



CSE 332: Data Structures & Parallelism

Lecture 5: Algorithm Analysis II

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Winter 2022

Today

- Analyzing Recursive Code
- Solving Recurrences

Analyzing code (“worst case”)

Basic operations take “some amount of” **constant time**

- Arithmetic
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful “lie”.)

Consecutive statements

Sum of time of each statement

Loops

Num iterations * time for loop body

Conditionals

Time of condition plus time of slower branch

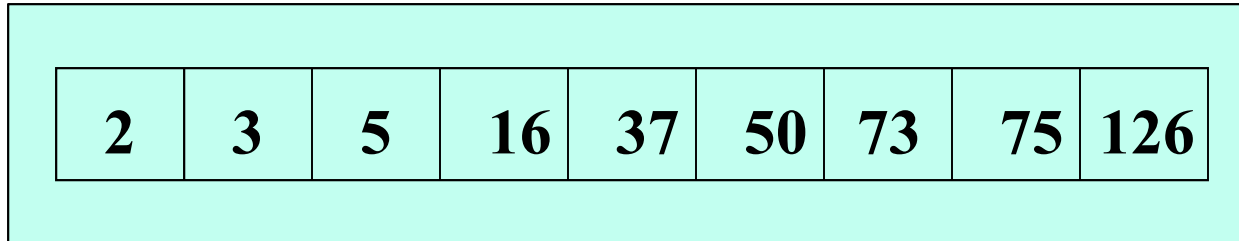
Function Calls

Time of function’s body

 Recursion

Solve *recurrence equation*

Linear search



Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k){
    for(int i=0; i < arr.length; ++i)
        if(arr[i] == k)
            return true;
    return false;
}
```

Best case: 6 “ish” steps = $O(1)$

Worst case: 5 “ish” * (arr.length)
= $O(\text{arr.length})$

$O(N)$

Analyzing Recursive Code

- Computing run-times gets interesting with recursion
- Say we want to perform some computation recursively on a list of size n
 - Conceptually, in each recursive call we:
 - Perform some amount of work, call it w(n)
 - Call the function recursively with a smaller portion of the list

- So, if we do w(n) work per step, and reduce the problem size in the next recursive call by 1, we do total work:

$$\underline{T(n)} = \underline{w(n)} + \underline{T(n-1)} \quad \leftarrow \text{Recurrence Relation}$$

- With some base case, like T(1) = 5 = O(1) c₁

Example Recursive code: sum array

Recursive:

- Recurrence is some constant amount of work $O(1)$ done n times

```
int sum(int[] arr) {  
    return help(arr, 0);  
}  
int help(int[] arr, int i) {  
    if (i == arr.length)  
        return 0;  
    return arr[i] + help(arr, i + 1);  
}
```

Base case: $T(0) = 3 = c_2 \Rightarrow O(1)$

Each time **help** is called, it does that $O(1)$ amount of work, and then calls **help** again on a problem one less than previous problem size.

Recurrence Relation: $T(n) = O(1) + T(n-1)$

Solving Recurrence Relations

- Say we have the following recurrence relation:

$$\rightarrow T(n) = 6 \text{ "isn"} + T(n-1)$$

$$T(1) = 9 \text{ "isn"}$$

← base case

- Now we just need to solve it; that is, reduce it to a closed form.
- Start by writing it out:

$$\begin{aligned} \Rightarrow T(n) &= 6 + T(n-1) \\ &= 6 + 6 + T(n-2) \\ &= 6 + 6 + 6 + T(n-3) \quad 9 \\ &= 6 + 6 + 6 + \dots + 6 + T(1) = 6 + 6 + 6 + \dots + 6 + 9 \\ &= 6k + T(n-k) \quad \leftarrow \\ &= 6k + 9, \text{ where } k \text{ is the \# of times we expanded } T() \end{aligned}$$

- We expanded it out $n-1$ times, so

$$\begin{aligned} T(n) &= 6k + T(n-k) \quad 9 \\ &= 6(n-1) + T(1) = 6(n-1) + 9 \\ &= 6n + 3 = O(n) \end{aligned}$$

Or When does $n-k=1$?
 Answer: when $k=n-1$

Binary search

Best case:

Worst case:

2	3	5	16	37	50	73	75	126
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Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k){
    return help(arr, k, 0, arr.length);
}

boolean help(int[] arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2; //i.e., lo+(hi-lo)/2
    if(lo==hi) return false;
    if(arr[mid]==k) return true;
    if(arr[mid]<k) return help(arr, k, mid+1, hi);
    else return help(arr, k, lo, mid);
}
```

only one of these will happen

Recurrence Relation: $T(n) = C_1 + T\left(\frac{n}{2}\right)$
Base case: $T(1) = C_2$

Binary search

Best case: 9 “ish” steps = $O(1)$

Worst case: $T(n) = 10$ “ish” + $T(n/2)$ where n is `hi-lo`

- $O(\log n)$ where n is `array.length`
- Solve *recurrence equation* to know that...

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
    return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2;
    if(lo==hi)        return false;
    if(arr[mid]==k)  return true;
    if(arr[mid]< k)  return help(arr,k,mid+1,hi);
    else             return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?

→ $T(n) = 10 + T(n/2)$ $T(1) = 15$

2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.

$$T(n) = 10 + 10 + T\left(\frac{n}{4}\right)$$

$$T(n) = 10 + 10 + 10 + T\left(\frac{n}{8}\right)$$

$$T(n) = 10 \cdot K + T\left(\frac{n}{2^k}\right)$$

when is $\frac{n}{2^k} = 1$?

$$\log_2(2^k) = \log_2(n)$$

$$K = \log_2 n$$

3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

$$T(n) = 10 \cdot \log_2 n + T(1)$$

→ $= 10 \log_2 n + 15 \rightarrow O(\log n)$

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
 - $T(n) = 10 + T(n/2)$ $T(1) = 15$
2. “Expand” the original relation to find an equivalent general expression *in terms of the number of expansions*.
 - $T(n) = 10 + 10 + T(n/4)$
= $10 + 10 + 10 + T(n/8)$
= ...
= $10k + T(n/(2^k))$ (where k is the number of expansions)
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
 - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
 - So $T(n) = 10 \log_2 n + 15$ (get to base case and do it)
 - So $T(n)$ is $O(\log n)$

sum array again

Two “obviously” linear algorithms: $T(n) = O(1) + T(n-1)$

Iterative:

```
int sum(int[] arr) {
    int ans = 0;
    for(int i=0; i<arr.length; ++i)
        ans += arr[i];
    return ans;
}
```

Recursive:

- Recurrence is
 $c + c + \dots + c$
for n times

```
int sum(int[] arr) {
    return help(arr,0);
}
int help(int[]arr,int i) {
    if(i==arr.length)
        return 0;
    return arr[i] + help(arr,i+1);
}
```

What about a binary version of sum?

```
int sum(int[] arr) {
    return help(arr, 0, arr.length);
}
int help(int[] arr, int lo, int hi) {
    if(lo==hi) return 0;
    if(lo==hi-1) return arr[lo];
    int mid = (hi+lo)/2;
    return help(arr, lo, mid) + help(arr, mid, hi);
}
```

What about a binary version of sum?

```
int sum(int[] arr) {
    return help(arr, 0, arr.length);
}
int help(int[] arr, int lo, int hi) {
    if(lo==hi) return 0;
    if(lo==hi-1) return arr[lo];
    int mid = (hi+lo)/2;
    return help(arr, lo, mid) + help(arr, mid, hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$

- $1 + 2 + 4 + 8 + \dots$ for $\log n$ times
- $2^{(\log n)} - 1$ which is proportional to n (by definition of logarithm)

Easier explanation: it adds each number once while doing little else

“Obvious”: You can’t do better than $O(n)$ – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
 - Like having a friend do half the work for you!

```
int sum(int[] arr) {
    return help(arr, 0, arr.length);
}
int help(int[] arr, int lo, int hi) {
    if (lo == hi) return 0;
    if (lo == hi - 1) return arr[lo];
    int mid = (hi + lo) / 2;
    return help(arr, lo, mid) + help(arr, mid, hi);
}
```

- If you have as many “friends of friends” as needed, the recurrence is now $T(n) = O(1) + 1T(n/2)$
 - $O(\log n)$: same recurrence as for **find**

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$\rightarrow T(n) = O(1) + T(n/2)$	logarithmic	$O(\log n)$
$\rightarrow T(n) = O(1) + 2T(n/2)$	linear	$O(n)$
$T(n) = O(1) + T(n-1)$	linear	$O(n)$
$T(n) = O(n) + T(n-1)$	quadratic	$O(n^2)$
$T(n) = O(1) + 2T(n-1)$	exponential	$O(2^n)$
$T(n) = O(n) + T(n/2)$	linear	$O(n)$
$T(n) = O(n) + 2T(n/2)$	loglinear	$O(n \log n)$

Note big-Oh can also use more than one variable

- Example: can sum all elements of an n -by- m matrix in $O(nm)$