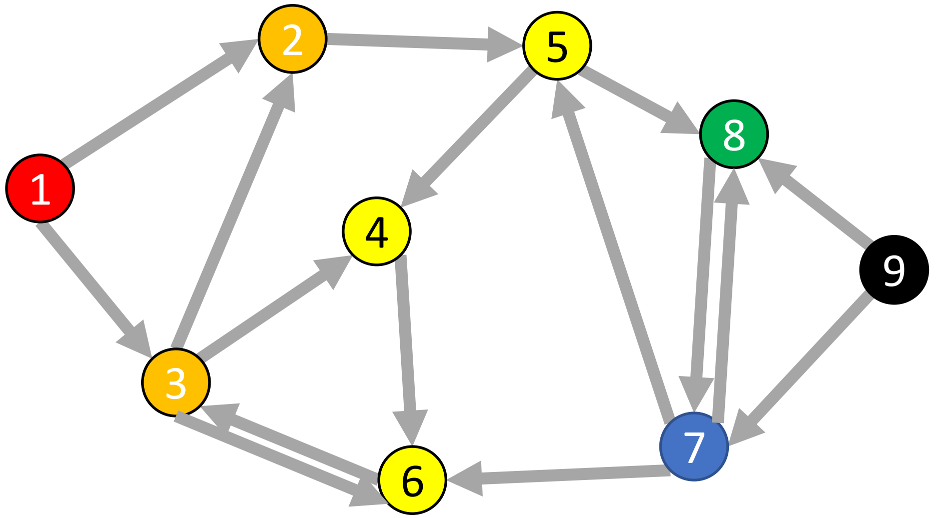
- Adjacency List:
  - Less memory when $|E| < |V|^2$
  - Operations with running time linear in degree of source node
    - Add an edge
    - Remove an edge
    - Check for edge
    - Get neighbors
- Adjacency Matrix:
  - Similar amount of memory when $|E| \approx |V|^2$
  - Constant time operations:
    - Add an edge
    - Remove an edge
    - Check for an edge
  - Operations running with linear time in $|V|$
    - Get neighbors

Adjacency List is more common in practice:

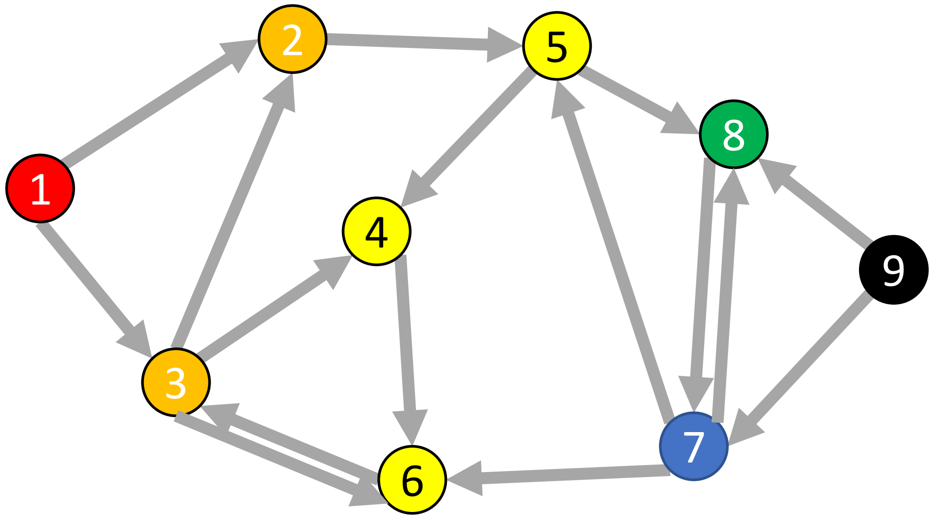
- Most graphs have $|E| \ll |V|^2$
  - Saves memory
  - Most nodes will have small degree
- Getting neighbors is a common operation
- Adjacency Matrix may be better if the graph is “dense” or if its edges change a lot

Breadth-First Search

- Input: a node s
- Behavior: Start with node s, visit all neighbors of s, then all neighbors of neighbors of s, …
- Visits every node reachable from s in order of distance
- Output:
  - How long is the shortest path?
  - Is the graph connected?



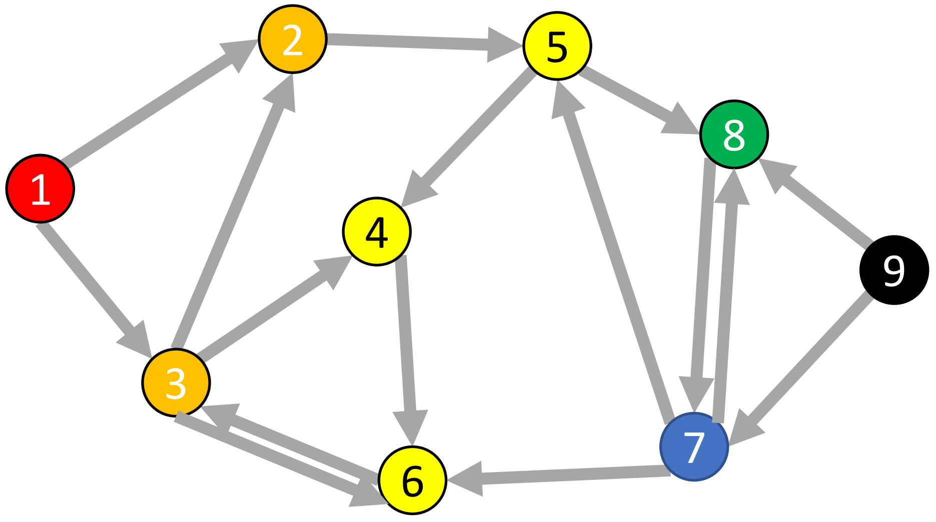
BFS



Running time: $\Theta(|V| + |E|)$

```
void bfs(graph, s){
        found = new Queue();
        found.enqueue(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.dequeue();
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                found.enqueue(v);
                        }
                }
        }
}
```

BFS – Worked Example

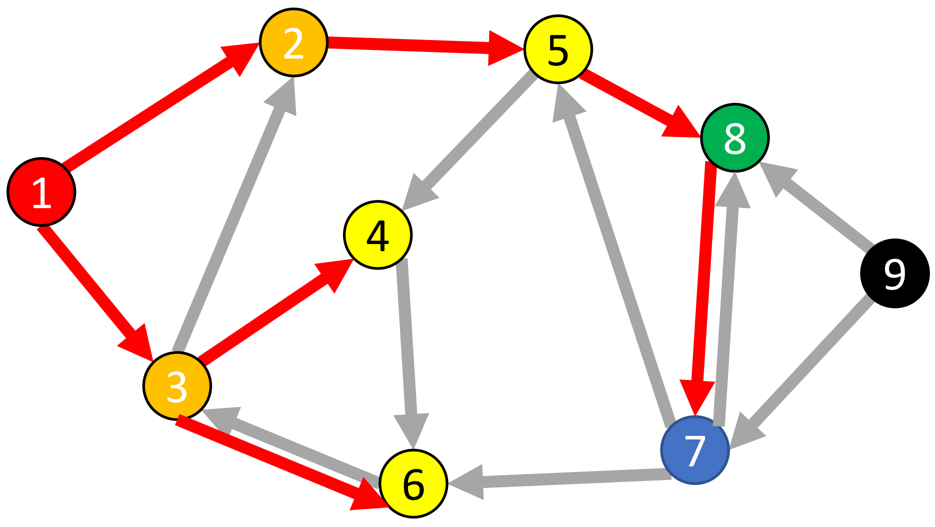


Node	Visited?	Other Info
1	True	
2		
3		
4		
5		
6		
7		
8		
9		

For each node:
- For each unvisited neighbor:
  - add that neighbor to a queue
  - mark that neighbor as visited

Queue:

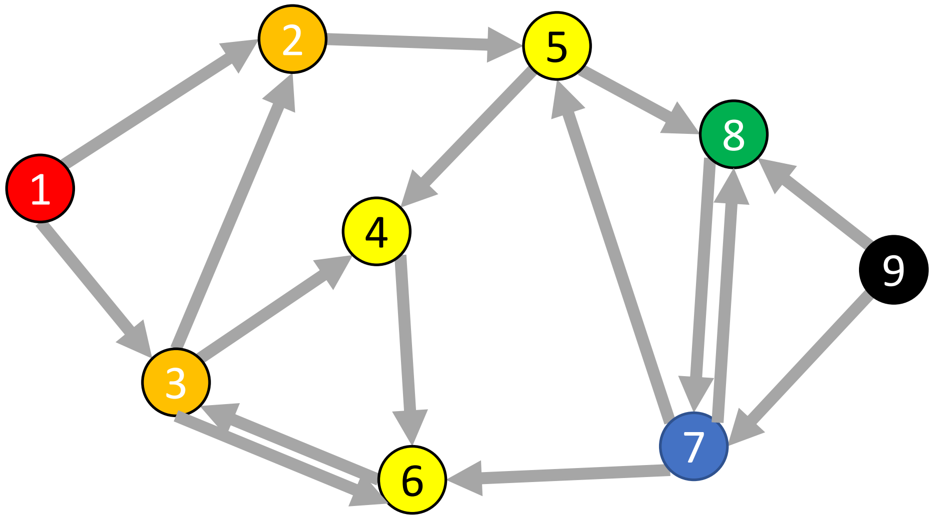
Find Distance (unweighted)



Idea: when it's seen, remember its "layer" depth!

```
int findDistance(graph, s, t){
	found = new Queue();
	layer = 0;
	found.enqueue(s);
	mark s as "visited";
	While (!found.isEmpty()){
		current = found.dequeue();
		layer = depth of current;
		for (v : neighbors(current)){
			if (! v marked "visited"){
				mark v as "visited";
				depth of v = layer + 1;
				found.enqueue(v);
			}
		}
	}
	return depth of t;
}
```

Find Distance – Worked Example



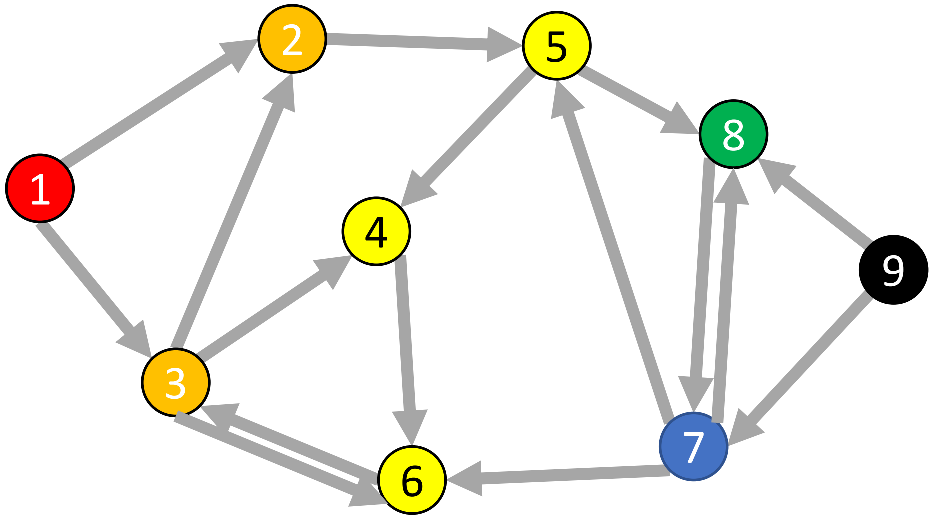
Node	Visited?	Layer
1		
2		
3		
4		
5		
6		
7		
8		
9		

For each node:

- update current layer
- For each unvisited neighbor:
  - add that neighbor to a queue
  - mark that neighbor as visited
  - set neighbor's layer to be current layer + 1

Queue:

Shortest Path - Idea



Node	Visited?	Previous
1		
2		
3		
4		
5		
6		
7		
8		
9		

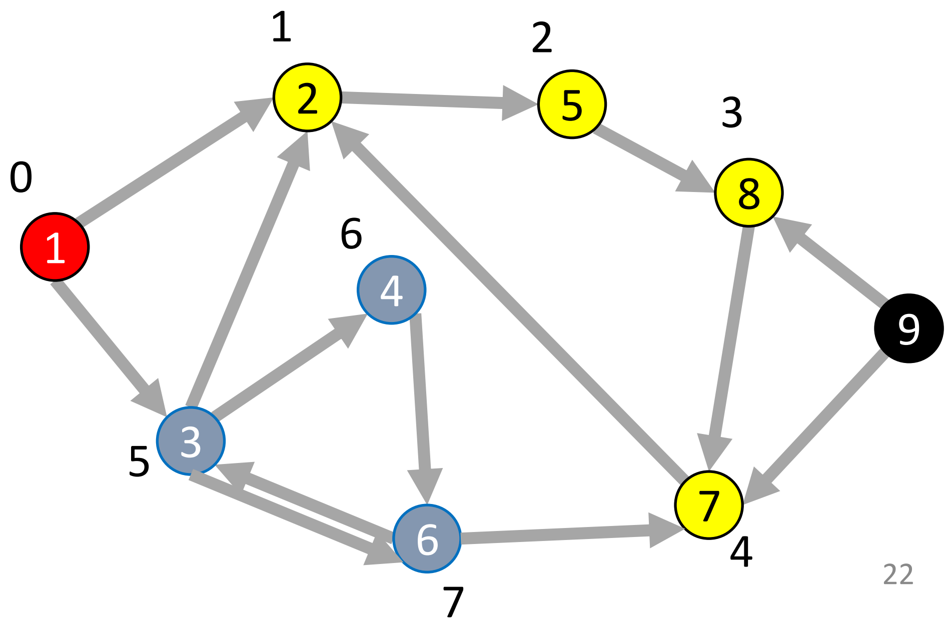
Queue:

For each node:
- For each unvisited neighbor:
  - add that neighbor to a queue
  - mark that neighbor as visited
  - set neighbor's previous to be the current node

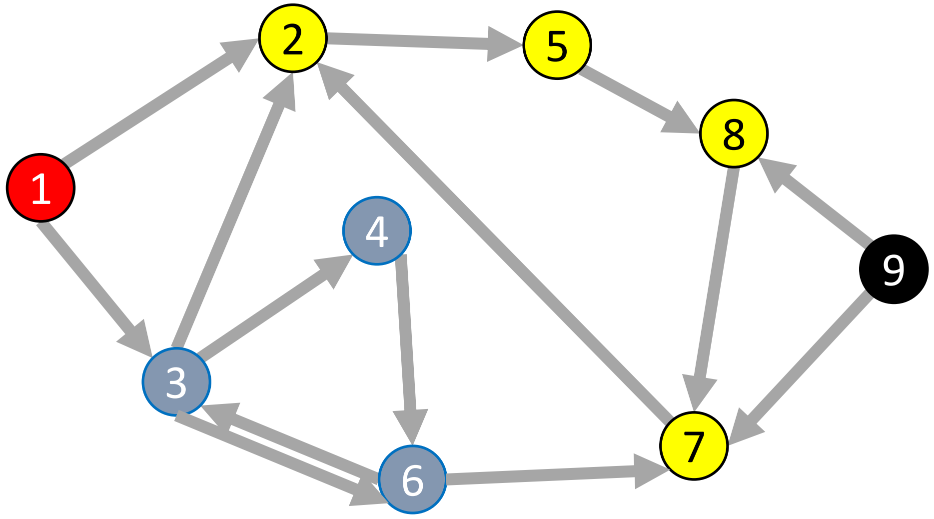
Depth-First Search

Depth-First Search

- Input: a node s
- Behavior: Start with node s, visit one neighbor of s, then all nodes reachable from that neighbor of s, then another neighbor of s,…
  - Before moving on to the second neighbor of s, visit everything reachable from the first neighbor of s
- Output:
  - Does the graph have a cycle?
  - A **topological sort** of the graph.



DFS (non-recursive)

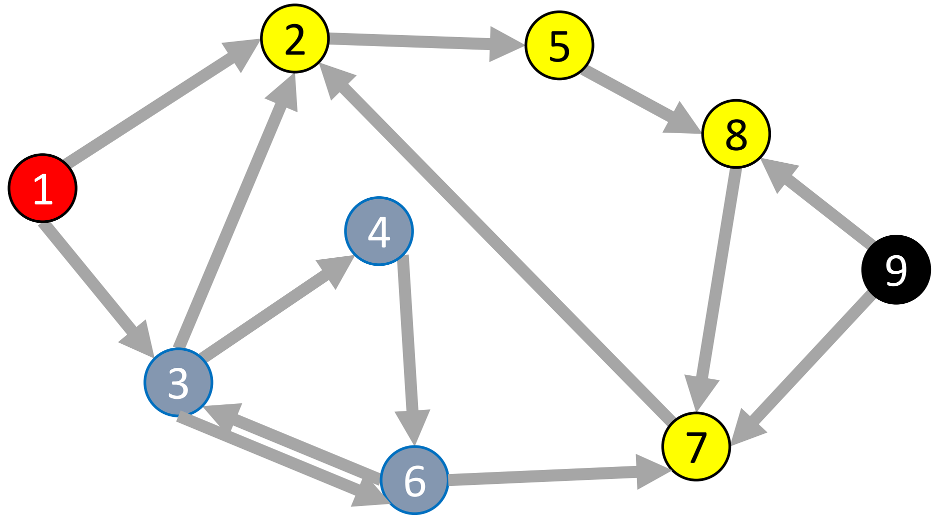


Running time: $\Theta(|V| + |E|)$

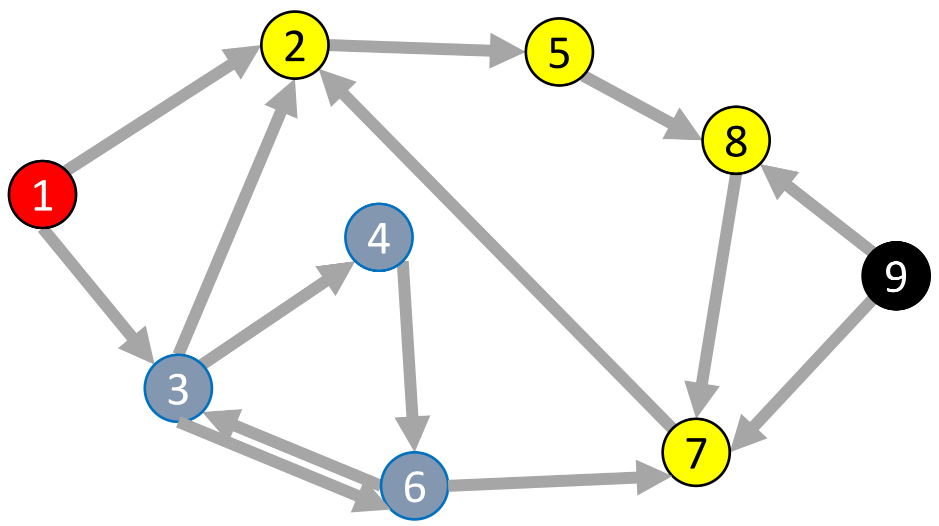
```
void dfs(graph, s){
        found = new Stack();
        found.pop(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.pop();
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                found.push(v);
                        }
                }
        }
}
```

DFS Recursively (more common)

```
void dfs(graph, curr){
        mark curr as “visited”;
        for (v : neighbors(current)){
                if (! v marked “visited”){
                        dfs(graph, v);
                }
        }
        mark curr as “done”;
}
```



DFS – Worked Example



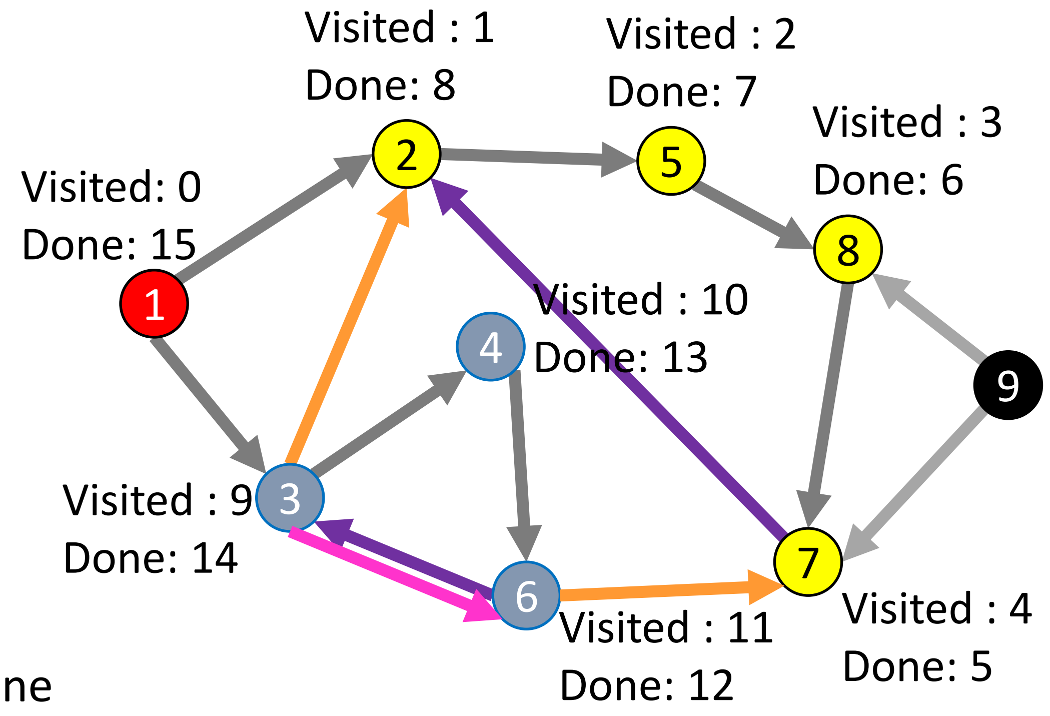
Node	Visited?	Done?	Other Info
1			
2			
3			
4			
5			
6			
7			
8			
9			

Starting from the current node:
- for each unvisited neighbor:
  - mark the neighbor as visited
  - do a DFS from the neighbor
- mark the current node as done

(Call) Stack:

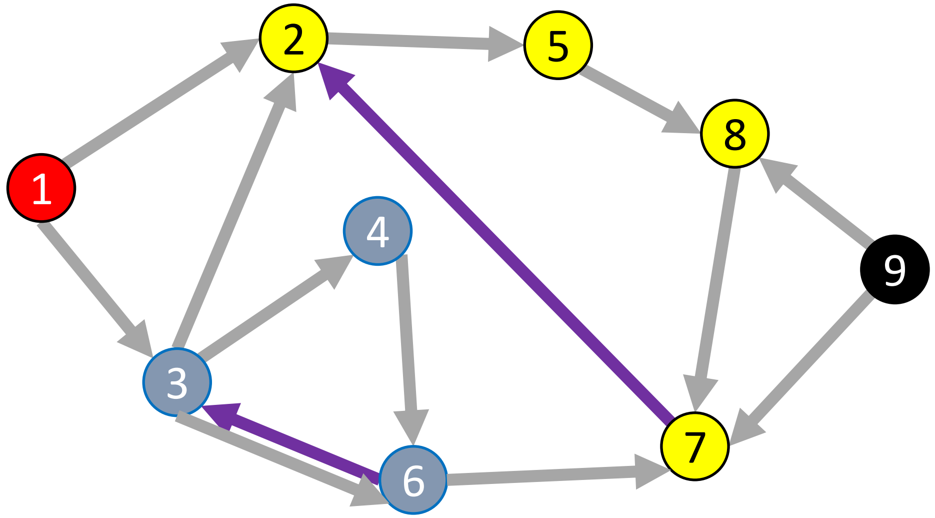
Using DFS

- Consider the “visited times” and “done times”
- Edges can be categorized:
  - Tree Edge
    - (a, b) was followed when pushing
    - (a, b) when b was unvisited when we were at a
  - Back Edge
    - (a, b) goes to an “ancestor”
    - a and b visited but not done when we saw (a, b)
    - $t_{visited}(b) < t_{visited}(a) < t_{done}(a) < t_{done}(b)$
  - Forward Edge
    - (a, b) goes to a “descendent”
    - b was visited and done between when a was visited and done
    - $t_{visited}(a) < t_{visited}(b) < t_{done}(b) < t_{done}(a)$
  - Cross Edge
    - (a, b) goes to a node that doesn’t connect to a
    - b was seen and done before a was ever visited
    - $t_{done}(b) < t_{visited}(a)$



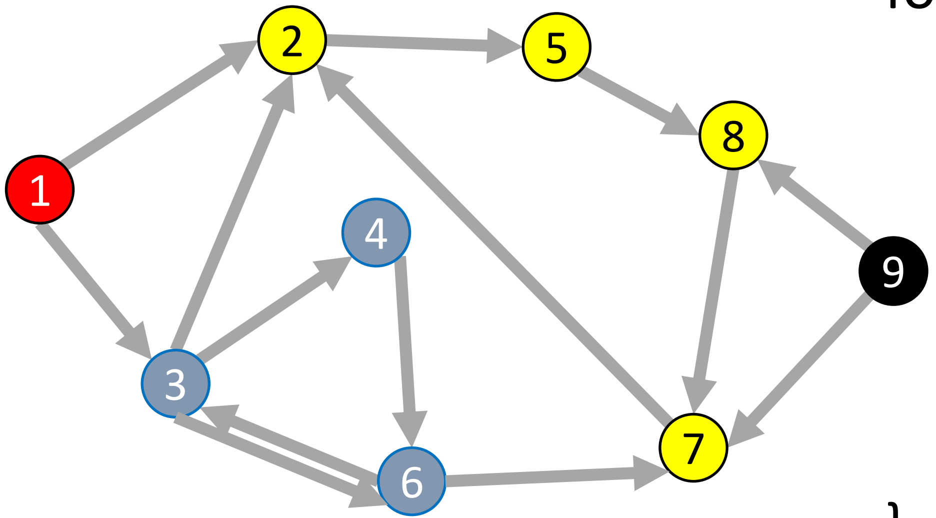
Back Edges

- Behavior of DFS:
  - “Visit everything reachable from the current node before going back”
- Back Edge:
  - The current node’s neighbor is an “in progress” node
  - Since that other node is “in progress”, the current node is reachable from it
  - The back edge is a path to that other node
  - **Cycle!**



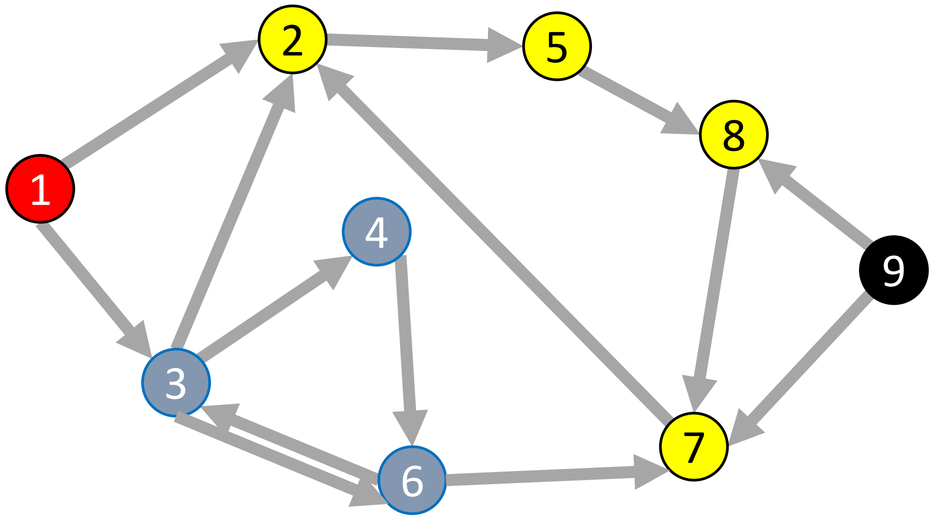
Cycle Detection

Idea: Look for a back edge!




```
boolean hasCycle(graph, curr){
        mark curr as "visited";
        cycleFound = false;
        for (v : neighbors(current)){
                if (v marked "visited" && ! v marked "done"){
                        cycleFound=true;
                }
                if (! v marked "visited" && !cycleFound){
                        cycleFound = hasCycle(graph, v);
                }
        }
        mark curr as "done";
        return cycleFound;
}
```

Cycle Detection – Worked Example



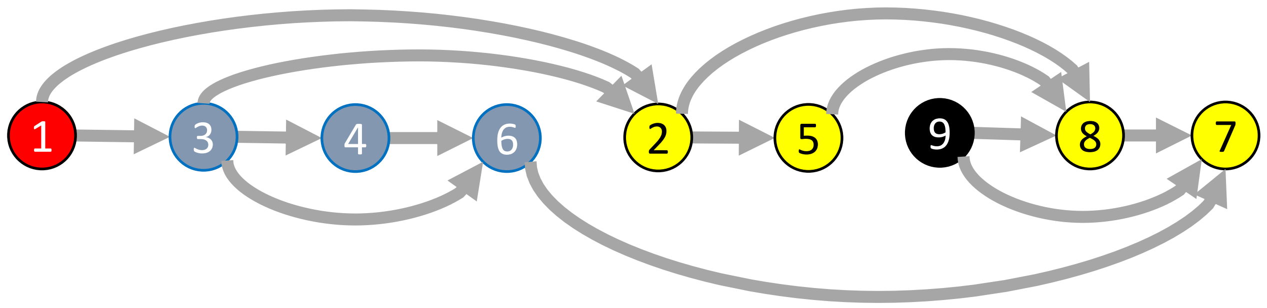
Starting from the current node:
  for each non-done neighbor:
    if the neighbor is visited:
      we found a cycle!
    else:
      mark the neighbor as visited
      do a DFS from the neighbor
  mark the current node as done

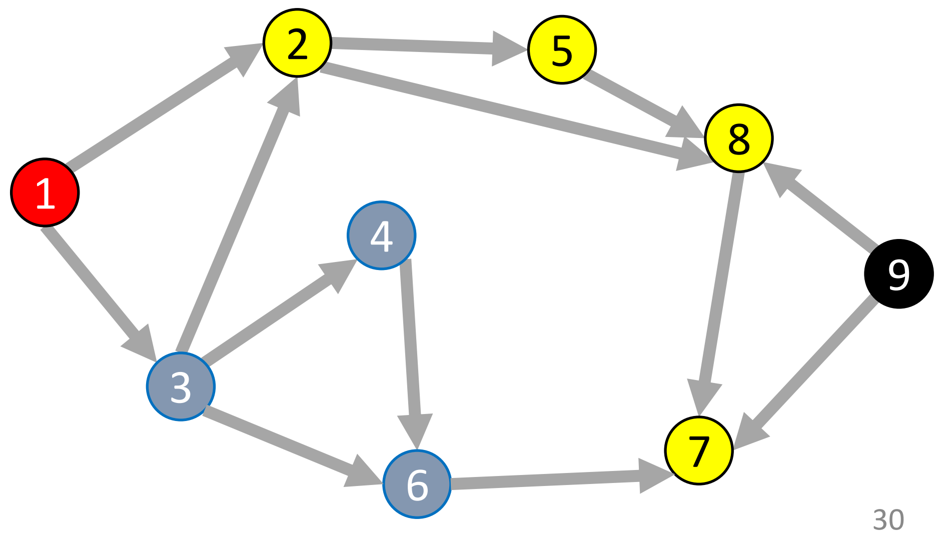
Node	Visited?	Done?	Other Info
1			
2			
3			
4			
5			
6			
7			
8			
9			

(Call) Stack:

Topological Sort

- A Topological Sort of a **directed acyclic graph** $\boldsymbol{G = (V, E)}$ is a permutation of V such that if $(u, v) \in E$ then u is before v in the permutation

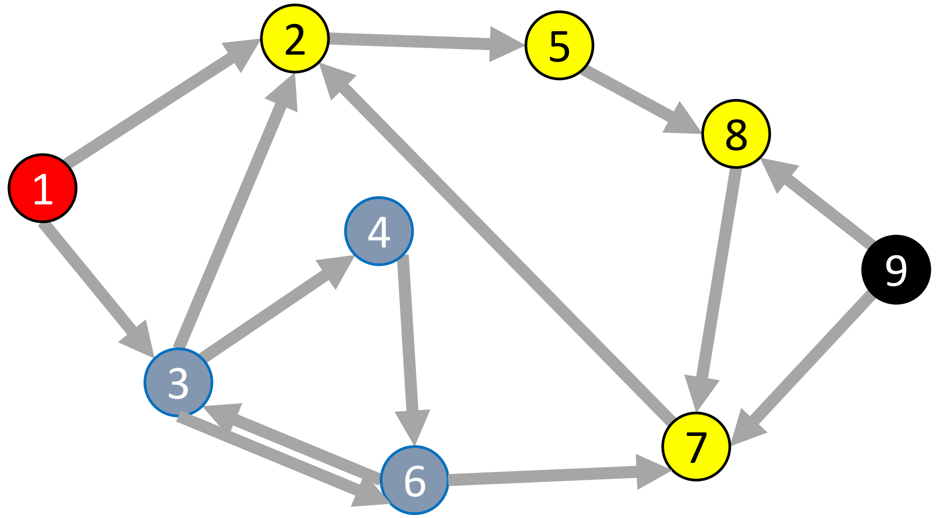




DFS Recursively

```
void dfs(graph, curr){
        mark curr as “visited”;
        for (v : neighbors(current)){
                if (! v marked “visited”){
                        dfs(graph, v);
                }
        }
        mark curr as “done”;
}
```

Idea: List in reverse order by “done” time

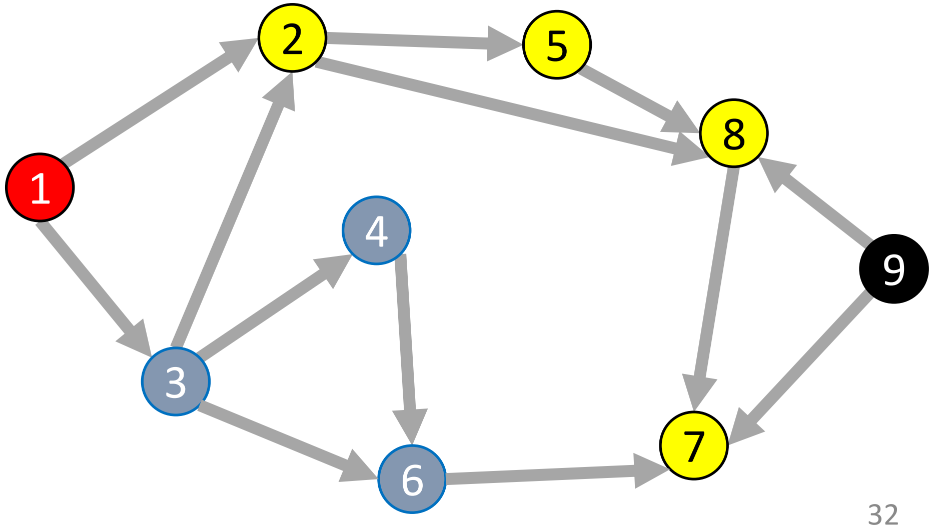


DFS: Topological sort

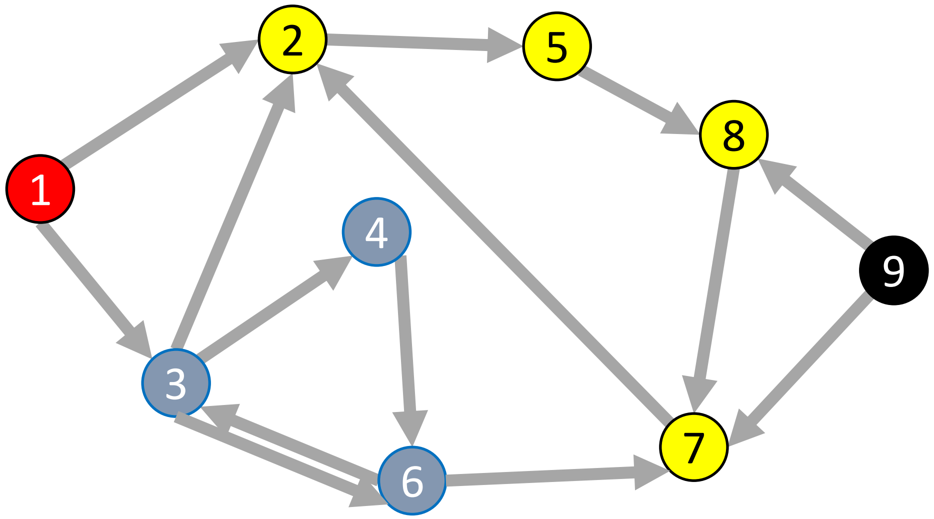
```
List topSort(graph){
    List<Nodes> done = new List<>();
    for (Node v : graph.vertices){
        if (!v.visited){
            finishTime(graph, v, finished);
        }
    }
    done.reverse();
    return done;
}

void finishTime(graph, curr, finished){
    curr.visited = true;
    for (Node v : curr.neighbors){
        if (!v.visited){
            finishTime(graph, v, finished);
        }
    }
    done.add(curr)
}
```

Idea: List in reverse order by "done" time

finished:



Topological Sort– Worked Example



Starting from the current node:
  for each non-done neighbor:
    if the neighbor is visited:
      we found a cycle!
    else:
      mark the neighbor as visited
      do a DFS from the neighbor
  mark the current node as done
  add current node to finished

Node	Visited?	Done?	Other Info
1			
2			
3			
4			
5			
6			
7			
8			
9			

(Call) Stack:

finished: