CSE 332 Autumn 2023
Lecture 15: Sorting

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Properties to consider

• Running time
  • What is the worst case running time?
  • What is the best case?
  • Does the algorithm run faster if the list is close to sorted?
    • If so, we call it Adaptive

• Memory Usage
  • How much memory does the algorithm use in addition to the array?
    • If $\Theta(1)$ then we call it In-Place
      • Sorts things by only swapping things in the same array we started with.

• What happens when there is a “tie”?
  • If “tied” elements are guaranteed to remain in the same relative order, this is called a Stable Sort
  • E.g. a stable sort guarantees that, after sorting by the first initial and then by last initial, “N.J.B.” will come before “S.C.B”
Properties of Selection Sort

• In-Place?
  • Yes!

• Adaptive?
  • No

• Stable?
  • Yes!
  • As long as you always pick the left-most element when there’s a “tie”
Properties of Insertion Sort

• In-Place?
  • Yes!

• Adaptive?
  • Yes!

• Stable?
  • Yes!
  • As long as you don’t swap when there’s a tie

• Online!
  • You can begin sorting the list before you have all the elements
  • “Insert” items as they arrive
Properties of Heap Sort

• Worst Case Running time:
  • $\Theta(n \log n)$

• In-Place?
  • Yes!

• Adaptive?
  • No

• Stable?
  • No
Divide And Conquer Sorting

• Divide and Conquer:
  • Recursive algorithm design technique
  • Solve a large problem by breaking it up into smaller versions of the same problem
Divide and Conquer

- **Base Case:**
  - If the problem is “small” then solve directly and return

- **Divide:**
  - Break the problem into subproblem(s), each smaller instances

- **Conquer:**
  - Solve subproblem(s) recursively

- **Combine:**
  - Use solutions to subproblems to solve original problem
Divide and Conquer Template Pseudocode

def my_DandC(problem):
    # Base Case
    if (problem.size() <= small_value):
        return solve(problem); # directly solve (e.g., brute force)
    
    # Divide
    List subproblems = divide(problem);

    # Conquer
    solutions = new List();
    for (sub : subproblems):
        subsolution = my_DandC(sub);
        solutions.add(subsolution);

    # Combine
    return combine(solutions);
Merge Sort

• **Base Case:**
  • If the list is of length 1 or 0, it’s already sorted, so just return it

• **Divide:**
  • Split the list into two “sublists” of (roughly) equal length

• **Conquer:**
  • Sort both lists recursively

• **Combine:**
  • Merge sorted sublists into one sorted list
Merge Sort In Action!

Sort between indices *low* and *high*

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<tr>
<td>5</td>
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<td>9</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

Base Case: if *low == high* then that range is already sorted!

Divide and Conquer: Otherwise call `mergesort` on ranges \( (low, \frac{low+high}{2}) \) and \( (\frac{low+high}{2} + 1, high) \)

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After Recursion:

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Merge (the combine part)

Create a new array to merge into, and 3 pointers/indices:
- **L_next**: the smallest “unmerged” thing on the left
- **R_next**: the smallest “unmerged” thing on the right
- **M_next**: where the next smallest thing goes in the merged array

One-by-one: put the smallest of **L_next** and **R_next** into **M_next**, then advance both **M_next** and whichever of **L/R** was used.
Merge Sort Pseudocode

```java
void mergesort(myArray){
    ms_helper(myArray, 0, myArray.length());
}

void mshelper(myArray, low, high){
    if (low == high){return;}  // Base Case
    mid = (low+high)/2;
    ms_helper(low, mid);
    ms_helper(mid+1, high);
    merge(myArray, low, mid, high);
}
```
void merge(myArray, low, mid, high){
    merged = new int[high-low+1]; // or whatever type is in myArray
    l_next = low;
    r_next = high;
    m_next = 0;
    while (l_next <= mid && r_next <= high){
        if (myArray[l_next] <= myArray[r_next]){  
            merged[m_next++] = myArray[l_next++];
        }
        else{  
            merged[m_next++] = myArray[r_next++];
        }
    }
    while (l_next <= mid){ merged[m_next++] = myArray[l_next++]; }  
    while (r_next <= high){ merged[m_next++] = myArray[r_next++]; } 
    for(i=0; i<=merged.length; i++){ myArray[i+low] = merged[i];} 
}
Analyzing Merge Sort

1. Identify time required to Divide and Combine
2. Identify all subproblems and their sizes
3. Use recurrence relation to express recursive running time
4. Solve and express running time asymptotically

- **Divide**: 0 comparisons
- **Conquer**: recursively sort two lists of size \( \frac{n}{2} \)
- **Combine**: \( n \) comparisons
- **Recurrence**:
  \[
  T(n) = 0 + T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n \\
  T(n) = 2T\left(\frac{n}{2}\right) + n
  \]
Red box represents a problem instance
Blue value represents time spent at that level of recursion

\[ T(n) = 2T\left(\frac{n}{2}\right) + n \]

\[ T(n) = \sum_{i=1}^{\log_2 n} n = n \log_2 n \]
Properties of Merge Sort

• Worst Case Running time:
  • \( \Theta(n \log n) \)

• In-Place?
  • No!

• Adaptive?
  • No!

• Stable?
  • Yes!
  • As long as in a tie you always pick l_next
Quicksort

• Like Mergesort:
  • Divide and conquer
  • $O(n \log n)$ run time (kind of...)

• Unlike Mergesort:
  • Divide step is the “hard” part
  • *Typically* faster than Mergesort
Quicksort

Idea: pick a pivot element, recursively sort two sublists around that element

• Divide: select pivot element $p$, $\text{Partition}(p)$
• Conquer: recursively sort left and right sublists
• Combine: Nothing!
Partition (Divide step)

Given: a list, a pivot $p$

Start: unordered list

| 8 | 5 | 7 | 3 | 12 | 10 | 1 | 2 | 4 | 9 | 6 | 11 |

Goal: All elements $< p$ on left, all $> p$ on right

| 5 | 7 | 3 | 1 | 2 | 4 | 6 | 8 | 12 | 10 | 9 | 11 |
Partition, Procedure

If Begin value < \( p \), move Begin right
Else swap Begin value with End value, move End Left
Done when Begin = End
Partition, Procedure

If \( \text{Begin value} < p \), move \( \text{Begin} \) right
Else swap \( \text{Begin} \) value with \( \text{End} \) value, move \( \text{End} \) Left
Done when \( \text{Begin} = \text{End} \)
Partition, Procedure

If $\text{Begin}$ value $< \ p$, move $\text{Begin}$ right
Else swap $\text{Begin}$ value with $\text{End}$ value, move $\text{End}$ Left
Done when $\text{Begin} = \text{End}$

Case 1: meet at element $< p$
Swap $p$ with pointer position (2 in this case)
Partition, Procedure

If Begin value < \( p \), move Begin right
Else swap Begin value with End value, move End Left
Done when Begin = End

Case 2: meet at element > \( p \)
Swap \( p \) with value to the left (2 in this case)
Partition Summary

1. Put $p$ at beginning of list
2. Put a pointer (Begin) just after $p$, and a pointer (End) at the end of the list
3. While Begin < End:
   1. If Begin value < $p$, move Begin right
   2. Else swap Begin value with End value, move End Left
4. If pointers meet at element < $p$: Swap $p$ with pointer position
5. Else If pointers meet at element > $p$: Swap $p$ with value to the left

Run time? $O(n)$
Conquer

Recursively sort **Left** and **Right** sublists

![Array diagram]

- All elements $< p$
- All elements $> p$

Exactly where it belongs!

Recursively sort **Left** and **Right** sublists
Quicksort Run Time (Best)

If the pivot is always the median:

Then we divide in half each time

\[ T(n) = 2T\left(\frac{n}{2}\right) + n \]

\[ T(n) = O(n \log n) \]
Quicksort Run Time (Worst)

If the pivot is always at the extreme:

```
1  5  2  3  6  4  7  8  10  9  11  12
```

Then we shorten by 1 each time

\[
T(n) = T(n - 1) + n
\]

\[
T(n) = O(n^2)
\]
Quicksort Run Time (Worst)

\[ T(n) = T(n - 1) + n \]

\[ T(n) = 1 + 2 + 3 + \cdots + n \]

\[ T(n) = \frac{n(n + 1)}{2} \]

\[ T(n) = O(n^2) \]
Quicksort on a (nearly) Sorted List

First element always yields unbalanced pivot

So we shorten by 1 each time

\[ T(n) = T(n - 1) + n \]

\[ T(n) = O(n^2) \]
Good Pivot

• What makes a good Pivot?
  • Roughly even split between left and right
  • Ideally: median
• There are ways to find the median in linear time, but it’s complicated and slow and you’re better off using mergesort
• In Practice:
  • Pick a random value as a pivot
  • Pick the middle of 3 random values as the pivot
Properties of Quick Sort

• Worst Case Running time:
  • $\Theta(n^2)$
  • But $\Theta(n \log n)$ average! And typically faster than mergesort!

• In-Place?
  • ....Debatable

• Adaptive?
  • No!

• Stable?
  • No!
More Formal Definition

• Input:
  • An array $A$ of items
  • A comparison function for these items
    • Given two items $x$ and $y$, we can determine whether $x < y$, $x > y$, or $x = y$

• Output:
  • A permutation of $A$ such that if $i \leq j$ then $A[i] \leq A[j]$
  • Permutation: a sequence of the same items but perhaps in a different order
Improving Running time

• Recall our definition of the sorting problem:
  • Input:
    • An array $A$ of items
    • A comparison function for these items
      • Given two items $x$ and $y$, we can determine whether $x < y$, $x > y$, or $x = y$
  • Output:
    • A permutation of $A$ such that if $i \leq j$ then $A[i] \leq A[j]$

• Under this definition, it is impossible to write an algorithm faster than $n \log n$ asymptotically.

• Observation:
  • Sometimes there might be ways to determine the position of values without comparisons!
“Linear Time” Sorting Algorithms

• Useable when you are able to make additional assumptions about the contents of your list (beyond the ability to compare)
  • Examples:
    • The list contains only positive integers less than $k$
    • The number of distinct values in the list is much smaller than the length of the list

• The running time expression will always have a term other than the list’s length to account for this assumption
  • Examples:
    • Running time might be $\Theta(k \cdot n)$ where $k$ is the range/count of values
BucketSort

• Assumes the array contains integers between 0 and \( k - 1 \) (or some other small range)

• Idea:
  • Use each value as an index into an array of size \( k \)
  • Add the item into the “bucket” at that index (e.g. linked list)
  • Get sorted array by “appending” all the buckets
BucketSort Running Time

• Create array of $k$ buckets
  • Either $\Theta(k)$ or $\Theta(1)$ depending on some things...
• Insert all $n$ things into buckets
  • $\Theta(n)$
• Empty buckets into an array
  • $\Theta(n + k)$
• Overall:
  • $\Theta(n + k)$
• When is this better than mergesort?
Properties of BucketSort

• In-Place?
  • No
• Adaptive?
  • No
• Stable?
  • Yes!
RadixSort

- Radix: The base of a number system
  - We’ll use base 10, most implementations will use larger bases

- Idea:
  - BucketSort by each digit, one at a time, from least significant to most significant

Place each element into a “bucket” according to its 1’s place

<table>
<thead>
<tr>
<th>103</th>
<th>801</th>
<th>401</th>
<th>323</th>
<th>255</th>
<th>823</th>
<th>999</th>
<th>101</th>
<th>113</th>
<th>901</th>
<th>555</th>
<th>512</th>
<th>245</th>
<th>800</th>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
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RadixSort

• Radix: The base of a number system
  • We’ll use base 10, most implementations will use larger bases

• Idea:
  • BucketSort by each digit, one at a time, from least significant to most significant

| 800 | 801 | 401 | 101 | 901 | 121 | 512 | 103 | 323 | 823 | 113 | 255 | 555 | 245 | 018 | 999 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |

Place each element into a “bucket” according to its 10’s place

| 800 | 801 | 401 | 101 | 901 | 103 | 512 | 113 | 018 | 121 | 323 | 823 | 245 | 255 | 555 | 999 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
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Place each element into a “bucket” according to its 10’s place
RadixSort

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• Idea:
  • BucketSort by each digit, one at a time, from least significant to most significant

Place each element into a “bucket” according to its 100’s place
RadixSort

• Radix: The base of a number system
  • We’ll use base 10, most implementations will use larger bases

• Idea:
  • BucketSort by each digit, one at a time, from least significant to most significant

Convert back into an array
RadixSort Running Time

• Suppose largest value is $m$
• Choose a radix (base of representation) $b$
• BucketSort all $n$ things using $b$ buckets
  • $\Theta(n + k)$
• Repeat once per each digit
  • $\log_b m$ iterations
• Overall:
  • $\Theta(n \log_b m + b \log_b m)$
• In practice, you can select the value of $b$ to optimize running time
• When is this better than mergesort?