# CSE 332: Data Structures & Parallelism
# Lecture 21: MST


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Summer 2022

# Announcements

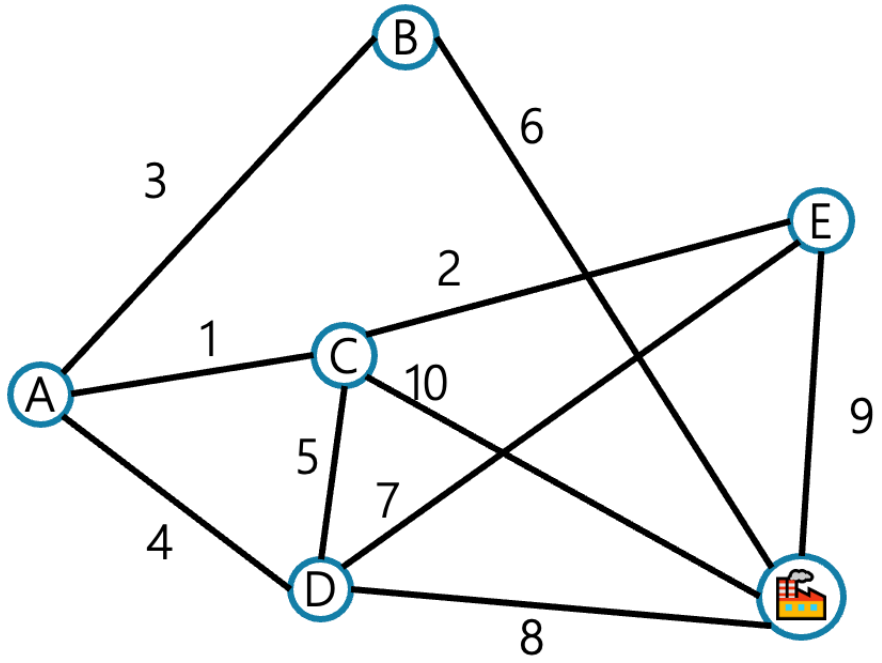
- Final Review Session: MOR 220 Wed 10/17 from 3:00-4:00pm
  - There might be snacks, so definitely come through 😎

# Outline for Today

- Intro to MST
- Prim’s algorithm
- Kruskal’s algorithm

# Time travel – 1920s

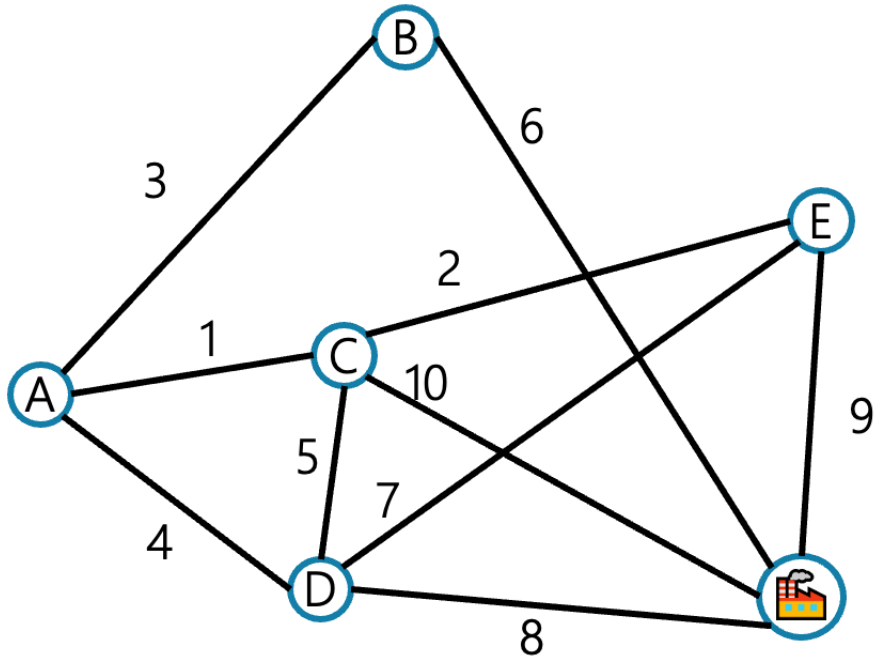
It's the 1920's. Your friend works at an electric company. They want to know where to build electrical wires to connect all cities to the powerplant.




They know how much it would cost to lay electric wires between any pair of locations, and they want the cheapest way to make sure there's electricity from the plant to every city.

# Time travel – 1950s

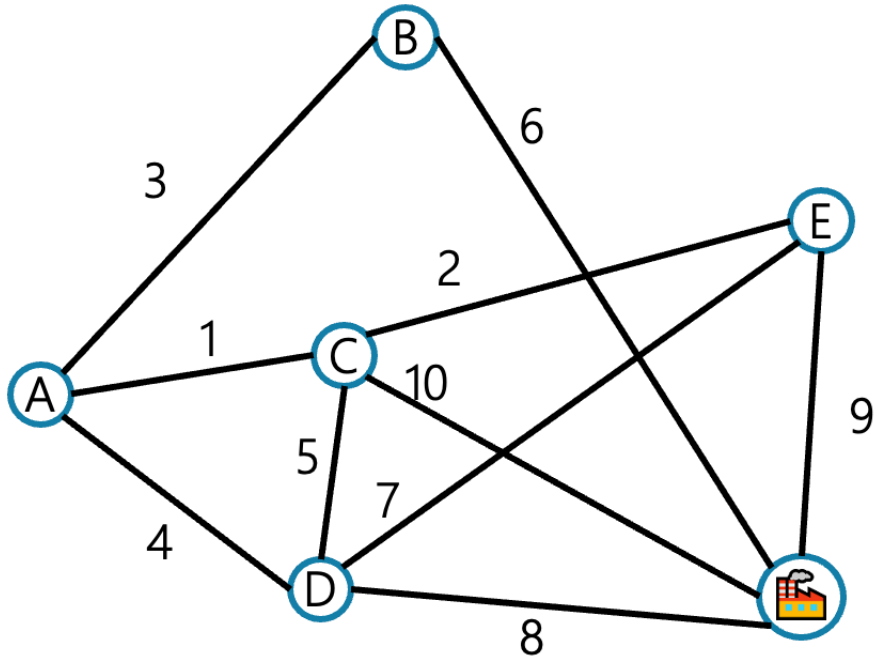
It's the 1950's. Your friend works at a phone company. They want to know where to build phone wires to connect phones to each other.




They know how much it would cost to lay phone wires between any pair of locations, and they want the cheapest way to make sure everyone can call everyone else.

# Time travel – 2020s

It's 2022! Your friend works at an internet company. They want to know where to build internet cables to connect all cities to the Internet.




They know much it would cost to lay internet cables between any pair of locations, and they want the cheapest way to make sure everyone can reach the server.

# What are we looking for?

A set of edges such that…

1. Every vertex touches at least one edge
2. Graph on these edges is connected
3. Sum of edge weights is minimized

Claim: The set of edges we pick never has a cycle

# Aside: Tree!

1. Don’t need a root
2. Varying numbers of children
3. Connected and no cycles

**Tree (when talking about undirected graphs)**

An undirected, connected acyclic graph.

# MST Problem

A set of edges such that…

1. Edges span the graph
2. Graph on these edges is connected
3. Sum of edge weights is minimized
4. Contains no cycle

We need a minimum spanning tree

**Minimum Spanning Tree Problem**

**Given:** an undirected, weighted graph G
**Find:** A minimum-weight set of edges such that you can get from any vertex of G to any other on only those edges.

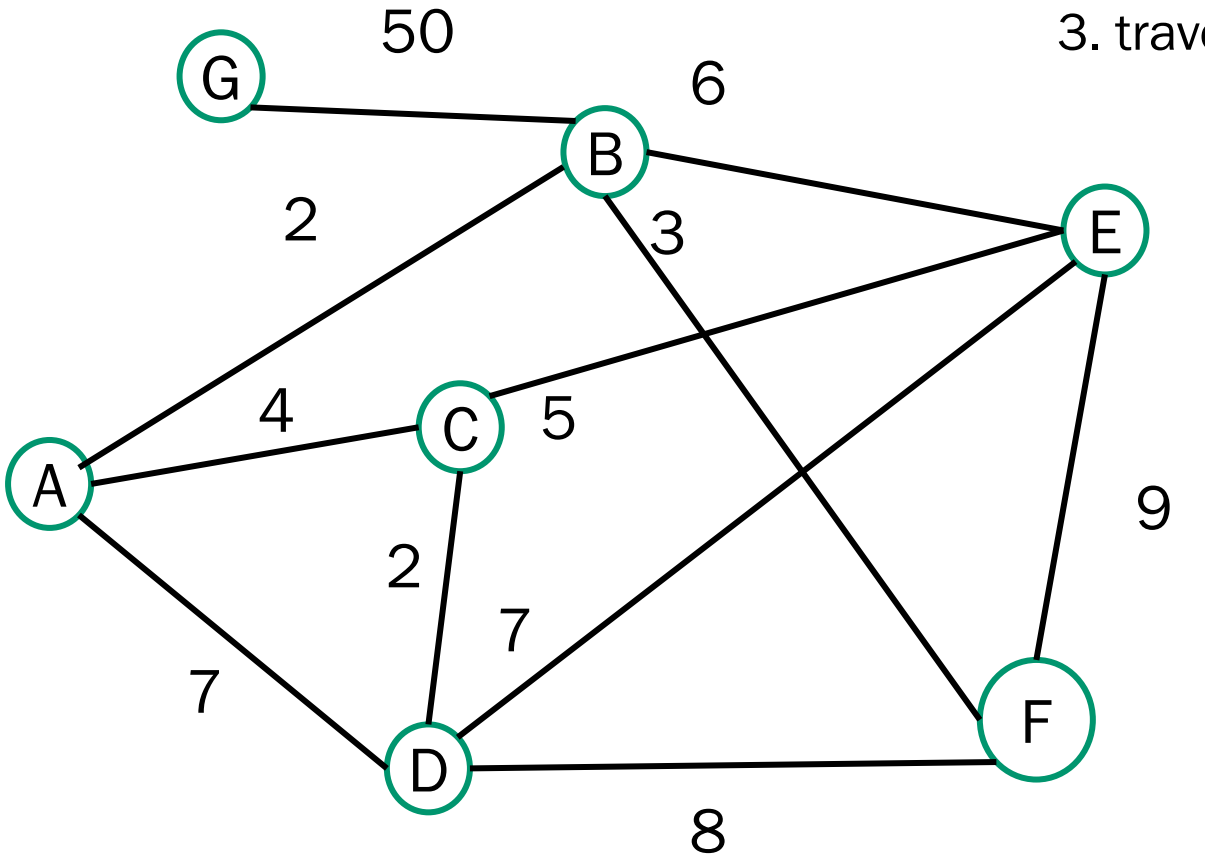
# Prim’s Algorithm

- Choose arbitrary starting point
- Add a new edge to the result each step
- How to choose the new edge?
  - Will let you reach more vertices
  - Is as light as possible

# Try it out

Prim's travelling strategy as a broke college student!

1. research neighboring cities (update table)
2. decide on the cheapest city to travel to
3. travel to that city (mark graph)!




| Vertex | Dist. | Best Edge | Processed |
|---|---|---|---|
| A | | | |
| B | | | |
| C | | | |
| D | | | |
| E | | | |
| F | | | |
| G | | | |

# Try it out

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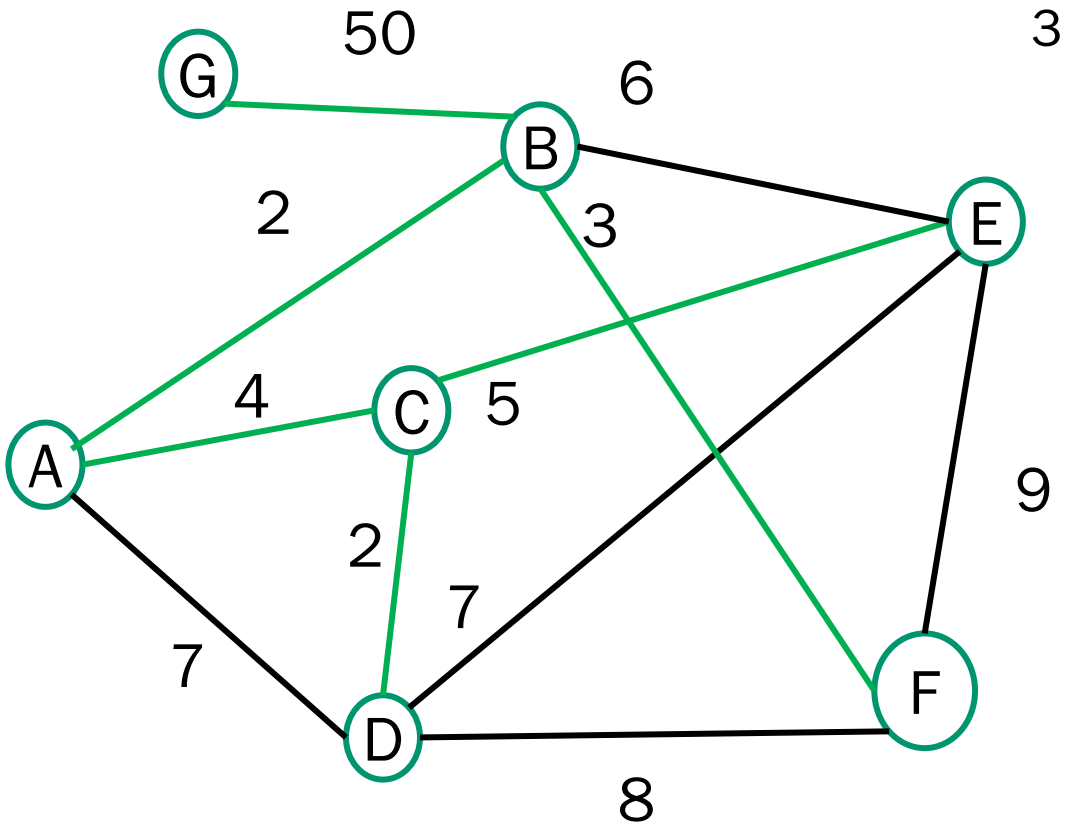



| Vertex | Dist. | Best Edge | Processed |
|---|---|---|---|
| A | -- | -- | Yes |
| B | 2 | (A,B) | Yes |
| C | 4 | (A,C) | Yes |
| D | ~~7~~ 2 | ~~(A,D)~~(C,D) | Yes |
| E | ~~6~~ 5 | ~~(B,E)~~(C,E) | Yes |
| F | 3 | (B,F) | Yes |
| G | 50 | (B,G) | Yes |

# Pseudocode

```
initialize distances to ∞
        mark source as distance 0
        mark all vertices unprocessed
        foreach(edge (source, v) )
                v.dist = w(source,v)
        while(there are unprocessed vertices){
                let u be the closest unprocessed vertex
                add u.bestEdge to spanning tree
                foreach(edge (u,v) leaving u){
                        if(w(u,v) < v.dist){
                                v.dist = w(u,v)
                                v.bestEdge = (u,v)
                        }
                }
        mark u as processed
}
```

Dijkstra's: keep track the **minimum distance** from the starting vertex to vertex **v**

Prim: keep track of the weight of the **minimum single edge** connecting vertex **v** to everything else that's already been connected

# Does Prim's Algorithm Always Work?

- Prim's Algorithm is a greedy algorithm
- Always select the best option available at the moment
- Once it decides to include an edge in the MST, it never reconsiders its decision
- Life lesson? Be greedy!
  - Always thinking about global optimum might stress you out
  - Instead, make the best decision everyday and hope for the best!
  - The information provided on this page does not, and is not intended to, constitute professional advice. When in doubt, stop being greedy.

# A different Approach

- Prim’s algorithm - vertex by vertex

- What if you think edge by edge instead?

- Start from the lightest edge
  - add it if it connects new things to each other
  - don’t add it if it would create a cycle

- This is Kruskal’s Algorithm

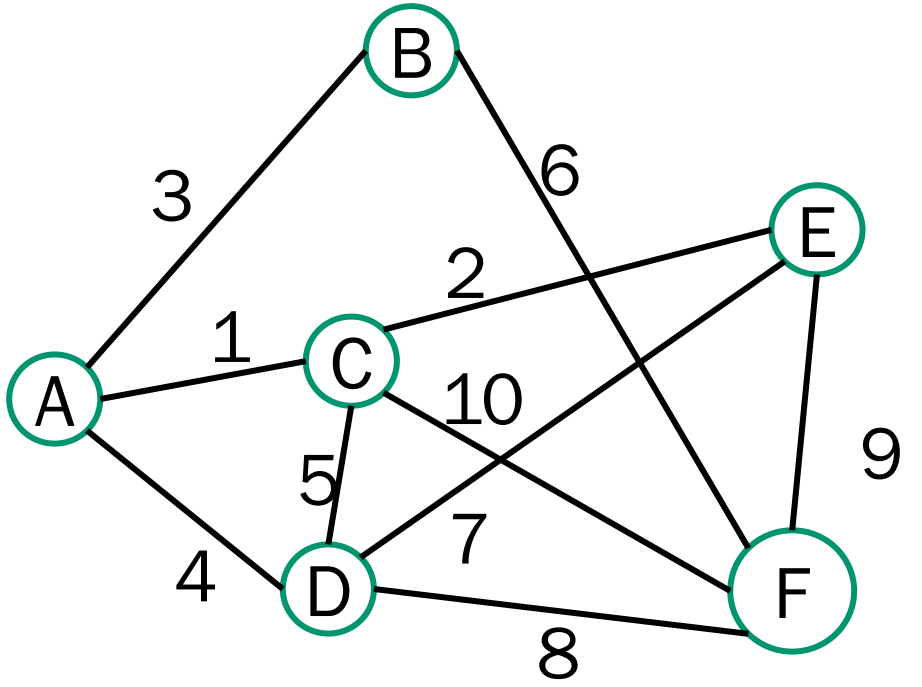
# Kruskal's Algorithm

```
KruskalMST(Graph G)
    initialize each vertex to be a connected component
      sort the edges by weight
      foreach(edge (u, v) in sorted order){
           if(u and v are in different components){
                add (u,v) to the MST
                Update u and v to be in the same
component
           }
      }
```

# Try It Out

Two simple steps:

1. If edge connects different clouds
   then add edge, combine clouds
2. Otherwise, ignore




| Edge | Include? | Reason |
|---|---|---|
| (A,C) | | |
| (C,E) | | |
| (A,B) | | |
| (A,D) | | |
| (C,D) | | |

| Edge | Include? | Reason |
|---|---|---|
| (B,F) | | |
| (D,E) | | |
| (D,F) | | |
| (E,F) | | |
| (C,F) | | |

# Kruskal's Algorithm: Running Time

```
KruskalMST(Graph G)
    initialize each vertex to be a connected component
      sort the edges by weight
      foreach(edge (u, v) in sorted order){
           if(u and v are in different components){
                add (u,v) to the MST
                Update u and v to be in the same
component
           }
      }
```

# Union-Find

## Union-Find ADT

**state**

Set of Sets

- **Disjoint:** No element appears in multiple sets
- No required order
- Each set has representative

**behavior**

makeSet(x) – creates a new set where the only member (and the representative) is x.

findSet(x) – looks up the set containing element x, returns name of that set

union(x, y) – combines sets named x and y. Picks new name for combined set.

# Union-Find Running Time

What's important for us?

Amortized running times! We care about the total time across the entire set of unions and finds, not the running time of just one.

| Operation | Amortized | Non-amortized |
|---|---|---|
| MakeSet() | $\Theta(1)$ | $\Theta(1)$ |
| Union() | $O(\log^* n)$ | $O(\log n)$ |
| Find() | $O(\log^* n)$ | $O(\log n)$ |

# *log * n*

$\log^* n$ : the number of times you need to apply $\log()$ to get a number at most 1.

E.g. $\log^*(16) = 3$

$$\log(16) = 4 \qquad \log(4) = 2 \qquad \log(2) = 1.$$

$\log^* n$ grows ridiculously slowly.

$$\log^*(10^{80}) = 5.$$

For all practical purposes, $\log^* n$ operations are constant time

# Using Union-Find

Have each disjoint set represent a connected component

When you add an edge, you **union** those connected components.

How do you know if two vertices belong to the same component?

# Kruskal's with Union Find

```
KruskalMST(Graph G)
  initialize each vertex to be a connected
component
    sort the edges by weight
    foreach(edge (u, v) in sorted order){
        if(find(u) != find(v)){
            add (u,v) to the MST
            union(find(u),find(v))          }
    }
```

# Full circle

Let’s time travel again!

# Time travel (again) – 1920s

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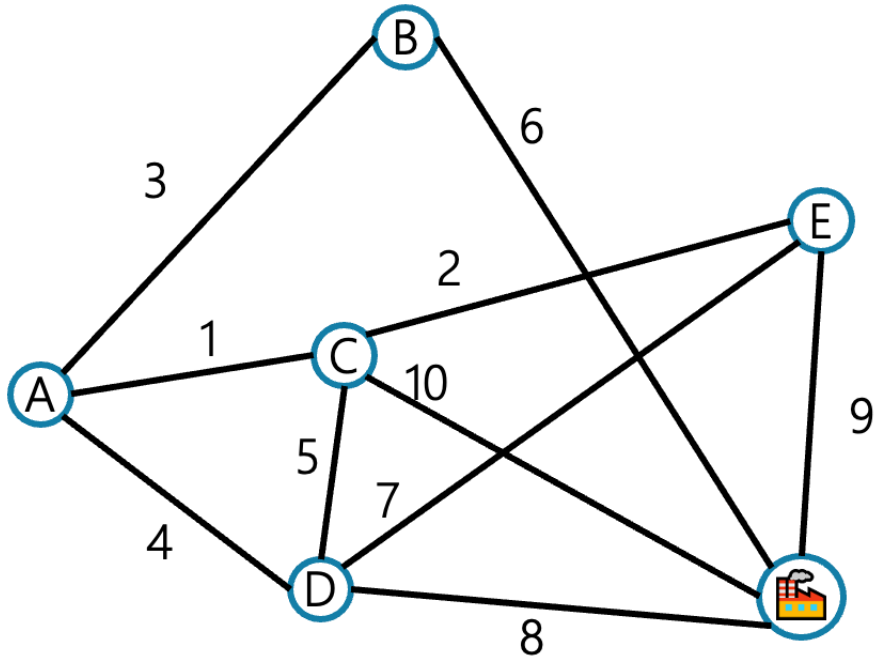


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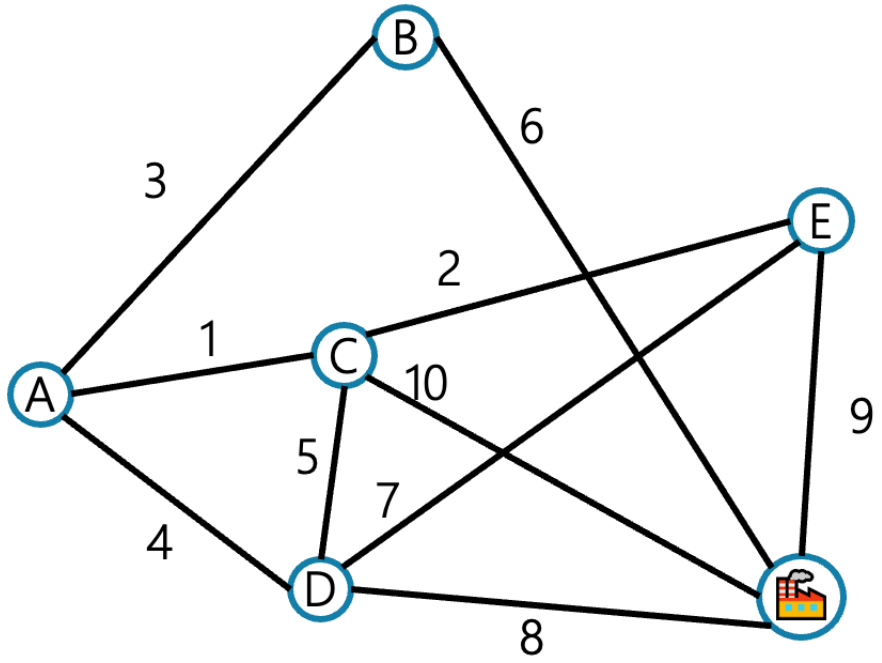
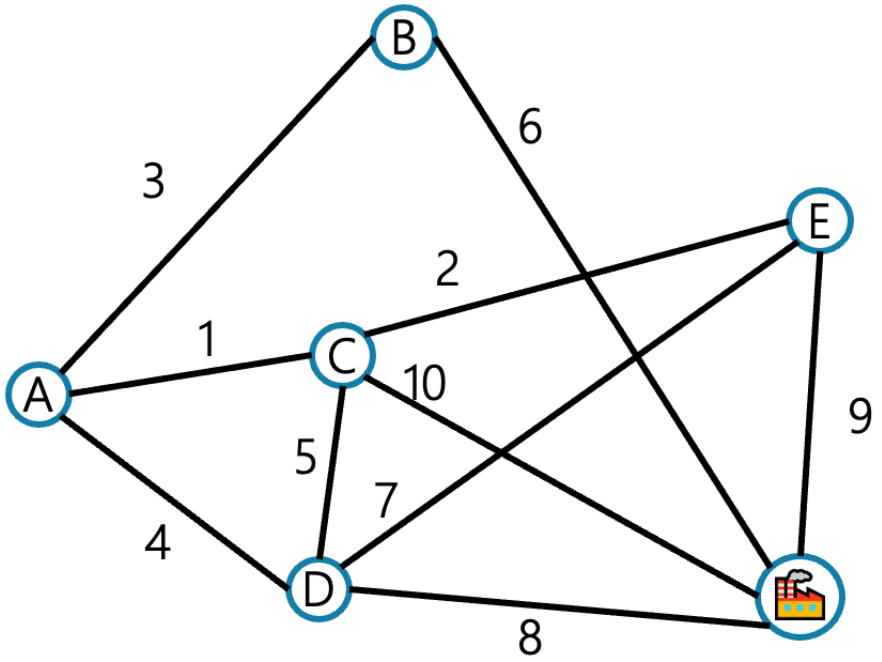



They know how much it would cost to lay phone wires between any pair of locations, and they want the cheapest way to make sure everyone can call everyone else.

# Time travel (again) – 2020s

It's 2022! You work at an internet company. They want to know where to build internet cables to connect all cities to the Internet. Maybe you will come up with the next MST algorithm!



# A Graph of Trees

- Recall a tree is an undirected, connected, and acyclic graph.
- How would we describe the graph Kruskal's builds?
  - It's not a tree until the end.

**It's a forest!**

**Forest**

any undirected and acyclic graph

EVERY TREE IS A FOREST.