CSE332: Data Structures & Parallelism

Lecture 2: Algorithm Analysis

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Winter 2022

Today – Algorithm Analysis

- What do we care about?
- How to compare two algorithms
- Analyzing Code
- Asymptotic Analysis
- Big-Oh Definition

What do we care about?

- Correctness:
  - Does the algorithm do what is intended.

- Performance:
  - Speed — time complexity
  - Memory — space complexity

- Why analyze?
  - To make good design decisions
  - Enable you to look at an algorithm (or code) and identify the bottlenecks, etc.

Q: How should we compare two algorithms?

A: How should we compare two algorithms?

- Uh, why NOT just run the program and time it??
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when *n* doubles in size)

- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation ("coding it up")

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is "plenty good" for small inputs (if *n* is 10, probably anything is fast enough)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"

- Can do analysis before coding!

Today – Algorithm Analysis

- What do we care about?
- How to compare two algorithms
- Analyzing Code
  - How to count different code constructs
  - Best Case vs. Worst Case
  - Ignoring Constant Factors
- Asymptotic Analysis
- Big-Oh Definition

Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of time of each statement
Loops	Num iterations * time for loop body
Conditionals	Time of condition plus time of slower branch
Function Calls	Time of function's body
Recursion	Solve *recurrence equation*

Examples

```
b = b + 5
c = b / a
b = c + 100
```

2 ops
2 ops
2 ops
6 ops

O(1)
constant

```
for (i = 0; i < n; i++) {
    sum++;
}
```

1
1
2
2
1

n * 5ops + 2
5n + 2 → O(n)
linear

```
if (j < 5) {
    sum++;
} else {
    for (i = 0; i < n; i++) {
        sum++;
    }
}
```

2ops

5n + 2

cond + branch
n + n → O(n)

Another Example

```
int coolFunction(int n, int sum) {
    int i, j;
    for (i = 0; i < n; i++) {
        for (j = 0; j < n; j++) {
            sum++;
        }
    }
    print "This program is great!"
    for (i = 0; i < n; i++) {
        sum++;
    }
    return sum
}
```

5n

n · 5n

1

5n

1

$n \cdot 5n + 1 + 5n + 1 \Longrightarrow O(n^2)$

$5n^2$

Using Summations for Loops

```
for (i = 0; i < n; i++) {
   sum++;
   }
```

$$1 + \sum_{i=0}^{n-1} 5 + 1 = 1 + \underbrace{5 + 5 + 5 + \cdots + 5}_{n \text{ times}} + 1$$

$$= 2 + 5n \Longrightarrow O(n)$$

Complexity cases

scenario

We'll start by focusing on two cases:

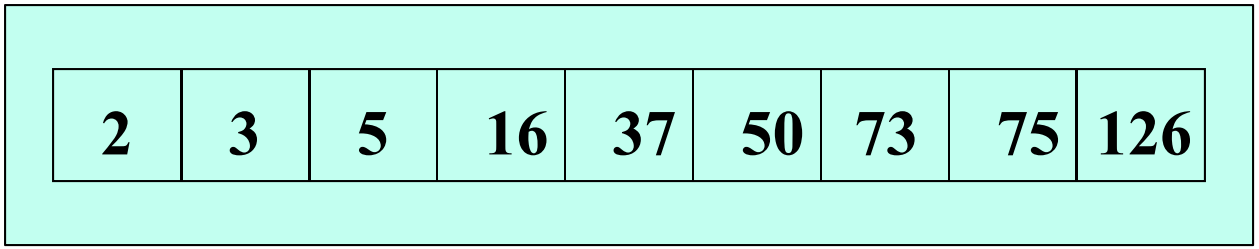
- **Worst-case complexity**: max # steps algorithm takes on "most challenging" input of size N

- **Best-case complexity**: min # steps algorithm takes on "easiest" input of size N

Average

Amortized - later

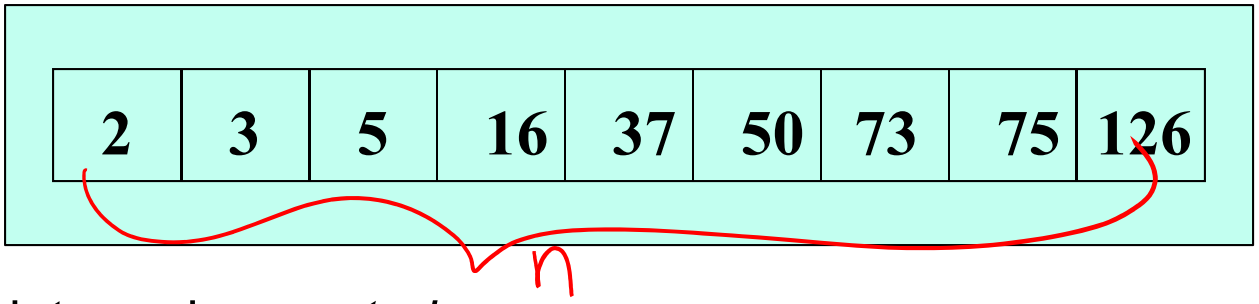
Example



Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
    ???
}
```

Linear search – Best Case & Worst Case



Find an integer in a *sorted* array

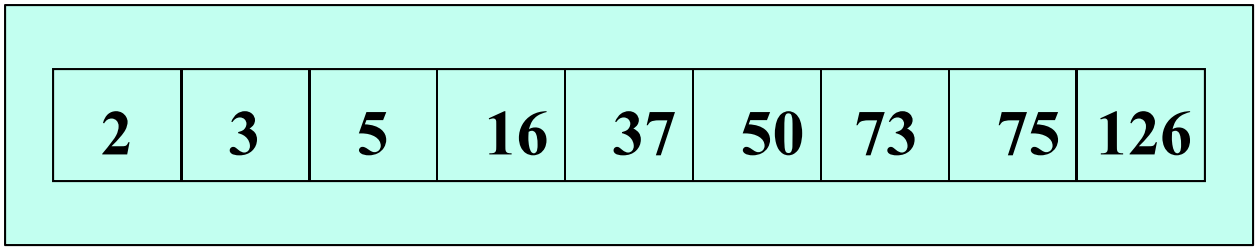
```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
    for(int i=0; i < arr.length; ++i)
        if(arr[i] == k)
            return true;
    return false;
}
```

60ps

Best case: Find(2) ~60ps O(1)
find 1st item in array

Worst case: Find(127) 6·n + 2
find something not present O(n)

Linear search – Running Times



Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6 "ish" steps = *O*(1)

Worst case: 5 "ish" * (arr.length)
= *O*(arr.length)

Remember a faster search algorithm?

Binary Search
O(log n)

- O(n)
- O(log n)
- O(1)

Linear Search
O(n)

Ignoring constant factors

- So binary search is $O(\log n)$ and linear is $O(n)$
  - But which will actually be faster?
  - Depending on **constant factors** and **size of n**, in a particular situation, linear search could be faster….

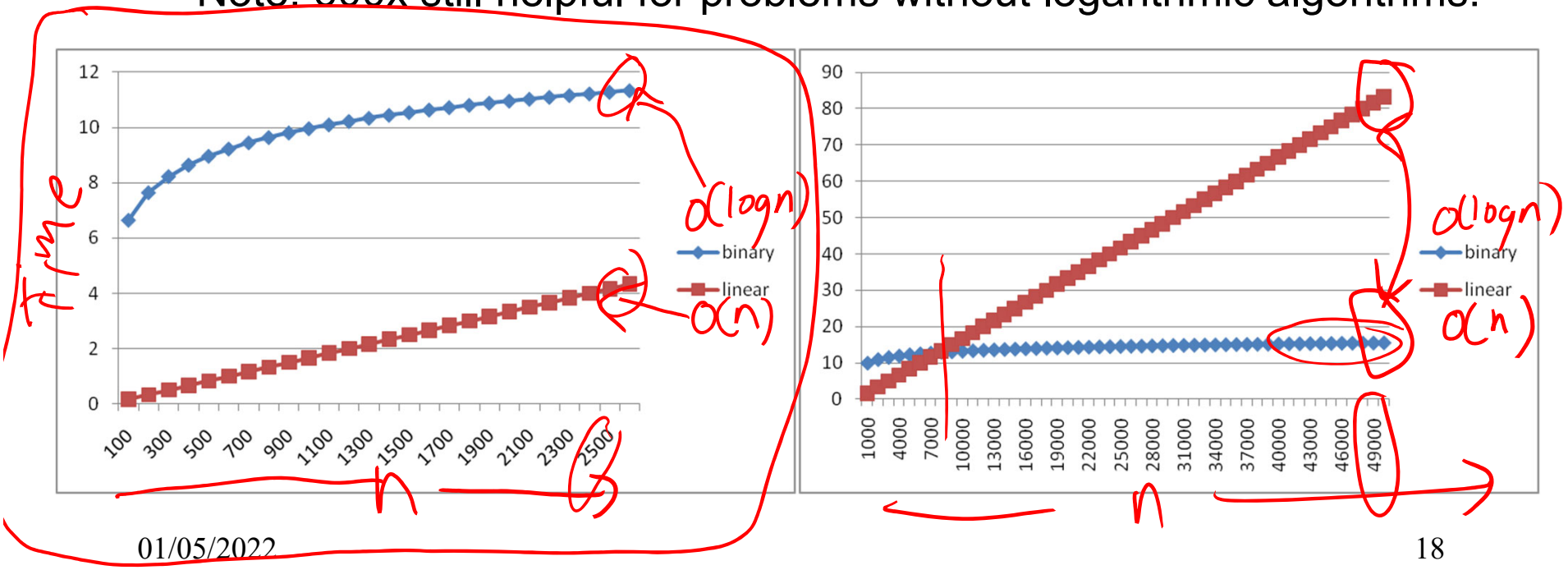
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each *n*
- And could depend on size of *n*

- **But** there exists some n_0 such that for all $n > n_0$ binary search "wins"

- Let's play with a couple plots to get some intuition…

Example

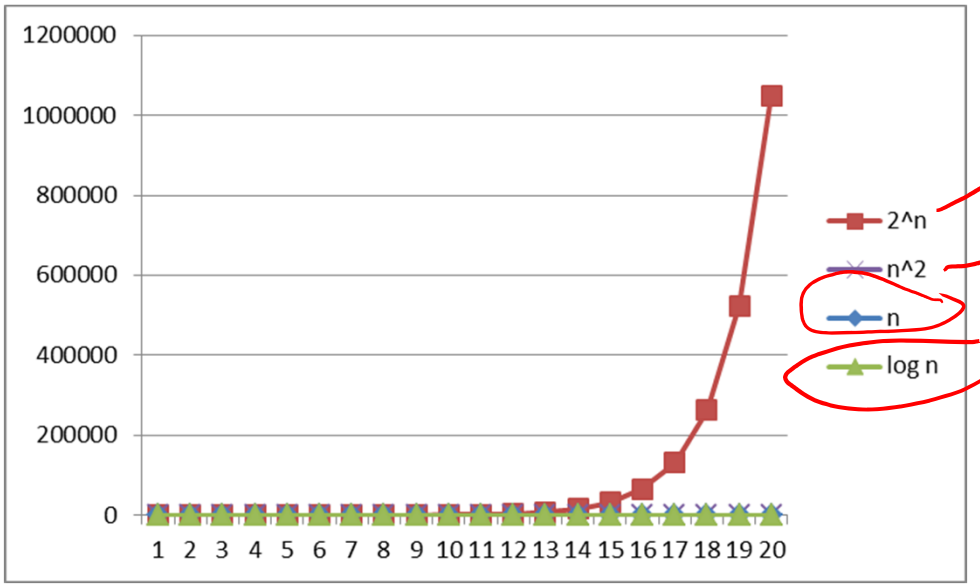
- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2018 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!



Logarithms and Exponents

- Since so much is binary in CS, **log** almost always means $\log_2$
- Definition: $\log_2 x = y$ if $x = 2^y$
- So, $\log_2$ 1,000,000 = "a little under 20"
- Just as exponents grow *very* quickly, logarithms grow *very* slowly

See Excel file for plot data – play with it!



Aside: Log base doesn't matter (much)

"Any base *B* log is equivalent to base 2 log within a constant factor"

- **And we are about to stop worrying about constant factors**!
- In particular, $\log_2 x = 3.22 \log_{10} x$
- In general, we can convert log bases via a constant multiplier
- Say, to convert from base B to base A:

$$\log_B x = (\log_A x) / (\log_A B)$$

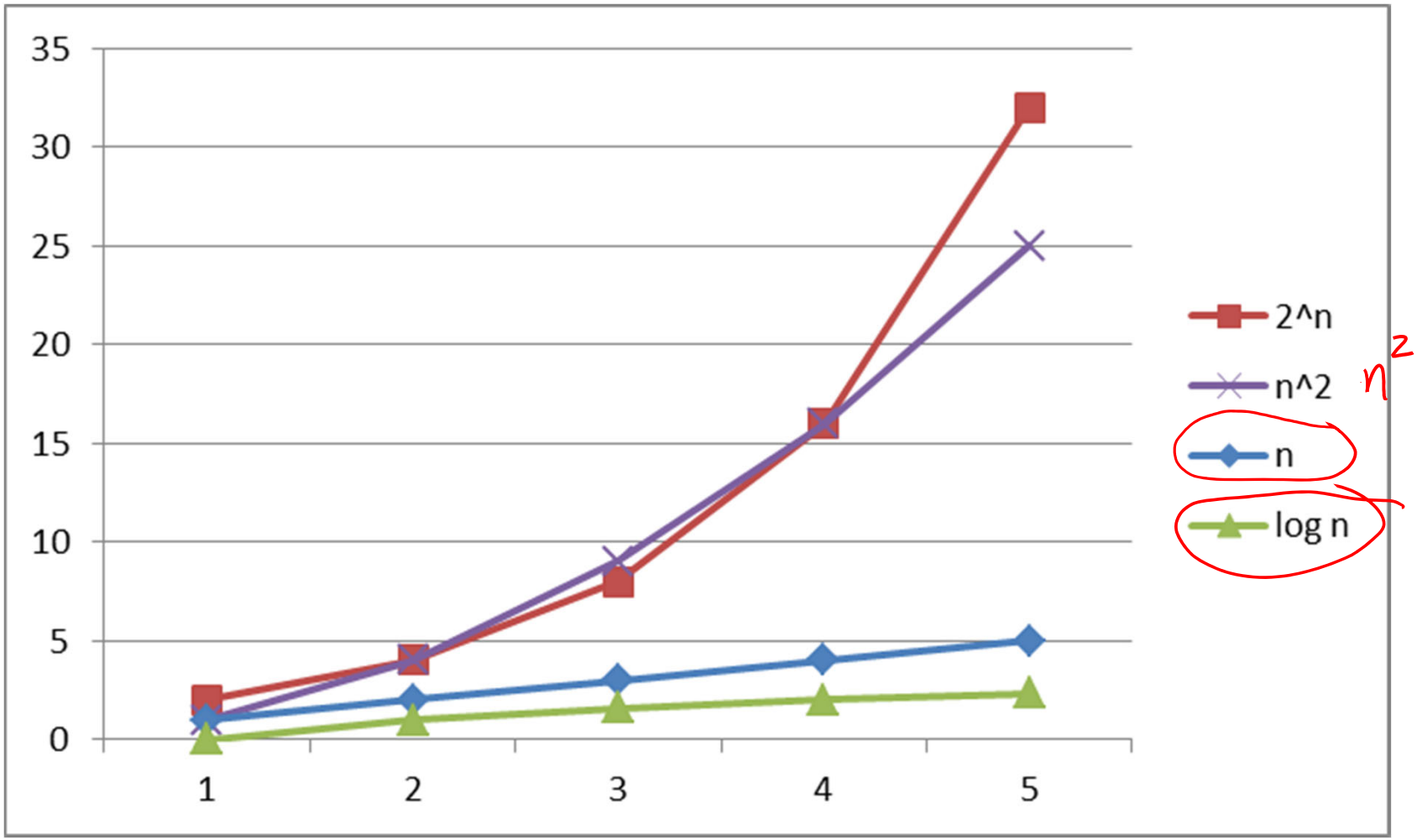
Review: Properties of logarithms

- **`log(A*B) = log A + log B`**
  - So **`log(N^k)= k log N`**
- **`log(A/B) = log A - log B`**
- **`X =`** $\log_2 2^x$
- **`log(log x)`** is written **`log log x`**
  - Grows as slowly as 2^{2^y} grows fast
  - Ex:

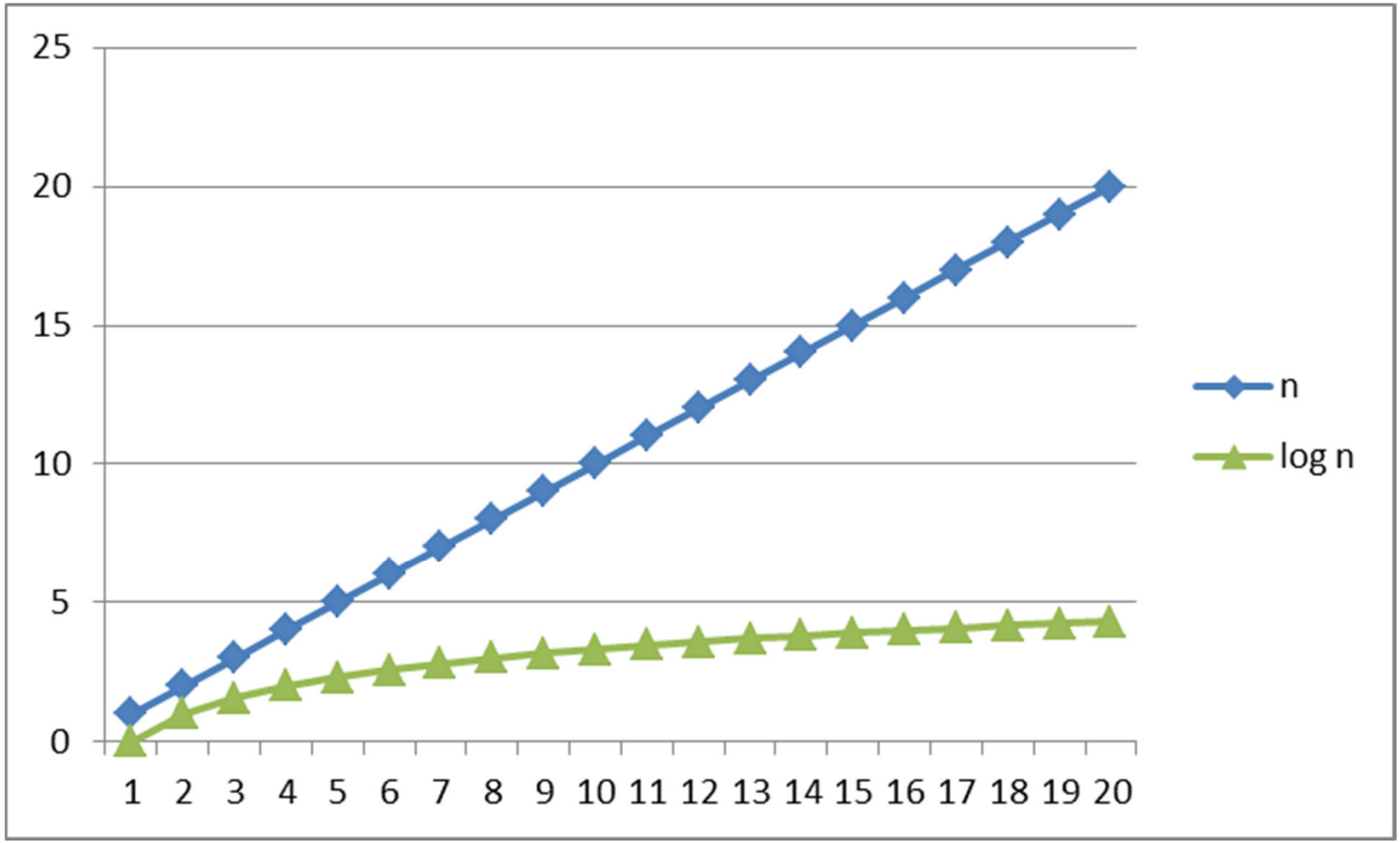
$$\log_2 \log_2 4billion \sim \log_2 \log_2 2^{32} = \log_2 32 = 5$$

- **`(log x)(log x)`** is written **`log²x`**
  - It is greater than **`log x`** for all **`x > 2`**

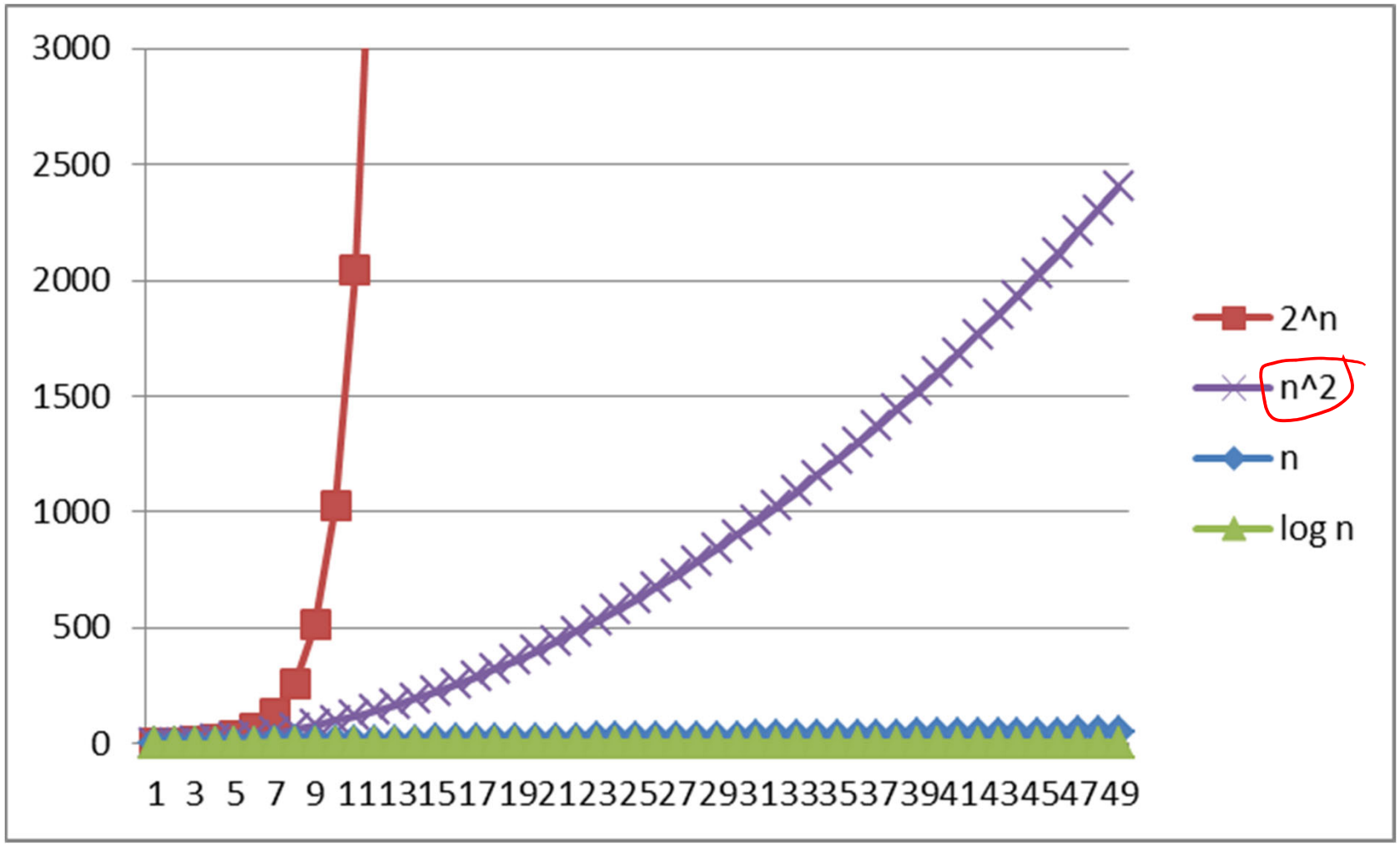
Logarithms and Exponents



Logarithms and Exponents



Logarithms and Exponents



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- Analyzing Code
- Asymptotic Analysis
- Big-Oh Definition

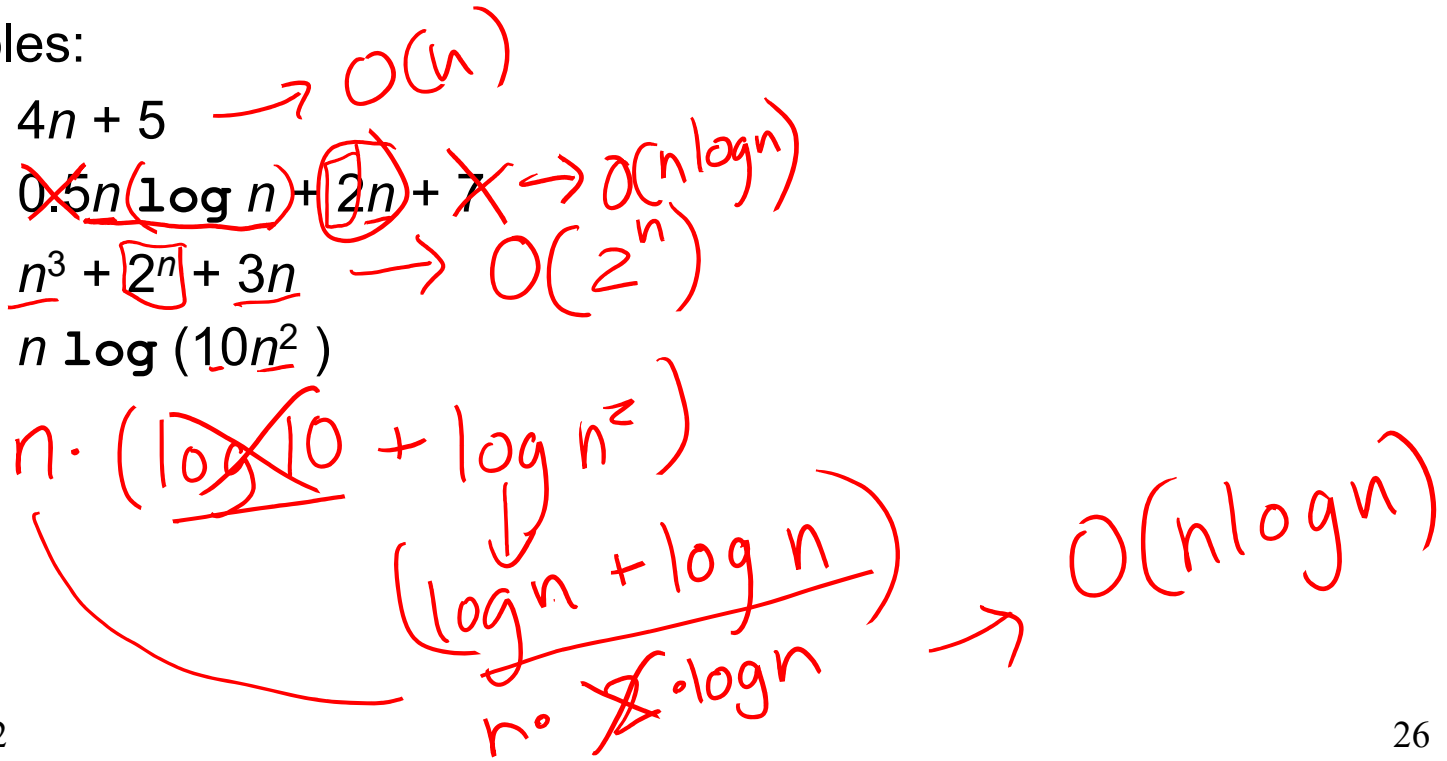
Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n \log n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \log (10n^2)$



Big-Oh relates functions

We use *O* on a function f(*n*) (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f(*n*)

So $(3n^2+17)$ **is in** $O(n^2)$

- $3n^2+17$ and n^2 have the same **asymptotic behavior**

Confusingly, we also say/write:

- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17)$ **=** $O(n^2)$

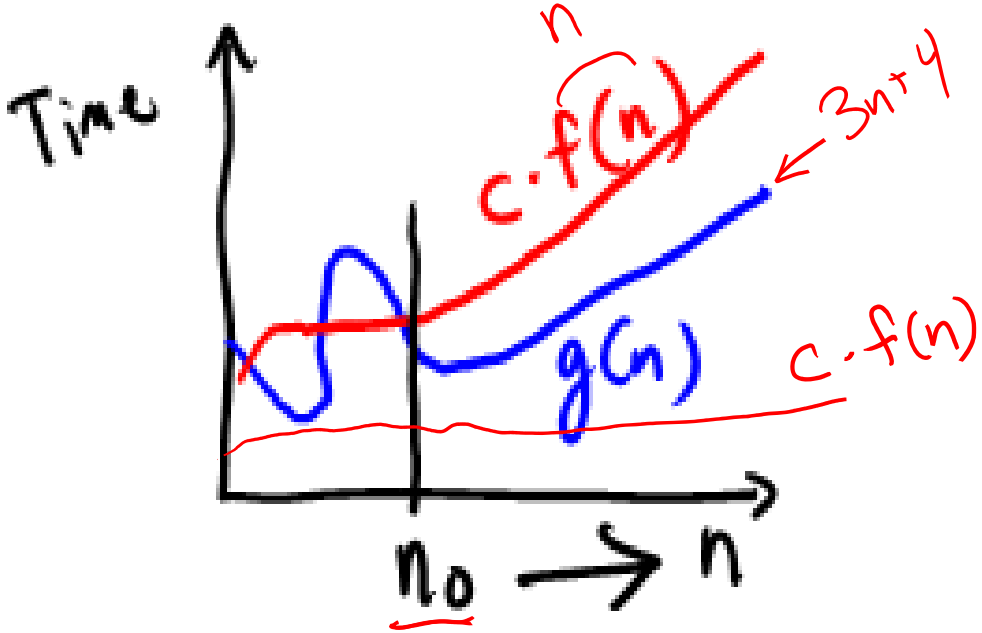
But we would never say $O(n^2) = (3n^2+17)$

Formally Big-Oh

Definition: **g(*n*)** is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

$$g(n) \leq c\, f(n) \quad \text{for all } n \geq n_0$$

Note: $n_0 \geq 1$ (and a natural number) and $c > 0$



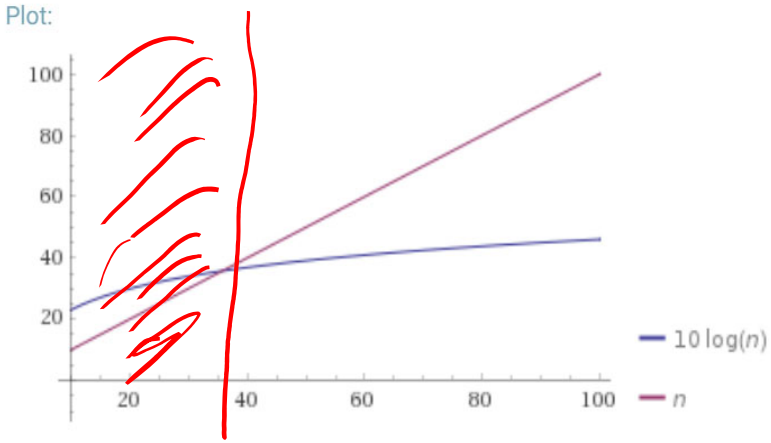
Why n_0? Why c?

Definition: **g(*n*)** is in O(**f(*n*)**) iff there exist positive constants ***c*** and ***n_0*** such that

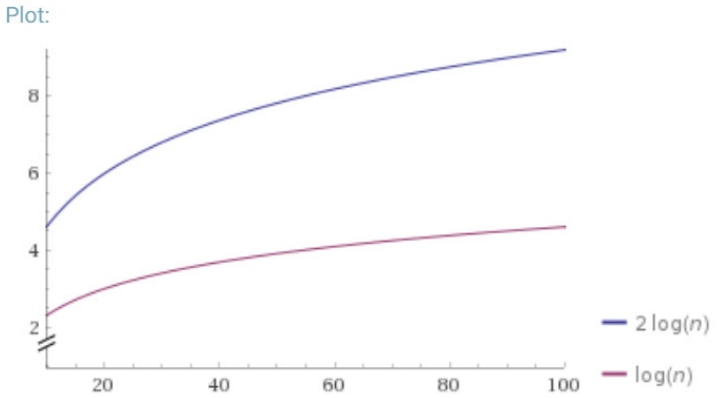
g(*n*) ≤ *c* f(*n*) for all ***n*** ≥ ***n_0***

Note: *n_0* ≥ 1 (and a natural number) and *c* > 0

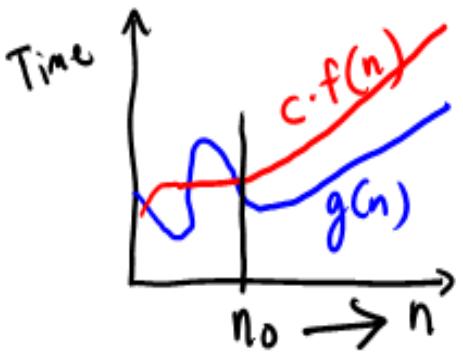
Why n_0?



Why c?



Formally Big-Oh



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Note: $n_0 \geq 1$ (and a natural number) and $c > 0$

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms".

Example: Let **g(*n*)** = $3n + 4$ and **f(*n*)** = n

$c = 4$ and $n_0 = 5$ is one possibility

$3n+4$ is in $O(n)$ for all $n \geq 5$

$3n+4 \leq 4 \cdot n$

$3 \cdot 5 + 4 \leq 4 \cdot 5$ ✓

$19 \leq 20$

$3 \cdot 6 + 4 \leq 4 \cdot 6$ ✓

$22 \leq 24$

This is "less than or equal to"

– So $3n + 4$ is also $O(n^5)$ and $O(2^n)$ etc.

*What's with the **c**?*

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called **c**)
- Consider:

  g(n) = 7n+5

  f(n) = n
- These have the same asymptotic behavior (linear), so **g(n)** is in O(**f(n)**) even though **g(n)** is always larger

- There is **no** positive n_0 such that **g(n)** ≤ **f(n)** for all n ≥ n_0
- The '**c**' in the definition allows for that:

  g(*n*) ≤ ***c*** **f(*n*)** for all $n \geq n_0$
- To show **g(n)** is in O(**f(n)**), have **c** = 12, n_0 = 1

An Example

c must be > 0
n_0 must be ≥ 1 (natural #)

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

- Example: Let **g(*n*)** = $4n^2 + 3n + 4$ and **f(*n*)** = n^3

We want to show $4n^2 + 3n + 4 \leq c \cdot n^3$ for all $n \geq n_0$

Note that:

$4n^2 \leq 4n^3$ $\quad n_0 = 1$

$3n \leq 3n^3$

$4 \leq 4n^3$

$4n^2 + 3n + 4 \leq 11 \cdot n^3$ $\quad c = 11$, $n_0 = 1$

Using the definition of Big-Oh (Example 2)

For **g(n)** = 4n & **f(n)** = n^2, show **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & $\mathbf{n_0}$
- When n=4, **g(n)** =16 & **f(n)** =16; this is the crossing over point
- So we can choose $\mathbf{n_0}$ = 4, and **c** = 1

- Note: There are many possible choices:
  ex: $\mathbf{n_0}$ = 78, and **c** = 42 works fine

The Definition: g(n) is in O(f(n)) iff there exist *positive* constants c and n_0 such that

$$g(n) \leq c\, f(n) \text{ for all } n \geq n_0.$$

Using the definition of Big-Oh (Example 3)

For **g(n)** = n^4 & **f(n)** = 2^n, show **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & $\mathbf{n_0}$
- One possible answer: $\mathbf{n_0}$ = 20, and **c** = 1

The Definition: g(n) is in O(f(n)) iff there exist *positive* constants c and n_0 such that

$$g(n) \leq c\, f(n) \text{ for all } n \geq n_0.$$

Examples

True or false?

1. $4+3n$ is $O(n)$ – True
2. $n+2\log n$ is $O(\log n)$ – False
3. $\log n+2$ is $O(1)$ – False
4. n^{50} is $O(1.1^n)$ – True

Notes:

- Do NOT ignore constants that are not multipliers:
  - n^3 is $O(n^2)$: FALSE
  - 3^n is $O(2^n)$: FALSE
- When in doubt, refer to the definition

Examples (Answers)

True or false?

1. $4+3n$ is $O(n)$ — True
2. $n+2\log n$ is $O(\log n)$ — False
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Notes:

- Do NOT ignore constants that are not multipliers:
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What you can drop

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we cannot count operations very accurately
- Eliminate low-order terms because they have vanishingly small impact as n grows
- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

(This all follows from the formal definition)

Big Oh: Common Categories

From fastest to slowest

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\log n)$	logarithmic
$O(n)$	linear
$O(n \log n)$	"n log n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant > 1)
$O(k^n)$	exponential (where k is any constant > 1)

$O(\log \log n)$

$O(\log^c n)$ for $c > 1$

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"

More Asymptotic Notation

- **Upper bound**: *O*(**f(n)**) is the set of all functions asymptotically less than or equal to **f(n)**
  - g(*n*) is in *O*(**f(n)**) if there exist constants ***c*** and n_0 such that

    g(*n*) $\leq$ ***c*** **f(n)** for all $n \geq n_0$

- **Lower bound**: Ω(**f(n)**) is the set of all functions asymptotically greater than or equal to **f(n)**
  - g(*n*) is in Ω(**f(n)**) if there exist constants ***c*** and n_0 such that

    g(*n*) $\geq$ ***c*** **f(n)** for all $n \geq n_0$

- **Tight bound**: θ(**f(n)**) is the set of all functions asymptotically equal to **f(n)**
  - Intersection of *O*(**f(n)**) and Ω(**f(n)**) (can use *different* ***c*** values)

g(n) is $\theta(f(n))$ if both g(n) is $O(f(n))$ AND g(n) is $\Omega(f(n))$

Regarding use of terms

A common error is to say $O(\ f(n)\ )$ when you mean $\theta(\ f(n)\ )$

- People often say O() to mean a tight bound
- Say we have f(n)=n; we could say f(n) is in O(n), which is true, but only conveys the upper-bound
- Since f(n)=n is *also* $O(n^5)$, it's tempting to say "this algorithm is *exactly* $O(n)$"
- Somewhat incomplete; instead say it is $\theta(n)$
- That means that it is not, for example $O(\log n)$

Less common notation:

- "little-oh": like "big-Oh" but strictly less than
  - Example: sum is $o(n^2)$ but not $o(n)$
- "little-omega": like "big-Omega" but strictly greater than
  - Example: sum is $\omega(\log n)$ but not $\omega(n)$

What we are analyzing

- The most common thing to do is give an O or θ bound to the worst-case running time of an algorithm

- Example: True statements about binary-search algorithm
  - Common: $\theta(\log n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is $\Omega(\log \log n)$ in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega(\log n)$ in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time

- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312

- Sometimes an *amortized guarantee*

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for **large *n*** and is independent of any computer / coding trick
  - But you can "abuse" it to be misled about trade-offs
  - Example: $n^{1/10}$ vs. `log` n
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the "cross-over" point is around $5 * 10^{17}$
    - So if you have input size less than 2^{58}, prefer $n^{1/10}$
- Comparing O() for ***small n*** values can be misleading
  - Quicksort: O(nlogn) (expected)
  - Insertion Sort: $O(n^2)$ (expected)
  - Yet in reality Insertion Sort is faster for small n's
  - We'll learn about these sorts later

Addendum: Timing vs. Big-Oh?

- At the core of CS is a backbone of theory & mathematics
  - Examine the algorithm itself, mathematically, not the implementation
  - Reason about performance as a function of n
  - Be able to mathematically prove things about performance
- Yet, timing has its place
  - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful