CSE 332: Data Structures & Parallelism

Lecture 21: Shortest Paths

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Today

- Graphs
  - Shortest Paths

Shortest Path Applications

- Network routing
- Driving directions
- Cheap flight tickets
- Critical paths in project management (see textbook)
- …

Single source shortest paths

- Done: BFS to find the minimum path length from **v** to **u** in $O(|E|+|V|)$

- Actually, can find the minimum path length from **v** to *every node*
  - Still $O(|E|+(|V|)$
  - No faster way for a "distinguished" destination in the worst-case

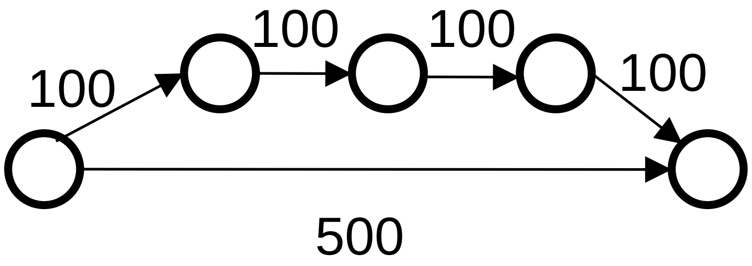
- Now: Weighted graphs

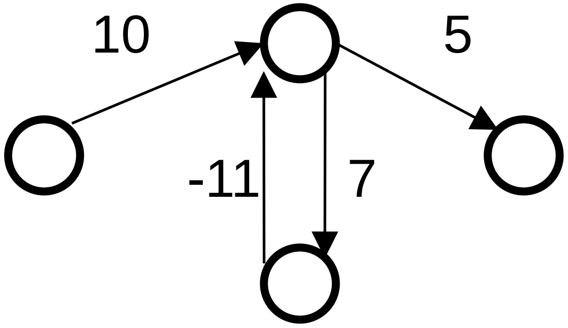
Given a weighted graph and node **v**,

find the minimum-cost path from **v** to every node

- As before, asymptotically no harder than for one destination
- Unlike before, BFS will not work

Not as easy





Why BFS won't work: Shortest path may not have the fewest edges

– Annoying when this happens with costs of flights

We will assume there are no negative weights

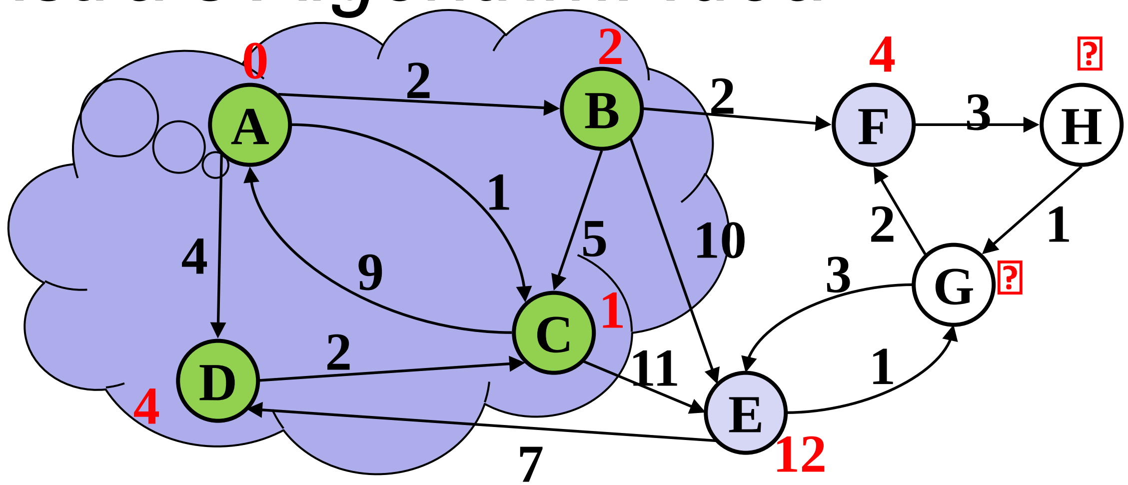
- *Problem* is *ill-defined* if there are negative-cost *cycles*
- *Today's algorithm* is *wrong* if *edges* can be negative

Dijkstra's Algorithm

- Named after its inventor Edsger Dijkstra (1930-2002)
  - Truly one of the "founders" of computer science;
    1972 Turing Award; this is just one of his many contributions
  - Sample quotation: "computer science is no more about computers than astronomy is about telescopes"

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"
  - A priority queue will turn out to be useful for efficiency

Dijkstra's Algorithm: Idea



- Initially, start node has cost 0 and all other nodes have cost ∞
- At each step:
  - Pick closest unknown vertex **v**
  - Add it to the "cloud" of known vertices
  - Update distances for nodes with edges from **v**
- That's it! (Have to prove it produces correct answers)

The Algorithm

1. For each node **`v`**, set **`v.cost =`** ∞ and **`v.known = false`**
2. Set **`source.cost = 0`**
3. While there are unknown nodes in the graph
   a) Select the unknown node **`v`** with lowest cost
   b) Mark **`v`** as known
   c) For each edge **`(v,u)`** with weight **`w`**, if **`u`** is unknown,

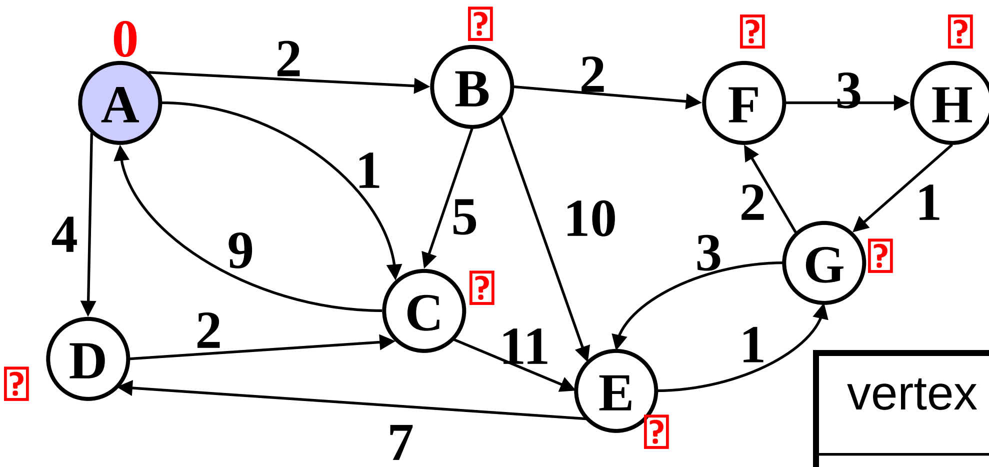
```
c1 = v.cost + w // cost of best path through v to u
c2 = u.cost   // cost of best path to u previously known
if(c1 < c2){ // if the path through v is better
  u.cost = c1
  u.path = v // for computing actual paths
}
```

Important features

- Once a vertex is marked known, the cost of the shortest path to that node is known
  - The path is also known by following back-pointers

- While a vertex is still not known, another shorter path to it might still be found

Example #1



Order Added to Known Set:

vertex	known?	cost	path
A			
B			
C			
D			
E			
F			
G			
H			

Features

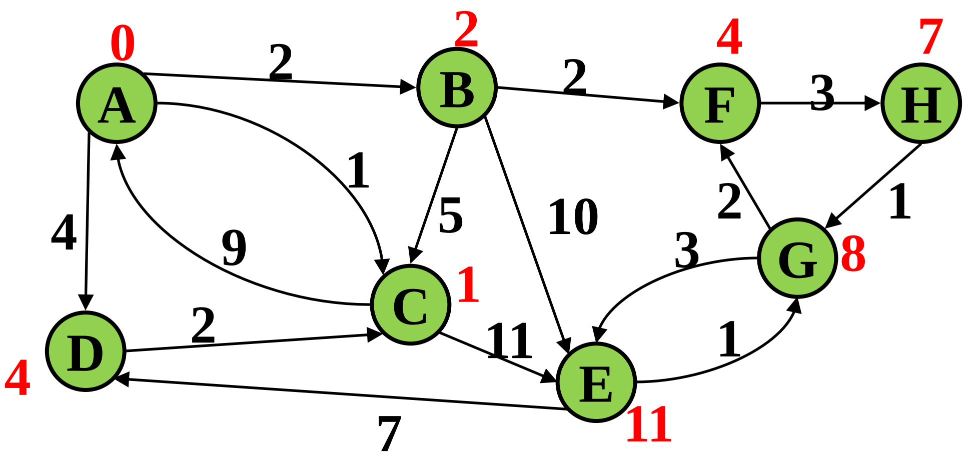
- When a vertex is marked known,
  the cost of the shortest path to that node is known
  - The path is also known by following back-pointers

- While a vertex is still not known,
  another shorter path to it might still be found

Note: The “Order Added to Known Set” is not important
- A detail about how the algorithm works (client doesn’t care)
- Not used by the algorithm (implementation doesn’t care)
- It is sorted by path-cost, resolving ties in some way

Interpreting the Results

- Now that we're done, how do we get the path from, say, A to E?



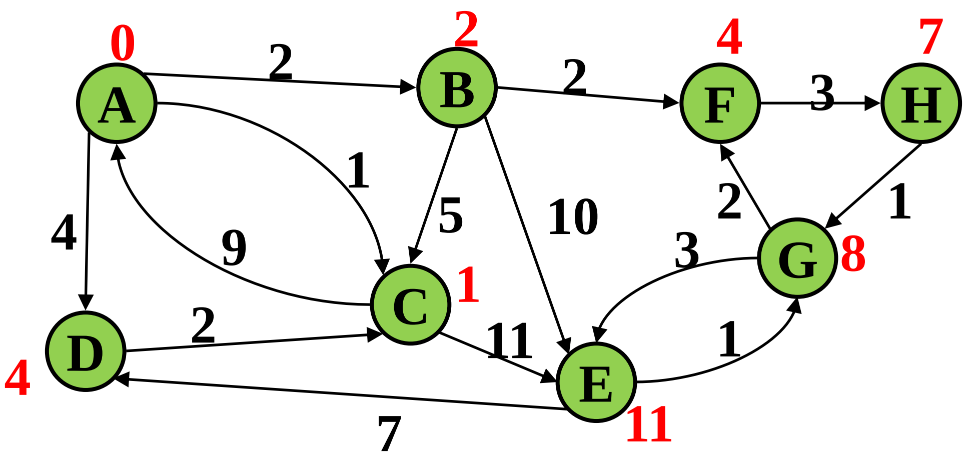
Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Stopping Short

- How would this have worked differently if we were only interested in:
  - The path from A to G?
  - The path from A to D?



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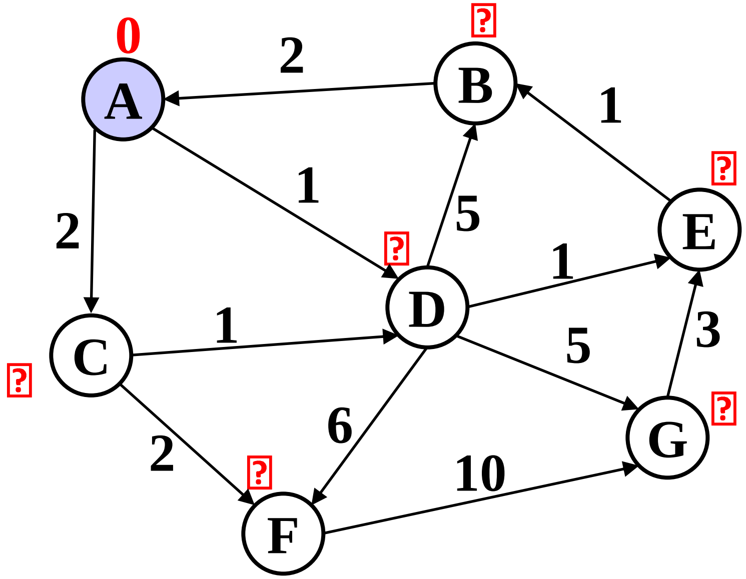
H	Y	7	F

<ins>Order Added to Known Set:</ins>

A, C, B, D, F, H, G, E

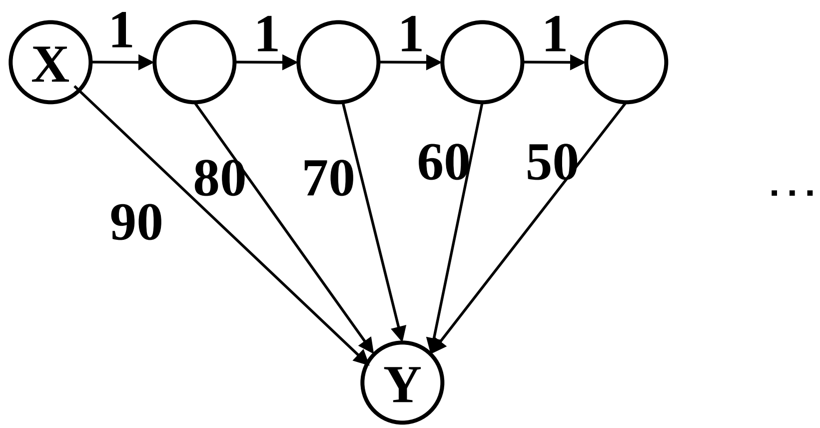
Example #2



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B			
C			
D			
E			
F			
G			

Example #3



How will the best-cost-so-far for Y proceed?

Is this expensive?

A Greedy Algorithm

- Dijkstra's algorithm
  - For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges

- An example of a *greedy algorithm*:
  - At each step, irrevocably does what seems best at that step
    - A locally optimal step, not necessarily globally optimal
  - Once a vertex is known, it is not revisited
    - Turns out to be globally optimal

Where are we?

- What should we do after learning an algorithm?
  - Prove it is correct
    - Not obvious!
    - We will sketch the key ideas
  - Analyze its efficiency
    - Will do better by using a data structure we learned earlier!

Correctness: Intuition

Rough intuition:

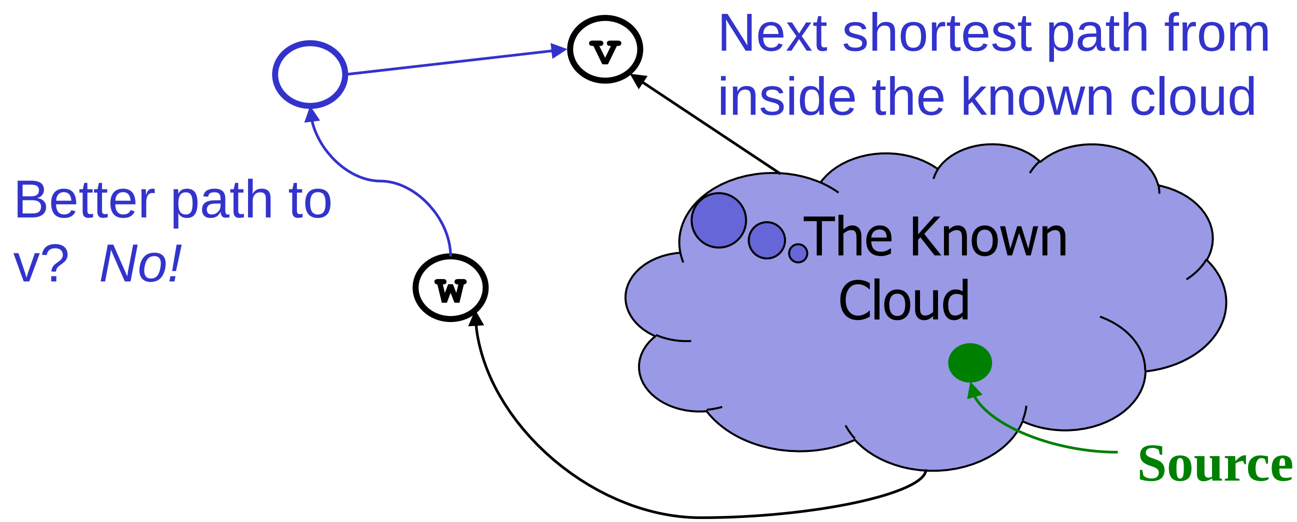
All the “known” vertices have the correct shortest path

- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node “known”, then by induction this holds and eventually everything is “known”

Key fact we need: When we mark a vertex “known” we won’t discover a shorter path later!

- This holds only because Dijkstra’s algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction…

Correctness: The Cloud (Rough Idea)



Suppose **v** is the next node to be marked known ("added to the cloud")

- The best-known path to **v** must have only nodes "in the cloud"
  - Since we've selected it, and we only know about paths through the cloud to a node right outside the cloud
- Assume the actual shortest path to **v** is different
  - It won't use only cloud nodes, (or we would know about it), so it must use non-cloud nodes
  - Let **w** be the *first* non-cloud node on this path.
  - The part of the path up to **w** is already known and must be shorter than the best-known path to **v**. So **v** would not have been picked. Contradiction!

Efficiency, first approach

Use pseudocode to determine asymptotic run-time

- Notice each edge is processed only once

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  while(not all nodes are known) {
    b = find unknown node with smallest cost
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
       if(b.cost + weight((b,a)) < a.cost){
         a.cost = b.cost + weight((b,a))
         a.path = b
       }
}
```

Improving asymptotic running time

- So far: $O(|V|^2 + |E|)$

- We had a similar "problem" with topological sort being $O(|V|^2 + |E|)$
- due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges

- Solution?

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

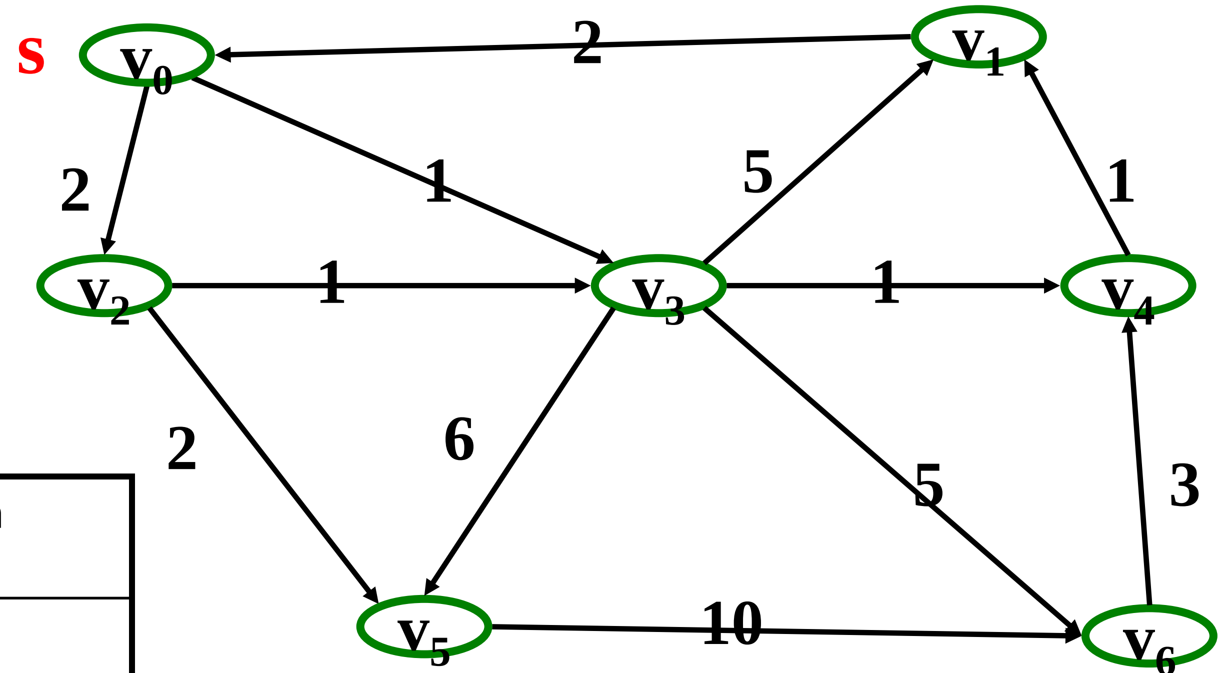
```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  build-heap with all nodes
  while(heap is not empty) {
    b = deleteMin()
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
      if(b.cost + weight((b,a)) < a.cost){
        decreaseKey(a,"new cost - old cost")
        a.path = b
      }
}
```

Dense vs. sparse again

- First approach: $O(|V|^2 + |E|)$ or: $O(|V|^2)$
- Second approach: $O(|V|\log|V| + |E|\log|V|)$

- So which is better?
  - Sparse: $O(|V|\log|V| + |E|\log|V|)$ (if $|E| > |V|$, then $O(|E|\log|V|)$)
  - Dense: $O(|V|^2 + |E|)$, or: $O(|V|^2)$
- But, remember these are worst-case and asymptotic
  - Priority queue might have slightly worse constant factors
  - On the other hand, for “normal graphs”, we might call **`decreaseKey`** rarely (or not percolate far), making $|E|\log|V|$ more like $|E|$

Find the shortest path to each vertex from v_0



V	Known	Dist from s	Path
v0			
v1			
v2			
v3			
v4			
v5			
v6			

Order declared Known: