



CSE 332: Data Structures & Parallelism

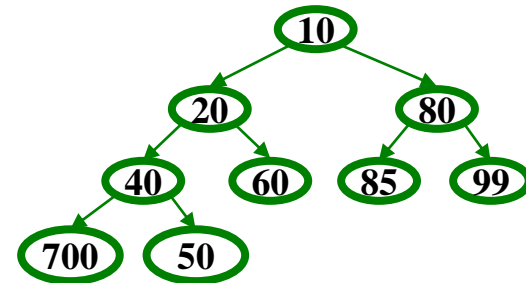
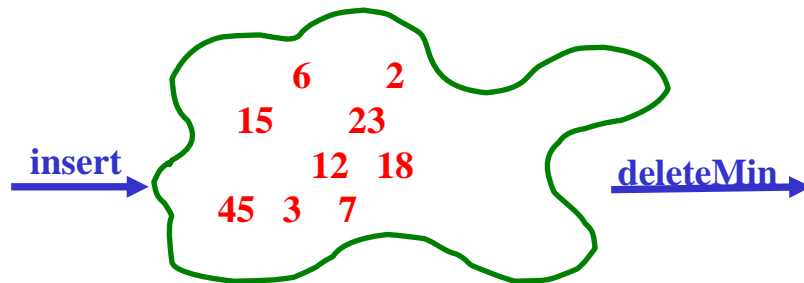
Lecture 4: Binary Heaps, Continued

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Today

- Binary Min Heap implementation
 - Insert
 - Deletemin
 - Buildheap

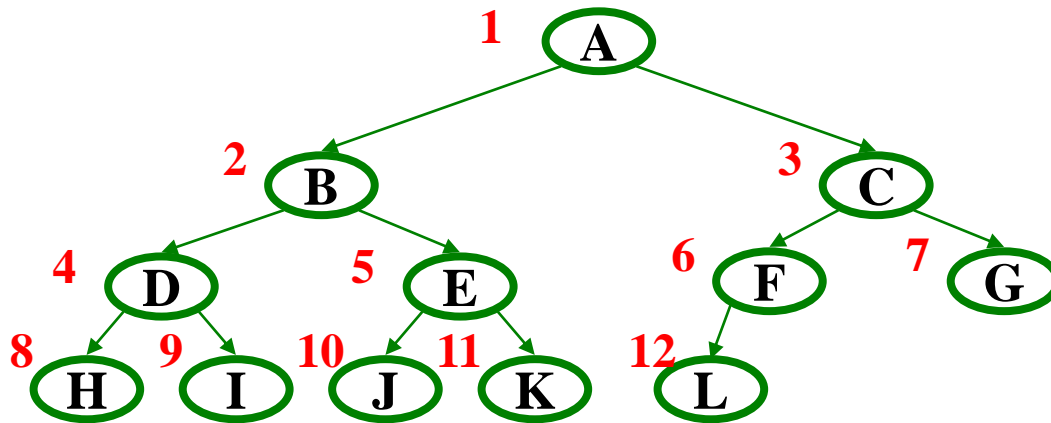
Review



- Priority Queue ADT: **insert** comparable object, **deleteMin**
- Binary heap data structure: Complete binary tree where each node has priority value greater than its parent
- $O(\text{height-of-tree}) = O(\log n)$ **insert** and **deleteMin** operations
 - **insert**: put at new last position in tree and percolate-up
 - **deleteMin**: remove root, put last element at root and percolate-down
- But: tracking the “last position” is painful and we can do better

Note: Exercises and P2 start counting from 0

Array Representation of Binary Trees



From node i :

left child: $i*2$

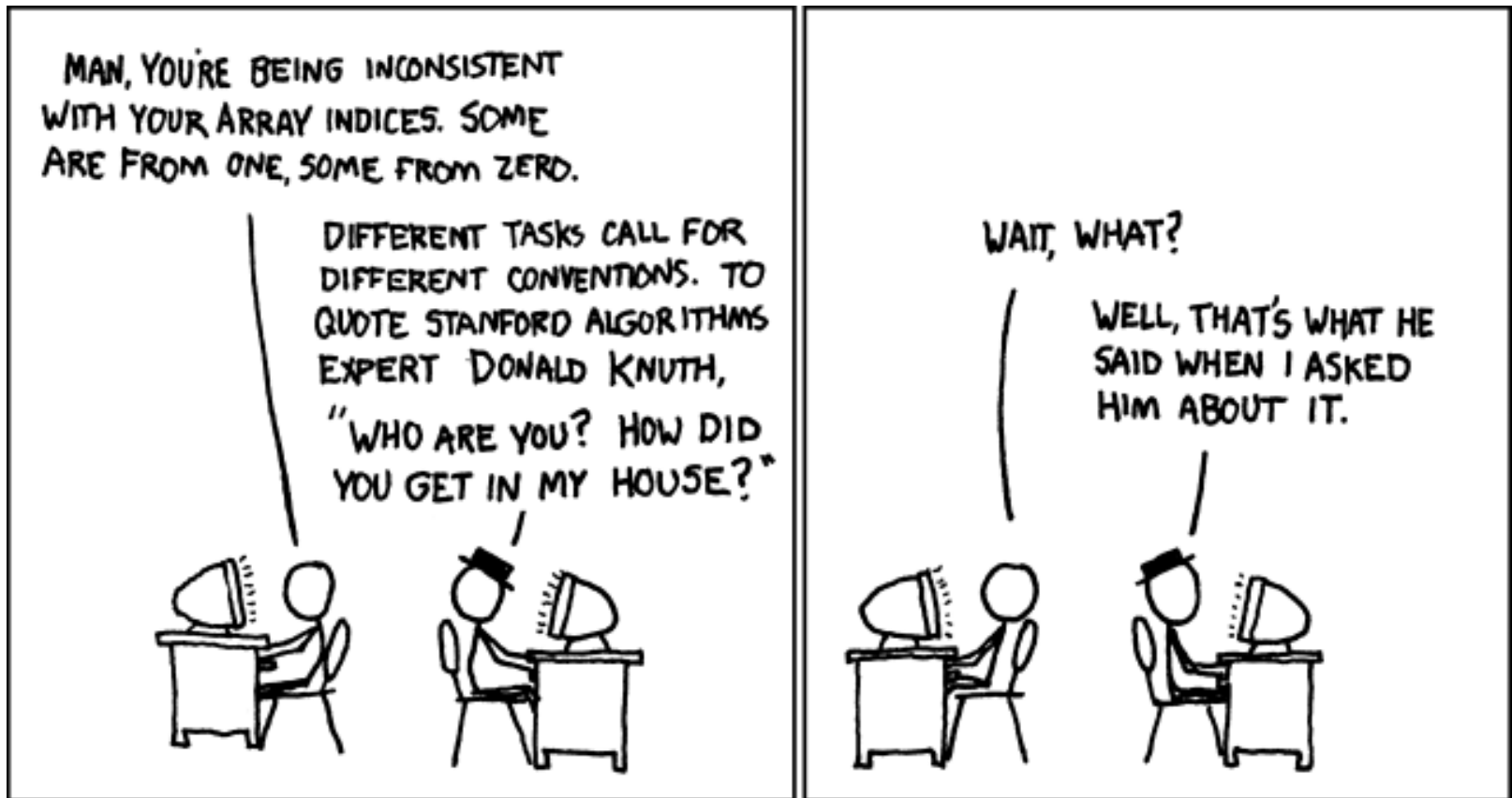
right child: $i*2+1$

parent: $i/2$

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13



<http://xkcd.com/163>

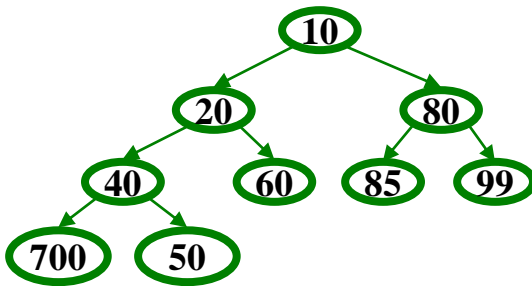
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Pseudocode: insert

```
void insert(int val) {  
    if(size==arr.length-1)  
        resize();  
    size++;  
    i=percolateUp(size, val);  
    arr[i] = val;  
}
```

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
int percolateUp(int hole,  
               int val) {  
    while(hole > 1 &&  
          val < arr[hole/2]){  
        arr[hole] = arr[hole/2];  
        hole = hole / 2;  
    }  
    return hole;  
}
```



	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

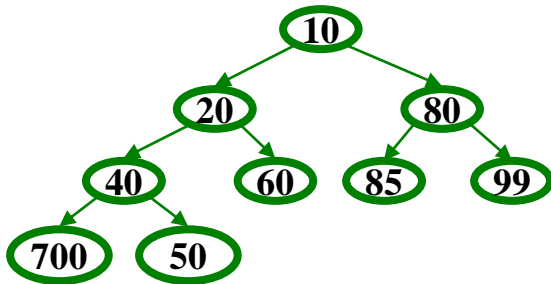
Note: Exercises and P2 start counting from 0

Pseudocode: deleteMin

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
int deleteMin() {
    if(isEmpty()) throw...
    ans = arr[1];
    hole = percolateDown
        (1, arr[size]);
    arr[hole] = arr[size];
    size--;
    return ans;
}
```

```
int percolateDown(int hole,
                  int val) {
    while(2*hole <= size) {
        left = 2*hole;
        right = left + 1;
        if(arr[left] < arr[right]
           || right > size)
            target = left;
        else
            target = right;
        if(arr[target] < val) {
            arr[hole] = arr[target];
            hole = target;
        } else
            break;
    }
    return hole;
}
```

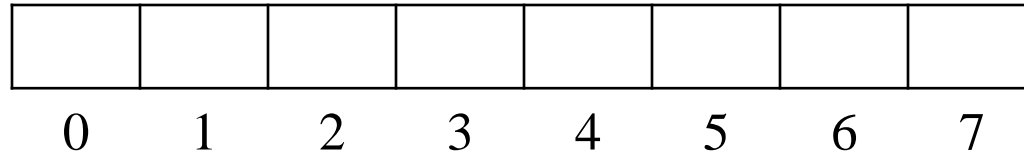


	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Note: Exercises and P2 start counting from 0

Example

1. insert: 16, 32, 4, 57, 80, 43, 2
2. deleteMin



Other operations

- **decreaseKey**: given pointer to object in priority queue (e.g., its array index), lower its priority value by p
 - Change priority and percolate up
- **increaseKey**: given pointer to object in priority queue (e.g., its array index), raise its priority value by p
 - Change priority and percolate down
- **remove**: given pointer to object in priority queue (e.g., its array index), remove it from the queue
 - **decreaseKey** with $p = \infty$, then **deleteMin**

Running time for all these operations?

Evaluating the Array Implementation...

Advantages:

Minimal amount of wasted space:

- Only index 0 and any unused space on right in the array
- No "holes" due to complete tree property
- No wasted space representing tree edges

Fast lookups:

- Benefit of array lookup speed
- Multiplying and dividing by 2 is extremely fast (can be done through bit shifting (see CSE 351))
- Last used position is easily found by using the PQueue's size for the index

Disadvantages:

- What if the array gets too full (or wastes space by being too empty)? Array will have to be resized.

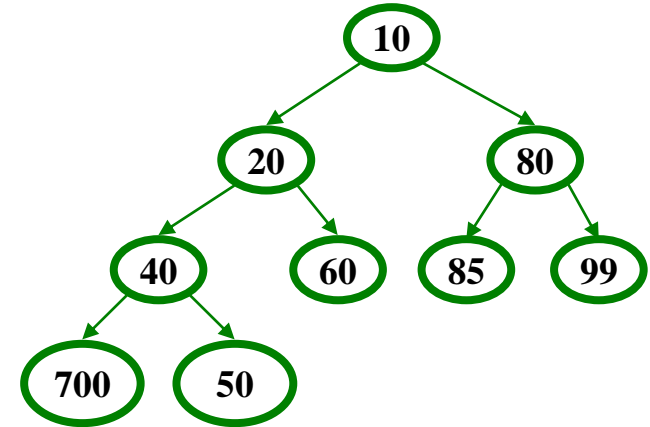
Advantages outweigh Disadvantages: This is how it is done!

So why $O(1)$ average-case insert?

- Yes, insert's **worst case** is $O(\log n)$
- The trick is that it all depends on the order the items are inserted (What is the worst case order?)
- Experimental studies of randomly ordered inputs shows the following:
 - Average 2.607 comparisons per insert (# of percolation passes)
 - An element usually moves up 1.607 levels
- deleteMin is average $O(\log n)$
 - Moving a leaf to the root usually requires re-percolating that value back to the bottom

Aside: Insert run-time: Take 2

- Insert: Place in next spot, percUp
- How high do we expect it to go?
- Aside: Complete Binary Tree
 - Each full row has 2x nodes of parent row
 - $1+2+4+8+\dots+2^k = 2^{k+1}-1$
 - Bottom level has $\sim 1/2$ of all nodes
 - Second to bottom has $\sim 1/4$ of all nodes
- PercUp Intuition:
 - Move up if value is less than parent
 - Inserting a random value, likely to have value not near highest, nor lowest; somewhere in middle
 - Given a random distribution of values in the heap, bottom row should have the upper half of values, 2nd from bottom row, next 1/4
 - Expect to only raise a level or 2, even if h is large
- Worst case: still $O(\log n)$
- Expected case: $O(1)$
- Of course, there's no guarantee; it may percUp to the root



Building a Heap

Suppose you have n items you want to put in a new priority queue

- A sequence of n **insert** operations works
- Runtime?

Can we do better?

- If we only have access to **insert** and **deleteMin** operations, then NO.
- There is a faster way - $O(n)$, but that requires the ADT to have a specialized **buildHeap** operation

Important issue in ADT design: how many specialized operations?

–Tradeoff: **Convenience, Efficiency, Simplicity**

Floyd's buildHeap Method

Recall our general strategy for working with the heap:

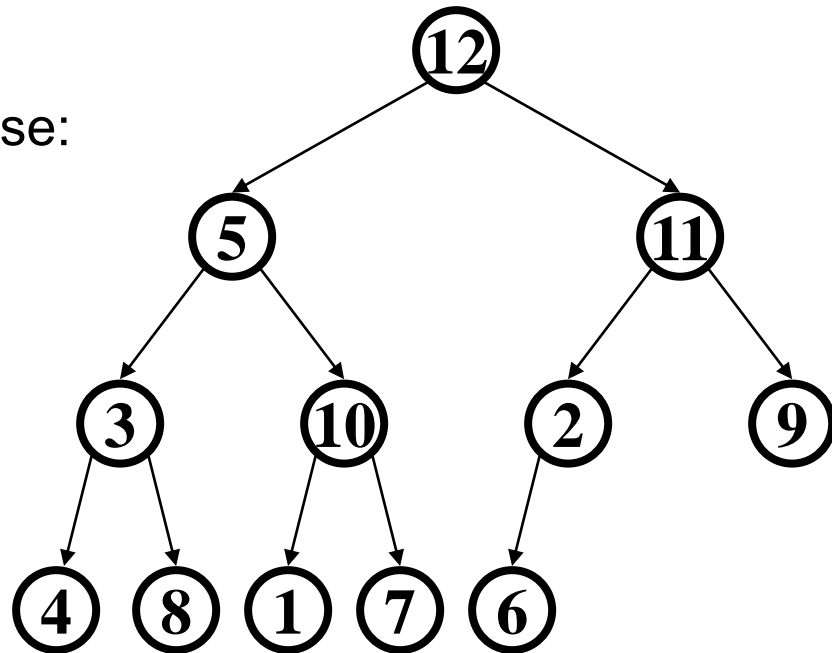
- Preserve structure property
- Break and restore heap ordering property

Floyd's *buildHeap*:

1. Create a complete tree by putting the n items in array indices $1, \dots, n$
2. Treat the array as a heap and fix the heap-order property
 - Exactly how we do this is where we gain efficiency

Thinking about *buildHeap*

- Say we start with this array:
[12,5,11,3,10,2,9,4,8,1,7,6]
- To “fix” the ordering can we use:
 - percolateUp?
 - percolateDown?



Floyd's buildHeap Method

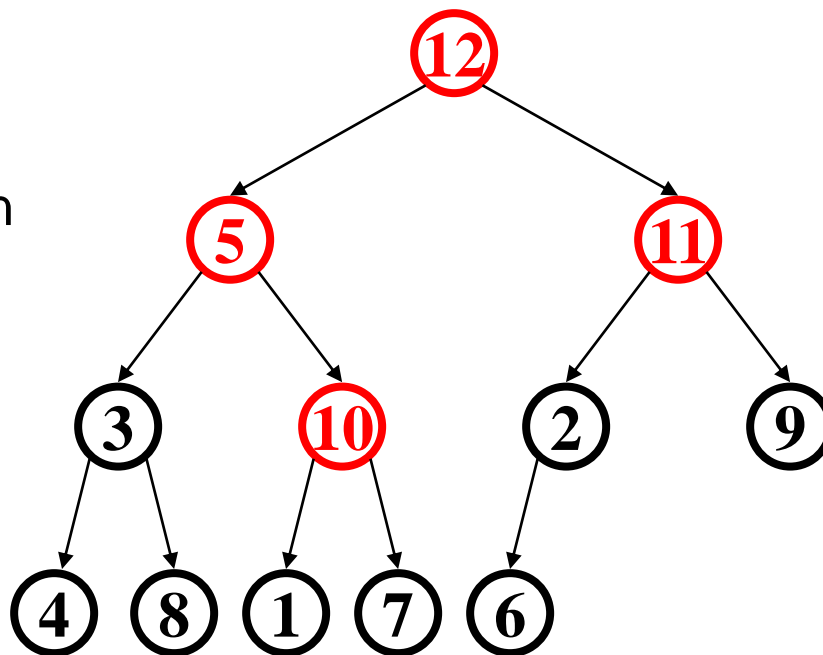
Bottom-up:

- Leaves are already in heap order
- Work up toward the root one level at a time

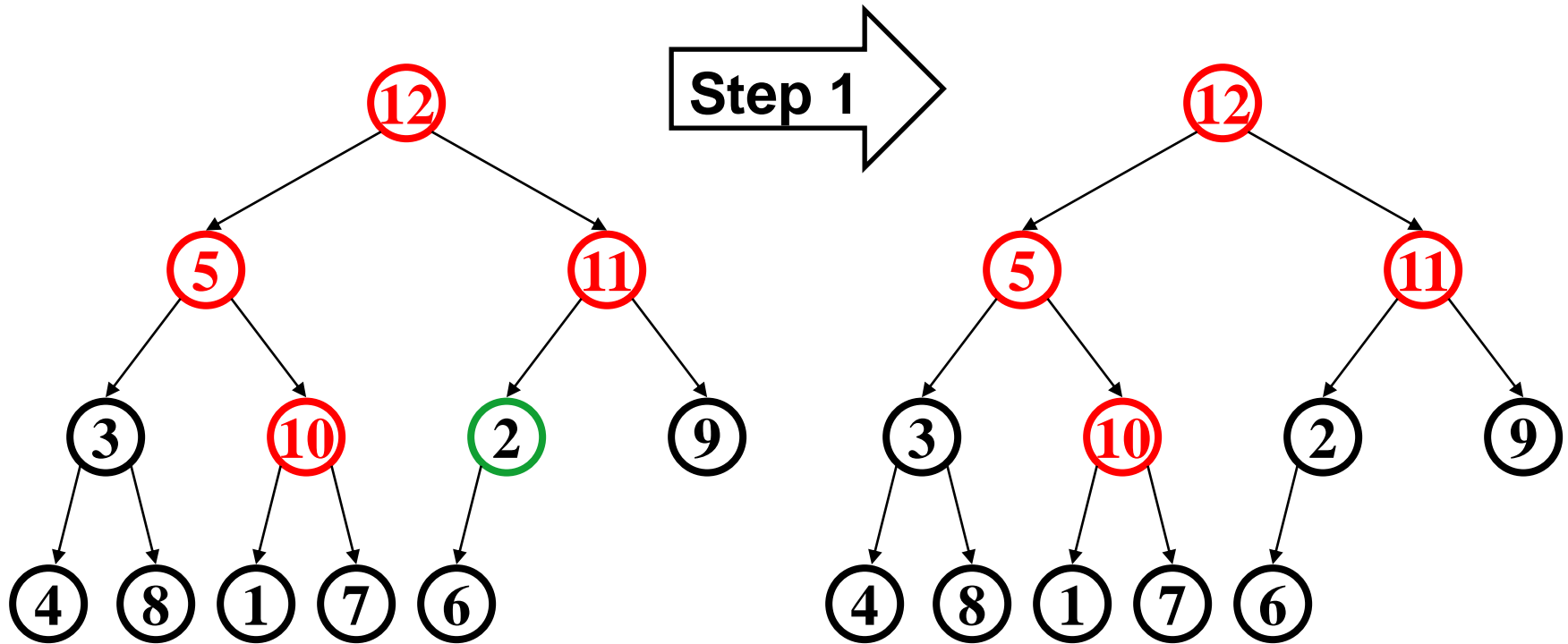
```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
        hole = percolateDown(i, val);  
        arr[hole] = val;  
    }  
}
```


buildHeap Example

- Say we start with this array:
[12,5,11,3,10,2,9,4,8,1,7,6]
- In tree form for readability
 - Red for node not less than descendants
 - heap-order problem
 - Notice no leaves are red
 - Check/fix each non-leaf bottom-up (6 steps here)

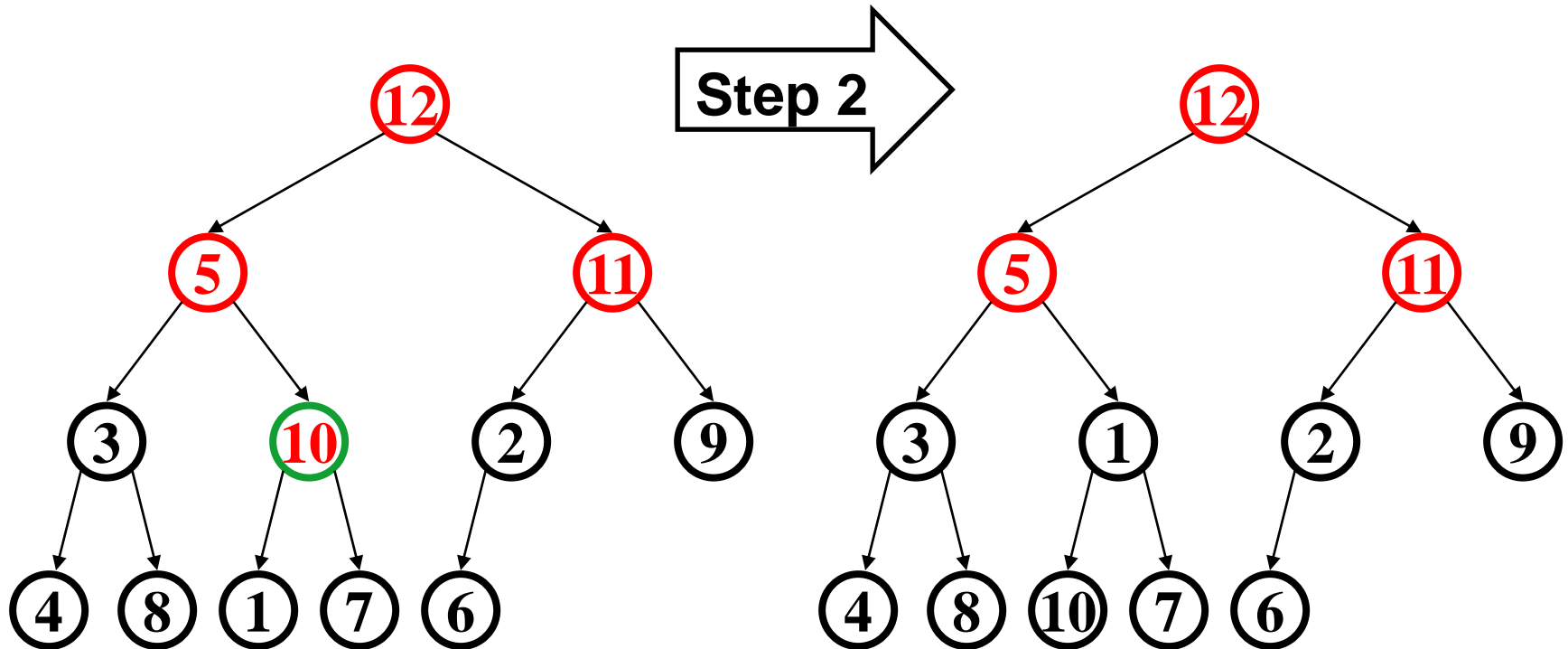


buildHeap Example



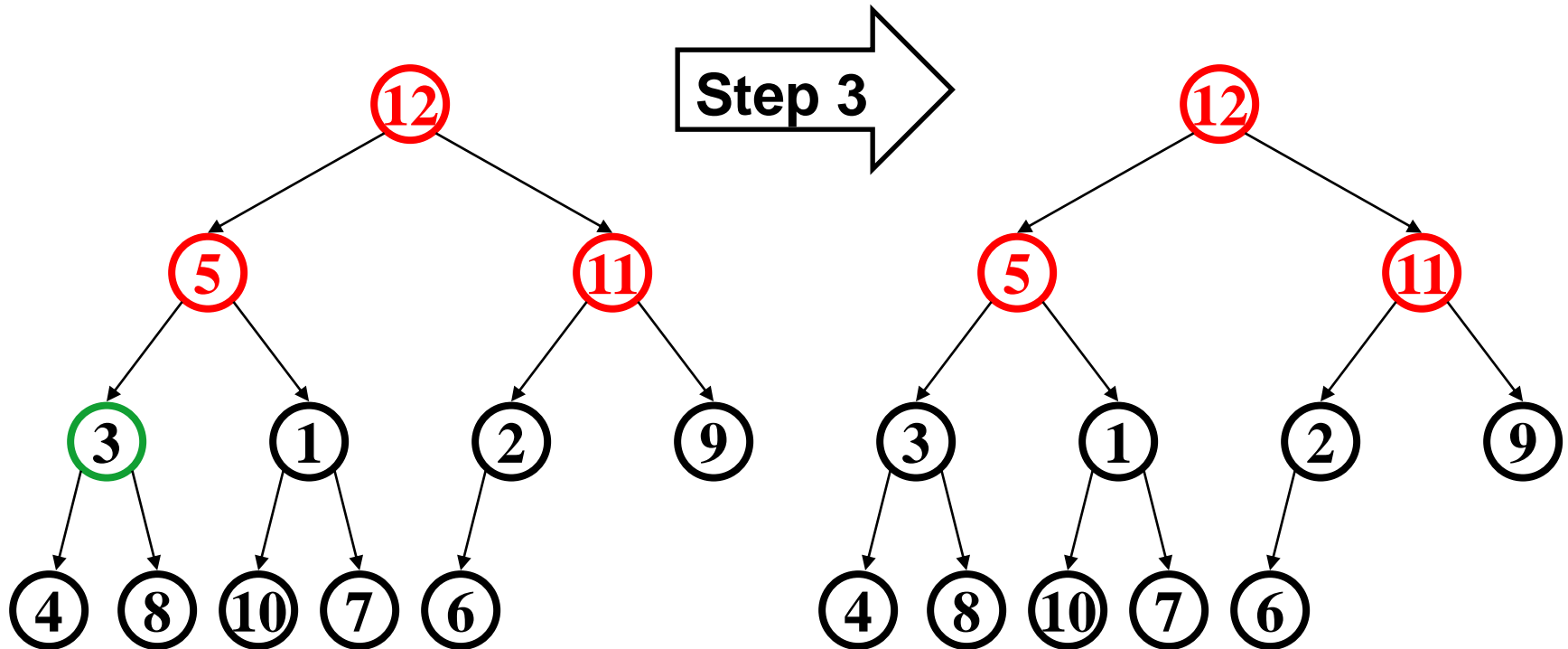
- Happens to already be less than child

buildHeap Example



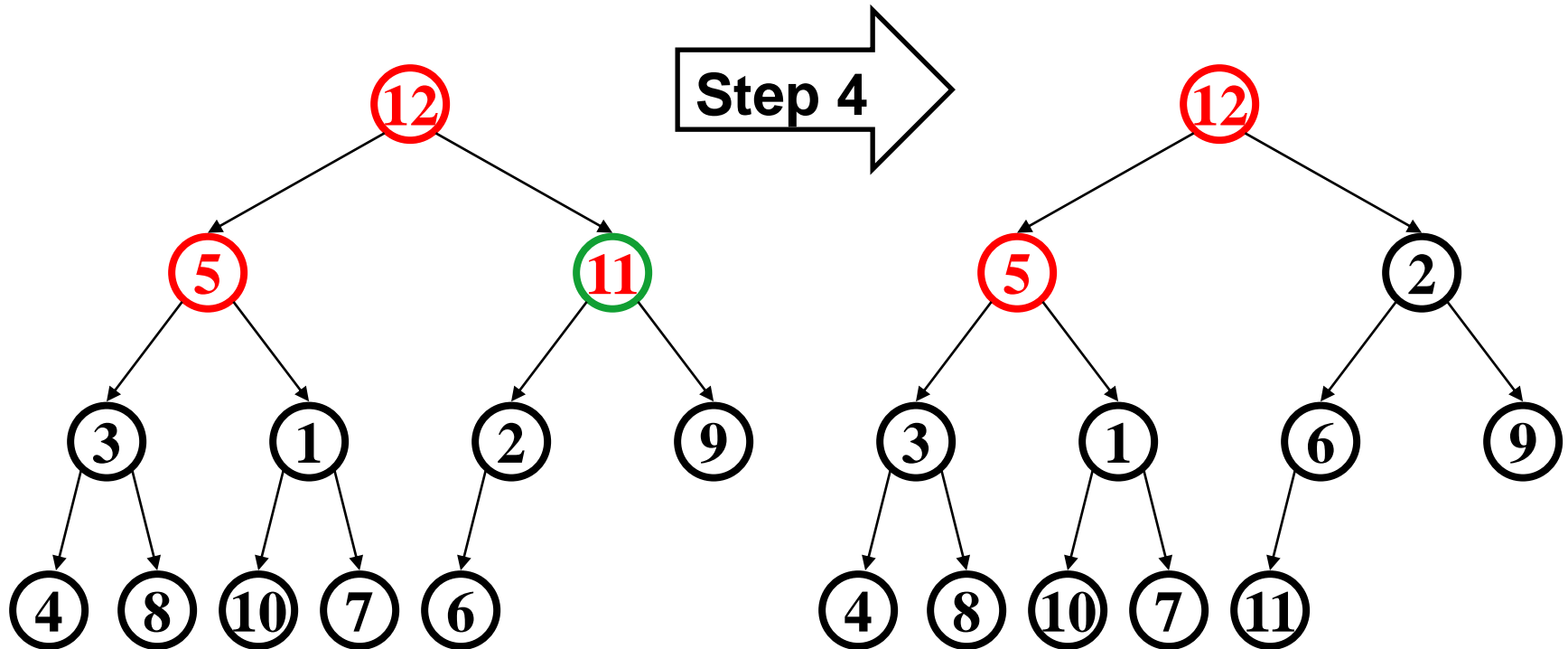
- Percolate down (notice that moves 1 up)

buildHeap Example



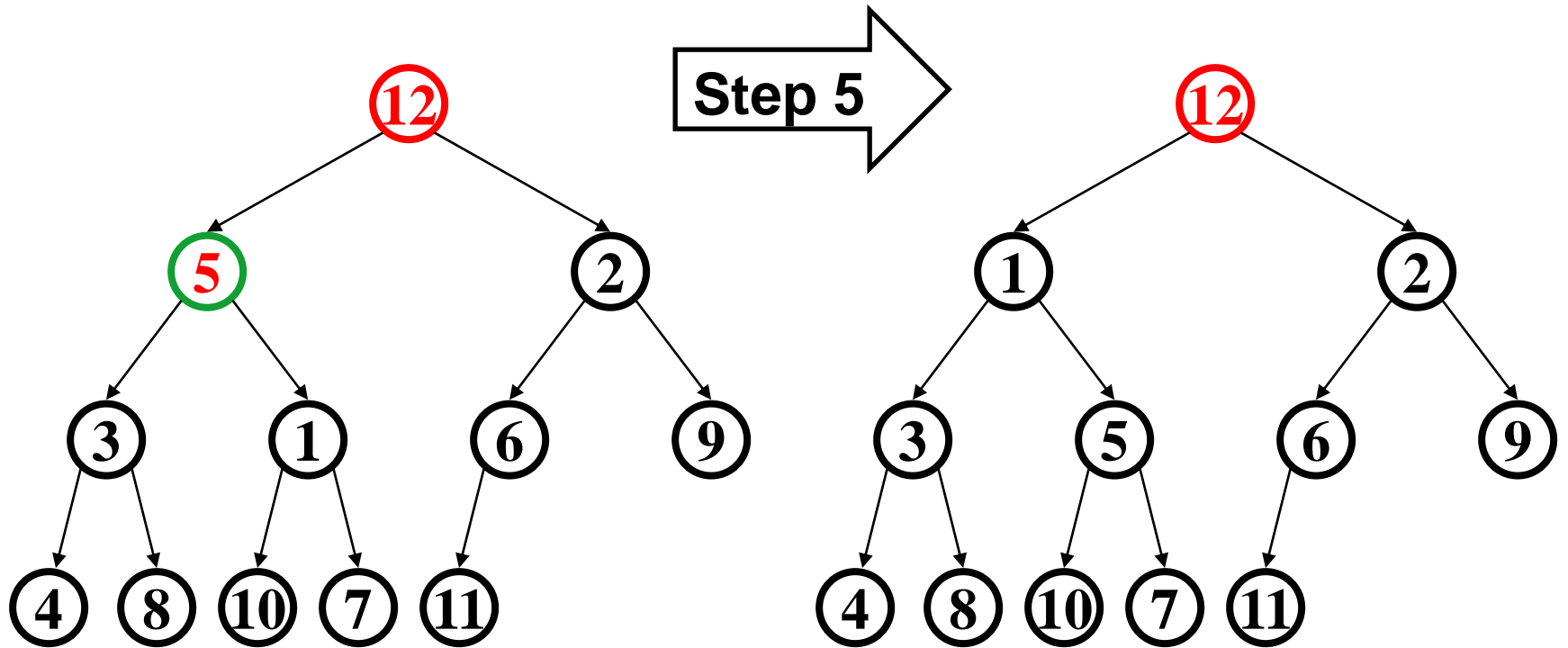
- Another nothing-to-do step

buildHeap Example

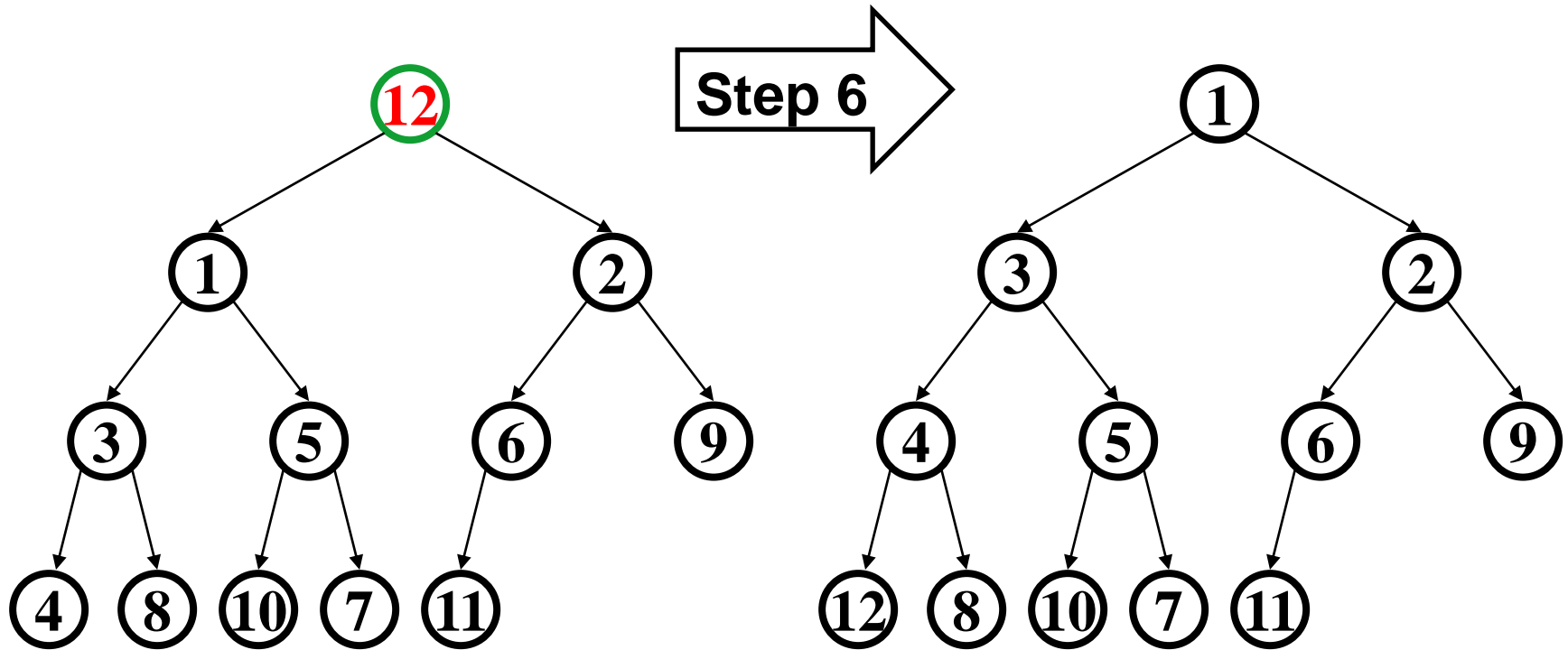


- Percolate down as necessary (steps 4a and 4b)

buildHeap Example



buildHeap Example



But is it right?

- “Seems to work”
 - Let’s *prove* it restores the heap property (correctness)
 - Then let’s *prove* its running time (efficiency)

```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
        hole = percolateDown(i, val);  
        arr[hole] = val;  
    }  
}
```


Correctness

```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
        hole = percolateDown(i, val);  
        arr[hole] = val;  
    }  
}
```

Loop Invariant: For all $j > i$, `arr[j]` is less than its children

- True initially: If $j > \text{size}/2$, then j is a leaf
 - Otherwise its left child would be at position $> \text{size}$
- True after one more iteration: loop body and `percolateDown` make `arr[i]` less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

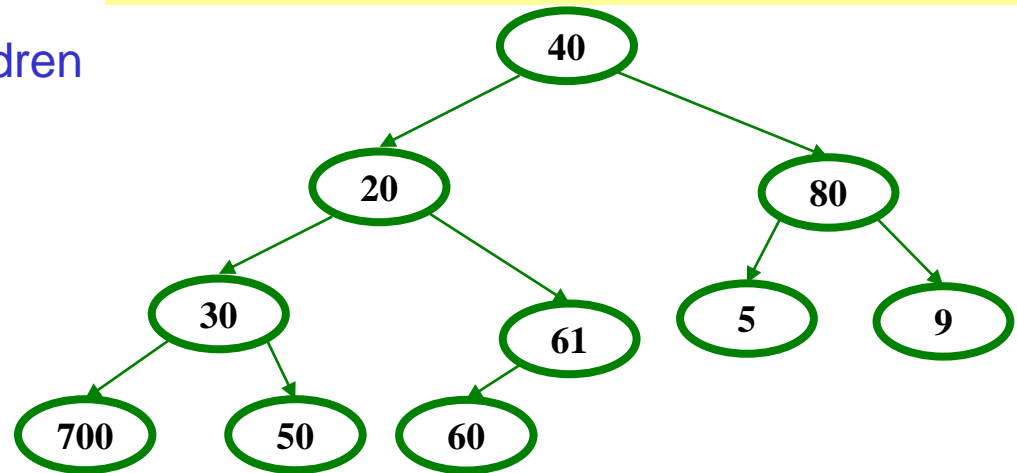
Loop Invariant:

For all $j > i$, $arr[j]$ is less than its children

- True initially:
If $j > size/2$, then j is a leaf
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void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
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        arr[hole] = val;  
    }  
}
```

So after the loop finishes,
all nodes are less than their children



	40	20	80	30	61	5	9	700	50	60			
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Efficiency

```
void buildHeap() {
    for(i = size/2; i>0; i--) {
        val = arr[i];
        hole = percolateDown(i, val);
        arr[hole] = val;
    }
}
```

Easy argument: `buildHeap` is $O(n \log n)$ where n is `size`

- `size/2` loop iterations
- Each iteration does one `percolateDown`, each is $O(\log n)$

This is correct, but there is a more precise (“tighter”) analysis of the algorithm...

Efficiency

```
void buildHeap() {
    for(i = size/2; i>0; i--) {
        val = arr[i];
        hole = percolateDown(i, val);
        arr[hole] = val;
    }
}
```

Better argument: `buildHeap` is $O(n)$ where n is `size`

- `size/2` total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most **1 step**
- 1/4 the loop iterations percolate at most **2 steps**
- 1/8 the loop iterations percolate at most **3 steps**... etc.
- $((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + \dots) = 2$ (page 4 of Weiss)
 - So at most **2** (`size/2`) *total* percolate steps: $O(n)$
 - Also see Weiss 6.3.4, sum of heights of nodes in a perfect tree

Lessons from `buildHeap`

- Without `buildHeap`, our ADT already let clients implement their own in $\theta(n \log n)$ worst case
 - Worst case is inserting lower priority values later
- By providing a specialized operation internally (with access to the data structure), we can do $O(n)$ worst case
 - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
 - Correctness: Non-trivial inductive proof using loop invariant
 - Efficiency:
 - First analysis easily proved it was $O(n \log n)$
 - A “tighter” analysis shows same algorithm is $O(n)$

More heaps (see Weiss if curious)

- ***d*-heaps**: have d children instead of 2 (Weiss 6.5)
 - Makes heaps shallower, useful for heaps too big for memory
 - How does this affect the asymptotic run-time (for small d 's)?
- **Leftist heaps, skew heaps, binomial queues** (Weiss 6.6-6.8)
 - Different data structures for priority queues that support a logarithmic time **merge** operation (impossible with binary heaps)
 - **merge**: given two priority queues, make one priority queue
 - Insert & deleteMin defined in terms of merge

Aside: How might you merge *binary* heaps:

- If one heap is much smaller than the other?
- If both are about the same size?