CSE 332 Autumn 2023
Lecture 9: AVL Trees

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Dictionary (Map) ADT

• Contents:
  • Sets of key+value pairs
  • Keys must be comparable

• Operations:
  • insert(key, value)
    • Adds the (key,value) pair into the dictionary
    • If the key already has a value, overwrite the old value
      • Consequence: Keys cannot be repeated
  • find(key)
    • Returns the value associated with the given key
  • delete(key)
    • Remove the key (and its associated value)
Less Naïve attempts

• Binary Search Trees (Friday)
• Tries (Project)
• AVL Trees (Today)
• B-Trees (this week)
• HashTables (next week)
• Red-Black Trees (not included in this course)
• Splay Trees (not included in this course)
## Dictionary Data Structures

<table>
<thead>
<tr>
<th>Data Structure</th>
<th>Time to insert</th>
<th>Time to find</th>
<th>Time to delete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unsorted Array</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Unsorted Linked List</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Sorted Array</td>
<td>$\Theta(n)$</td>
<td>$\Theta(\log n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Sorted Linked List</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>Binary Search Tree</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
<tr>
<td>AVL Tree</td>
<td>$\Theta(\log n)$</td>
<td>$\Theta(\log n)$</td>
<td>$\Theta(\log n)$</td>
</tr>
</tbody>
</table>
Binary Search Tree

• Binary Tree
  • Definition:
    • Every node has at most 2 children

• Order Property
  • All keys in the left subtree are smaller than the root
  • All keys in the right subtree are larger than the root
  • Apply recursively

• Why?
  • Makes searching quicker
    • Worst case: tree’s height
Find Operation (recursive)

```java
find(key, root){
    if (root == Null){
        return Null;
    }
    if (key == root.key){
        return root.value;
    }
    if (key < root.key){
        return find(key, root.left);
    }
    if (key > root.key){
        return find(key, root.right);
    }
    return Null;
}
```
Find Operation (iterative)

```java
find(key, root){
    while (root != Null && key != root.key){
        if (key < root.key){
            root = root.left;
        }
        else if (key > root.key){
            root = root.right;
        }
    }
    if (root == Null){
        return Null;
    }
    return root.value;
}
```
Insert Operation (iterative)

```java
insert(key, value, root){
    if (root == Null){ this.root = new Node(key, value); } 
    parent = Null;
    while (root != Null && key != root.key){
        parent = root;
        if (key < root.key){ root = root.left; }
        else if (key > root.key){ root = root.right; }
    }
    if (root != Null){ root.value = value; }
    else if (key < parent.key){ parent.left = new Node(key, value); }
    else{ parent.right = new Node (key, value); }
}
```

Note: Insert happens only at the leaves!
Delete Operation (iterative)

delete(key, root)
    while (root != Null && key != root.key)
        if (key < root.key){ root = root.left; }
        else if (key > root.key){ root = root.right; }

    if (root == Null){ return; }

    // Now root is the node to delete, what happens next?
Delete – 3 Cases

• 0 Children (i.e. it’s a leaf)

• 1 Child

• 2 Children
Finding the Max and Min

- **Max of a BST:**
  - Right-most Thing

- **Min of a BST:**
  - Left-most Thing

```java
maxNode(root)
    if (root == Null) { return Null; }
    while (root.right != Null) {
        root = root.right;
    }
    return root;

minNode(root)
    if (root == Null) { return Null; }
    while (root.left != Null) {
        root = root.left;
    }
    return root;
```
Delete Operation (iterative)

def delete(key, root):
    while (root != Null && key != root.key):
        if (key < root.key):
            root = root.left;
        else if (key > root.key):
            root = root.right;
    if (root == Null):
        return;
    if (root has no children):
        make parent point to Null Instead;
    if (root has one child):
        make parent point to that child instead;
    if (root has two children):
        make parent point to either the max from the left or min from the right


Improving the worst case

• How can we get a better worst case running time?
“Balanced” Binary Search Trees

• We get better running times by having “shorter” trees
• Trees get tall due to them being “sparse” (many one-child nodes)
• Idea: modify how we insert/delete to keep the tree more “full”
Idea 1: Both Subtrees of Root have same # Nodes
Idea 2: Both Subtrees of Root have same height
Idea 3: Both Subtrees of every Node have same # Nodes
Idea 4: Both Subtrees of every Node have same height
AVL Tree

• A Binary Search tree that maintains that the left and right subtrees of every node have heights that differ by at most one.
  • height of left subtree and height of right subtree off by at most 1
  • Not too weak (ensures trees are short)
  • Not too strong (works for any number of nodes)

• Idea of AVL Tree:
  • When you insert/delete nodes, if tree is “out of balance” then modify the tree
  • Modification = “rotation”
Is it an AVL Tree?
Using AVL Trees

- Each node has:
  - Key
  - Value
  - Height
  - Left child
  - Right child

Key = 9
Value = “hello”
Height = 3
Left = Node 3
Right = Node 10
Inserting into an AVL Tree

• Starts out the same way as BST:
  • “Find” where the new node should go
  • Put it in the right place (it will be a leaf)

• Next check the balance
  • If the tree is still balanced, you’re done!
  • Otherwise we need to do rotations
Insert Example

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Diagram of a binary tree with nodes labeled 0, 1, 2, 3, 6, 9, 10, 16, and 7. The root node is labeled 9, with children labeled 3 and 10. Node 3 has children labeled 1 and 6, and node 1 has children labeled 0 and 2. Node 6 has a child labeled 7.
Insert Example

```
-1
```
Not Balanced!

Solution: rotate the whole tree to the right

Height = 3

Height = 1
Balanced!
Right Rotation

- Make the left child the new root
- Make the old root the right child of the new root
- Make the new root’s right subtree the old root’s left subtree
Insert Example

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Not Balanced!

Solution: rotate the deepest imbalance to the left
Balanced!
Left Rotation

- Make the right child the new root
- Make the old root the left child of the new
- Make the new root’s left subtree the old root’s right subtree
Insertion Story So Far

• After insertion, update the heights of the node’s ancestors
• Check for imbalance
• If there’s imbalance then at the deepest root of imbalance:
  • If the left subtree was deeper then rotate right
  • If the right subtree was deeper then rotate left

This is incomplete! There are some cases where this doesn’t work!
Insertion Story So Far

• After insertion, update the heights of the node’s ancestors
• Check for imbalance
• If there’s imbalance then at the deepest root of imbalance:
  • Case LL: If we inserted in the left subtree of the left child then rotate right
  • Case RR: If we inserted in the right subtree of the right child then rotate left
  • Case LR: If we inserted into the right subtree of the left child then ???
  • Case RL: If we inserted into the left subtree of the right child then ???

Cases LR and RL require 2 rotations!
Case LR

• From “root” of the deepest imbalance:
  • Rotate left at the left child
  • Rotate right at the root
Case LR in General

• Imbalance caused by inserting in the left child’s right subtree
• Rotate left at the left child
• Rotate right at the imbalanced node
Case RL in General

- Imbalance caused by inserting in the right child’s left subtree
- Rotate right at the right child
- Rotate left at the imbalanced node
Insert Summary

• After a BST insertion, update the heights of the node’s ancestors
• Check for imbalance
• If there’s imbalance then at the deepest root of imbalance:
  • Case LL: If we inserted in the left subtree of the left child then: rotate right
  • Case RR: If we inserted in the right subtree of the right child then: rotate left
  • Case LR: If we inserted into the right subtree of the left child then: rotate left at the left child and then rotate right at the root
  • Case RL: If we inserted into the left subtree of the right child then: rotate right at the right child and then rotate left at the root