



CSE 332: Data Structures & Parallelism

Lecture 13: Beyond Comparison Sorting

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Spring 2023

Today

- Sorting
 - Comparison sorting
 - Beyond comparison sorting

The Big Picture

Simple algorithms:
 $O(n^2)$

Insertion sort
Selection sort
Shell sort
...

Fancier algorithms:
 $O(n \log n)$

Heap sort
Merge sort
Quick sort (avg)
...

Comparison lower bound:
 $\Omega(n \log n)$

Specialized algorithms:
 $O(n)$

Bucket sort
Radix sort

Handling huge data sets

External sorting

How fast can we sort?

- Heapsort & mergesort have $O(n \log n)$ worst-case running time
- Quicksort has $O(n \log n)$ average-case running times
- These bounds are all tight, actually $\Theta(n \log n)$
- So maybe we need to dream up another algorithm with a lower asymptotic complexity, such as $O(n)$ or $O(n \log \log n)$
 - Instead: *prove* that this is *impossible*
 - *Assuming* our comparison *model*: The only operation an algorithm can perform on data items is a 2-element comparison

A Different View of Sorting

- Assume we have n elements to sort
 - And for simplicity, none are equal (no duplicates)
- How many **permutations** (possible orderings) of the elements?
- Example, $n=3$,

A Different View of Sorting

- Assume we have n elements to sort
 - And for simplicity, none are equal (no duplicates)
- How many **permutations** (possible orderings) of the elements?
- Example, $n=3$, six possibilities
 - $a[0] < a[1] < a[2]$ $a[0] < a[2] < a[1]$ $a[1] < a[0] < a[2]$
 - $a[1] < a[2] < a[0]$ $a[2] < a[0] < a[1]$ $a[2] < a[1] < a[0]$
- In general, n choices for least element, then $n-1$ for next, then $n-2$ for next, ...
 - $n(n-1)(n-2)\dots(2)(1) = n!$ possible orderings

Describing every comparison sort

- A different way of thinking of sorting is that the sorting algorithm has to “find” the right answer among the $n!$ possible answers
 - Starts “knowing nothing”, “anything is possible”
 - Gains information with each comparison, eliminating some possibilities
 - Intuition: At best, each comparison can eliminate half of the remaining possibilities
 - In the end narrows down to a single possibility

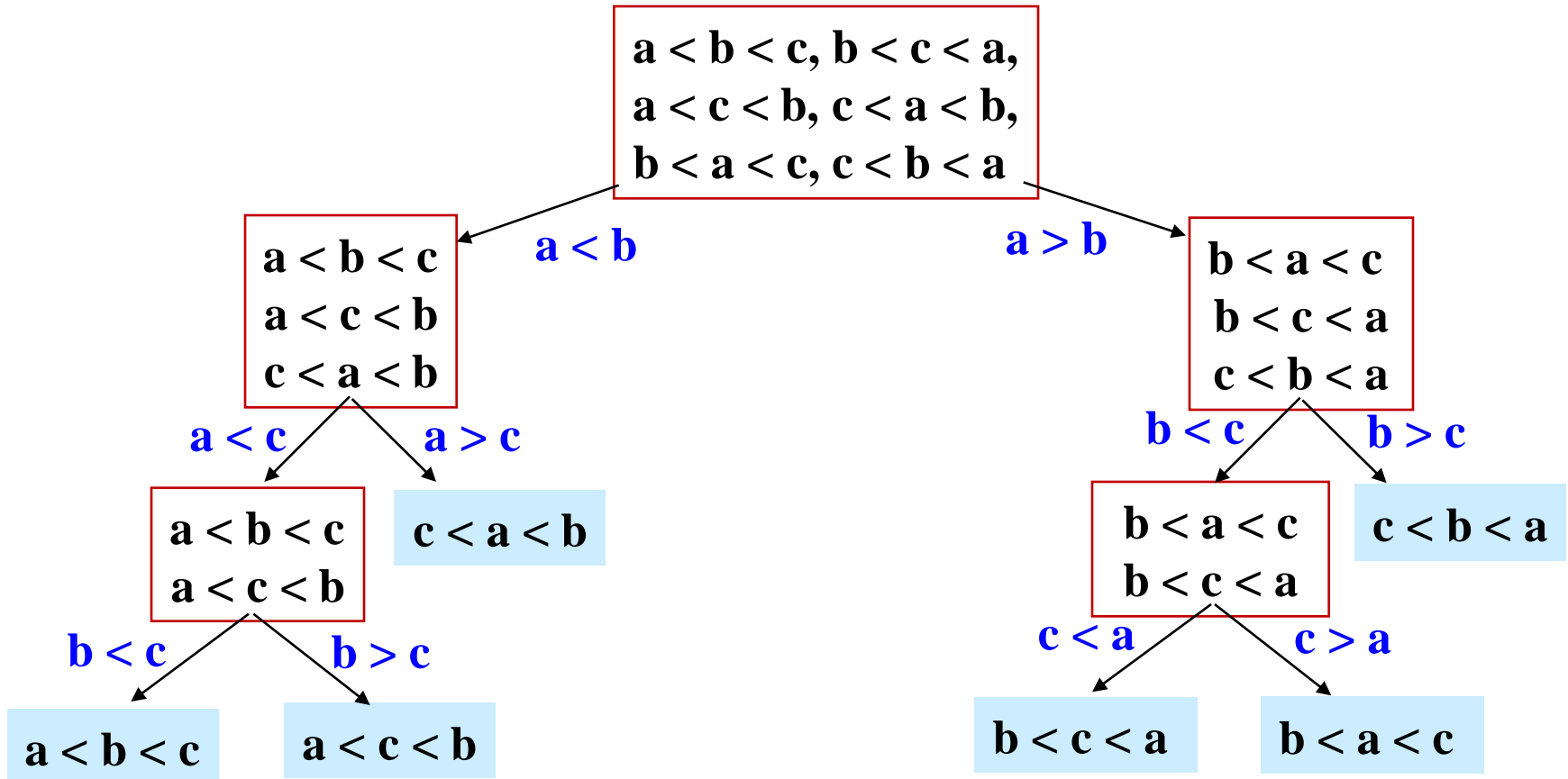
Counting Comparisons

- Don't know what the algorithm is, but it cannot make progress without doing comparisons
 - Eventually does a first comparison “is $a < b$?”
 - Can use the result to decide what second comparison to do
 - Etc.: comparison k can be chosen based on first $k-1$ results
- What is the first comparison in:
 - Selection Sort?
 - Insertion Sort?
 - Quicksort?
 - Mergesort?

Counting Comparisons

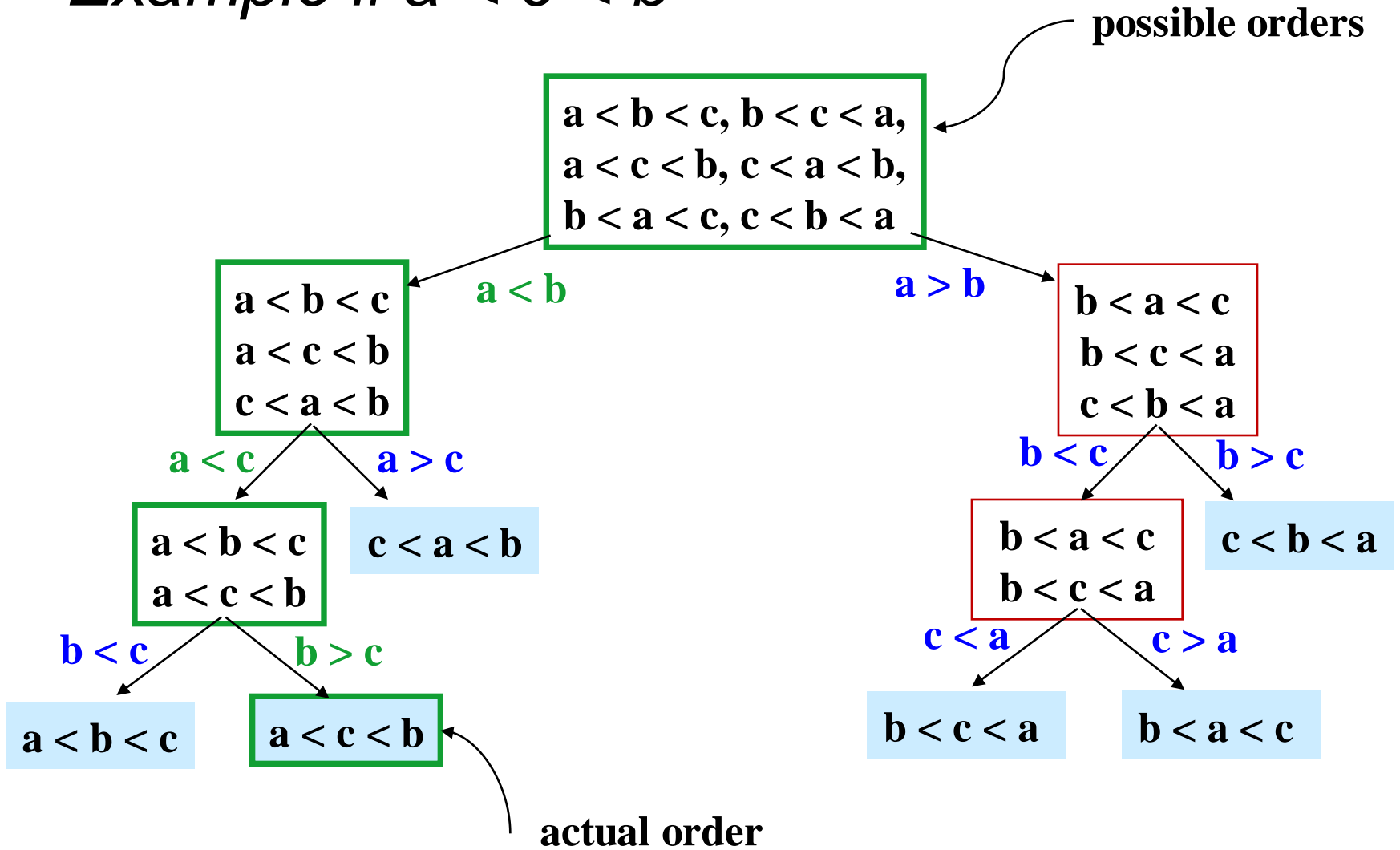
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 - Eventually does a first comparison “is $a < b$?”
 - Can use the result to decide what second comparison to do
 - Etc.: comparison k can be chosen based on first $k-1$ results
- Can represent this process as a *decision tree*
 - Nodes contain “set of remaining possibilities”
 - At root, anything is possible; no option eliminated
 - Edges are “answers from a comparison”
 - The algorithm does not actually build the tree; it's what our *proof* uses to represent “the most the algorithm could know so far” as the algorithm progresses

One Decision Tree for $n=3$



- The leaves contain all the possible orderings of a, b, c
- A different algorithm would lead to a different tree

Example if $a < c < b$



What the decision tree tells us

- A *binary* tree because each comparison has 2 outcomes
 - Perform only comparisons between 2 elements; binary result
 - Ex: Is $a < b$? Yes or no?
 - We assume no duplicate elements
 - Assume algorithm doesn't ask redundant questions
- Because any data is possible, any algorithm needs to ask enough questions to produce all $n!$ answers
 - Each answer is a different leaf
 - So the tree must be big enough to have $n!$ leaves
 - Running *any* algorithm on *any* input will at best correspond to a root-to-leaf path in *some* decision tree with $n!$ leaves
 - So no algorithm can have worst-case running time better than the height of a tree with $n!$ leaves
 - Worst-case number-of-comparisons for an algorithm is an input leading to a longest path in algorithm's decision tree

Where are we

Proven: No comparison sort can have worst-case running time better than: **the height of a binary tree with $n!$ leaves**

- Turns out average-case is same asymptotically
- A comparison sort could be worse than this height, but it cannot be better
- Fine, *how tall is a binary tree with $n!$ leaves?*

Now: Show that a binary tree with $n!$ leaves has height $\Omega(n \log n)$

- That is, $n \log n$ is the lower bound, the height must be at least this, could be more, (in other words your comparison sorting algorithm could take longer than this, but it won't be faster)
- Factorial function grows very quickly

Then we'll conclude that: **(Comparison) Sorting is $\Omega(n \log n)$**

- This is an amazing computer-science result: proves all the clever programming in the world can't sort in linear time!

Lower bound on Height

- A binary tree of height h has **at most** *how many* leaves?

$$L \leq \underline{\hspace{2cm}}$$

- A binary tree with L leaves has height **at least**:

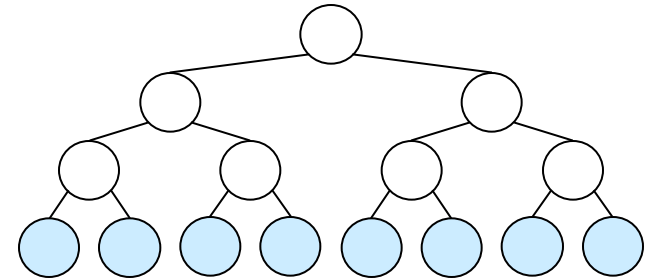
$$h \geq \underline{\hspace{2cm}}$$

- The decision tree has how many leaves:

- So the decision tree has height:

$$h \geq \underline{\hspace{2cm}}$$

Lower bound on height



- The height of a binary tree with L leaves is at least $\log_2 L$
- So the height of our decision tree, h :

$$h \geq \log_2 (n!)$$

$$= \log_2 (n \cdot (n-1) \cdot (n-2) \cdots (2) \cdot (1))$$

$$= \log_2 n + \log_2 (n-1) + \dots + \log_2 1$$

$$\geq \log_2 n + \log_2 (n-1) + \dots + \log_2 (n/2)$$

$$\geq (n/2) \log_2 (n/2)$$

each of the $n/2$ terms left is $\geq \log_2 (n/2)$

$$= (n/2)(\log_2 n - \log_2 2)$$

property of binary trees

definition of factorial

property of logarithms

keep first $n/2$ terms

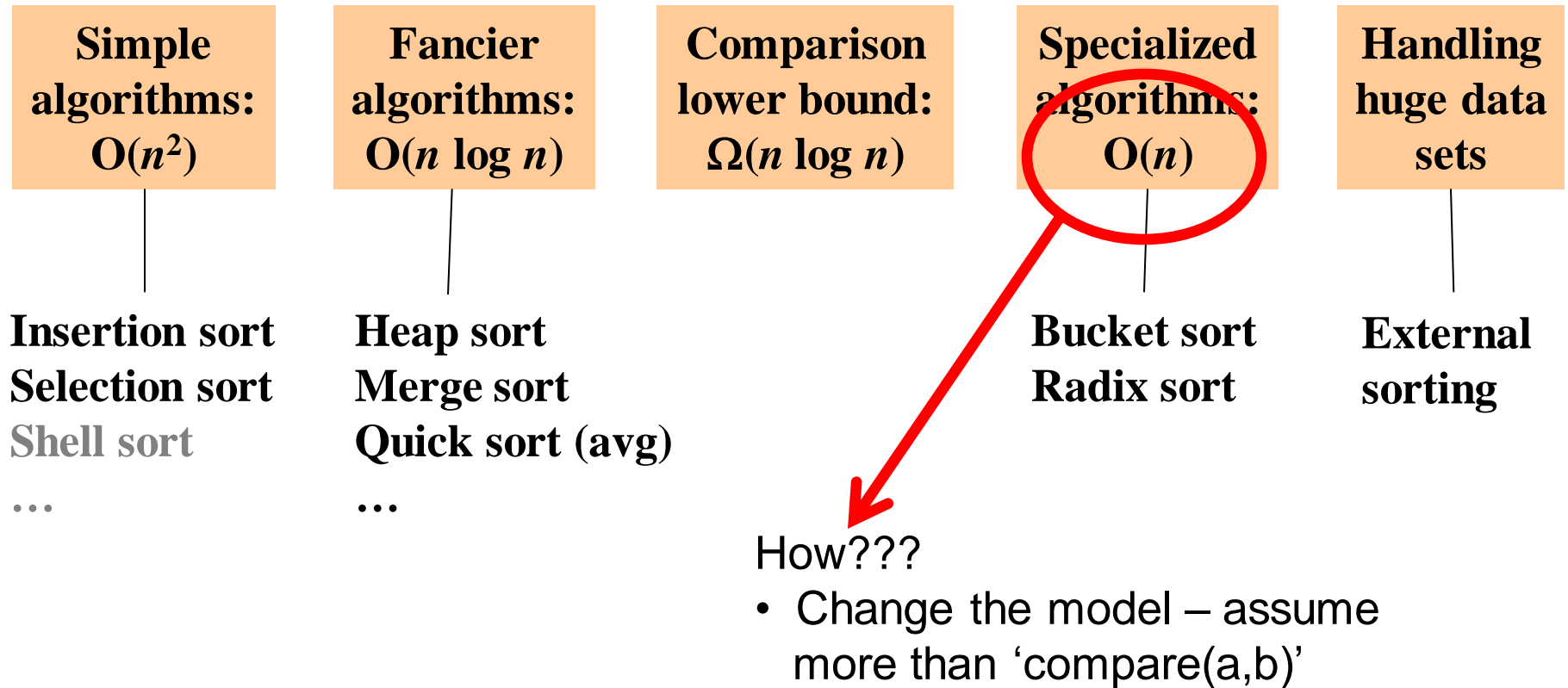
property of logarithms

$$= (1/2)n \log_2 n - (1/2)n$$

arithmetic

$$\text{"="} \Omega (n \log n)$$

The Big Picture



BucketSort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and K (or any small range),
 - Create an array of size K , and put each element in its proper bucket (a.k.a. bin)
 - If data is only integers, no need to store more than a *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

count array	
1	
2	
3	
4	
5	

- Example:
K=5
Input: (5,1,3,4,3,2,1,1,5,4,5)
output:

Analyzing bucket sort

- Overall: $O(n+K)$
 - Linear in n , but also linear in K
 - $\Omega(n \log n)$ lower bound does not apply because **this is not a comparison sort**
- Good when range, K , is smaller (or not much larger) than n
 - (We don't spend time doing lots of comparisons of duplicates!)
- Bad when K is much larger than n
 - Wasted space; wasted time during final linear $O(K)$ pass
- For data in addition to integer keys, use list at each bucket

Bucket Sort with Data

- Most real lists aren't just #'s; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, place at end $O(1)$ (keep pointer to last element)

Bucket sort illustrates a more general trick: How might you implement a heap for a small range of integer priorities in a similar manner...

count array	
1	→ Rocky V
2	
3	→ Harry Potter
4	
5	→ Casablanca → Star Wars

- Example: Movie ratings: 1=bad,... 5=excellent
- Input=
 - 5: Casablanca
 - 3: Harry Potter movies
 - 1: Rocky V
 - 5: Star Wars

Result: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars
This result is **stable**; Casablanca still before Star Wars

Radix sort

- Radix = “the base of a number system”
 - Examples will use 10 because we are used to that
 - In implementations use larger numbers
 - For example, for ASCII strings, might use 128
- Idea:
 - Bucket sort on one digit at a time
 - Number of buckets = radix
 - Starting with *least* significant digit, sort with Bucket Sort
 - Keeping sort *stable*
 - Do one pass per digit
- **Invariant:** After k passes, the last k digits are sorted
- Aside: Origins go back to the 1890 U.S. census

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	721		3				537	478	9
			143				67	38	

Input: 478
537
9
721
3
38
143
67

First pass:

1. bucket sort by ones digit
2. Iterate thru and collect into a list
 - List is sorted by first digit

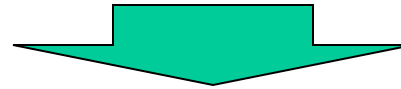
Order now: 721

3
143
537
67
478
38
9

Example

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

Radix = 10



0	1	2	3	4	5	6	7	8	9
3 9		721	537 38	143		67	478		

Order was: 721
3
143
537
67
478
38
9

Second pass:

stable bucket sort by tens digit

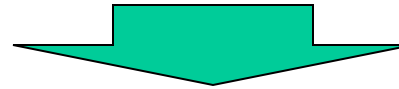
If we chop off the 100's place,
these #s are sorted

Order now: 3
9
721
537
38
143
67
478

Example

0	1	2	3	4	5	6	7	8	9
3		721	537	143		67	478		
9			38						

Radix = 10



0	1	2	3	4	5	6	7	8	9
3	143			478	537		721		
9									
38									
67									

Order was:

3
9
721
537
38
143
67
478

Order now:

3
9
38
67
143
478
537
721

Third pass:

stable bucket sort by 100s digit

Only 3 digits: We're done!

RadixSort

- Input: 126, 328, 636, 341, 416, 131, 328

BucketSort on lsd:

0	1	2	3	4	5	6	7	8	9

BucketSort on next-higher digit:

0	1	2	3	4	5	6	7	8	9

BucketSort on msd:

0	1	2	3	4	5	6	7	8	9

Analysis of Radix Sort

Performance depends on:

- Input size: n
- Number of buckets = Radix: B
 - e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
- Number of passes = “Digits”: P
 - e.g. Ages of people: 3; Phone #: 10; Person’s name: ?
- Work per pass is 1 bucket sort: _____
 - Each pass is a Bucket Sort
- Total work is _____
 - We do ‘P’ passes, each of which is a Bucket Sort

Comparison to Comparison Sorts

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
 - Approximate run-time: $15 \cdot (52 + n)$
 - This is less than $n \log n$ only if $n > 33,000$
 - Of course, cross-over point depends on constant factors of the implementations plus P and B
 - And radix sort can have poor locality properties
- Not really practical for many classes of keys
 - Strings: Lots of buckets

Recap: Features of Sorting Algorithms

In-place

- Sorted items occupy the same space as the original items. (No copying required, only $O(1)$ extra space if any.)

Stable

- Items in input with the same value end up in the same order as when they began.

Examples:

- Merge Sort - not in place, stable
- Quick Sort - in place, not stable

Sorting massive data: External Sorting

Need sorting algorithms that **minimize disk/tape access** time:

- Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
- Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access

Basic Idea:

- Load chunk of data into Memory, sort, store this “run” on disk/tape
- Use the Merge routine from Mergesort to merge runs
- Repeat until you have only one run (one sorted chunk)

- Mergesort can leverage multiple disks
- Weiss gives some examples

Sorting Summary

- Simple $O(n^2)$ sorts can be fastest for small n
 - selection sort, insertion sort (latter linear for mostly-sorted)
 - good for “below a cut-off” to help divide-and-conquer sorts
- $O(n \log n)$ sorts
 - heap sort, in-place but not stable nor parallelizable
 - merge sort, not in place but stable and works as external sort
 - quick sort, in place but not stable and $O(n^2)$ in worst-case
 - often fastest, but depends on costs of comparisons/copies
- $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
 - Bucket sort good for small number of key values
 - Radix sort uses fewer buckets and more phases
- Best way to sort? It depends!