Note: Exercises and P2 start counting from 0

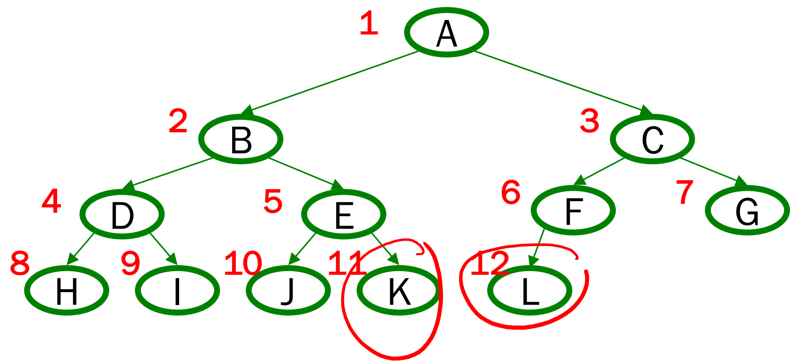
Array Representation of a Binary Heap

From node i:

left child: 2i

right child: 2i+1

parent: floor(i/2)



	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

- We skip index 0 to make the math simpler
- Actually, it can be a good place to store the current size of the heap

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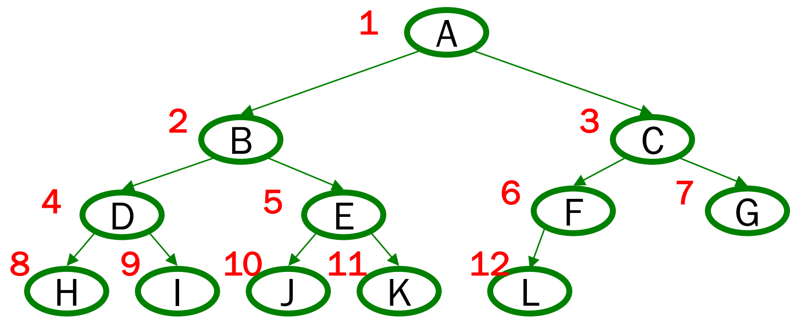


	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

- We skip index 0 to make the math simpler
- Actually, it can be a good place to store the current size of the heap

Note: Exercises and P2 start counting from 0

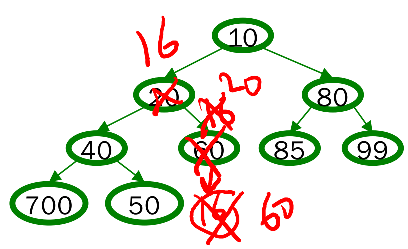
Pseudocode: insert

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
void insert(int val) {
  if(size==arr.length-1)
    resize();
  size++;
  i=percolateUp(size,val);
  arr[i] = val;
}
```

```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
        val < arr[hole/2]){
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```

insert(16)



	10	~~20~~ 16	80	40	~~60~~ ~~16~~ 20	85	99	700	50	~~16~~ 60			
0	1	2	3	4	5	6	7	8	9	10	11	12	13

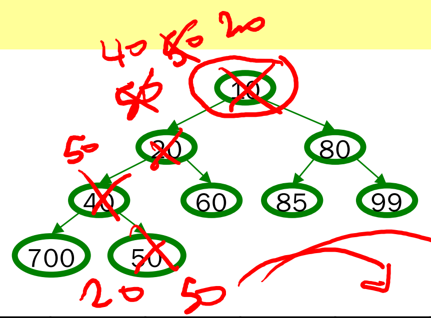
Note: Exercises and P2 start counting from 0

Pseudocode: deleteMin

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
int deleteMin() {
  if(isEmpty()) throw…
  ans = arr[1];
  hole = percolateDown
           (1,arr[size]);
  arr[hole] = arr[size];
  size--;
  return ans;
}
```

```
int percolateDown(int hole,
                  int val) {
 while(2*hole <= size) {
  left  = 2*hole;
  right = left + 1;
  if(arr[left] < arr[right]
     || right > size)
    target = left;
  else
    target = right;
  if(arr[target] < val) {
    arr[hole] = arr[target];
    hole = target;
  } else
      break;
 }
 return hole;
}
```



	~~10~~	~~20~~	80	40	60	85	99	700	~~50~~				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

50

Poll Everywhere

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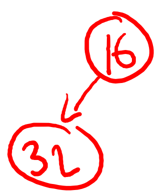
Note: Exercises and P2 start counting from 0

Example

1. insert: 16, 32, 4, 57, 80, 43, 2
2. deleteMin

3. What are the 2 properties of minheaps



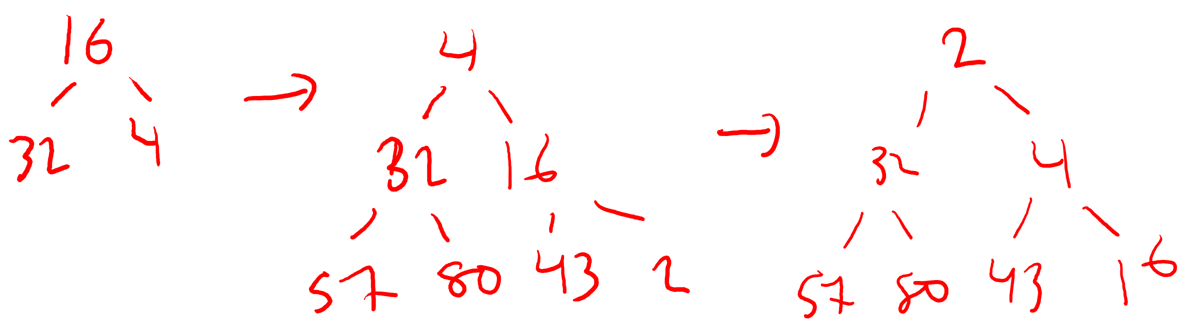
0	1	2	3	4	5	6	7

Note: Exercises and P2 start counting from 0

Example

1. insert: ~~16~~, ~~32~~, ~~4~~, ~~57~~, ~~80~~, ~~43~~, 2
2. deleteMin

	2	32	4	57	80	43	16
0	1	2	3	4	5	6	7

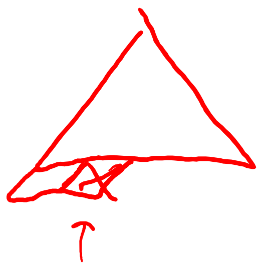


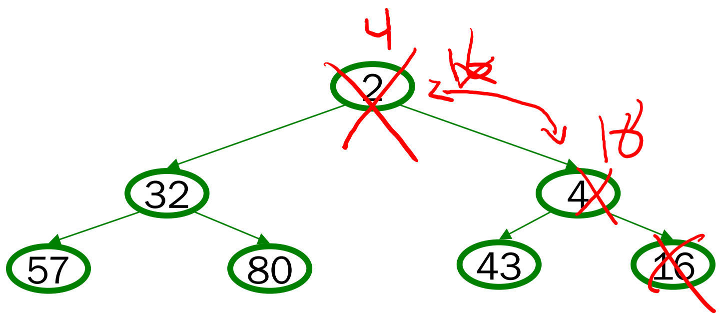
Note: Exercises and P2 start counting from 0

Example: After insertion

1. insert: 16, 32, 4, 57, 80, 43, 2
2. deleteMin

	4		16				
	~~2~~	32	~~4~~	57	80	43	~~16~~
0	1	2	3	4	5	6	7



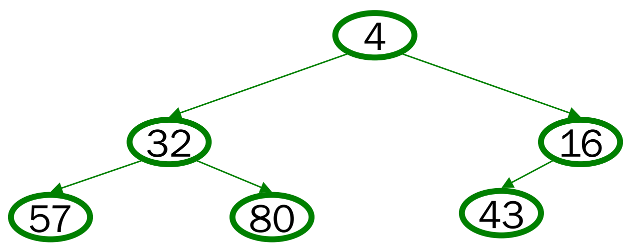


Note: Exercises and P2 start counting from 0

Example: After deletion

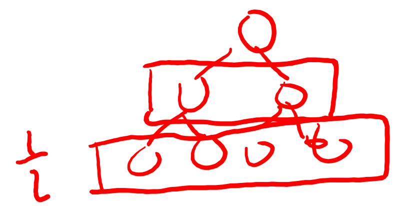
1. insert: 16, 32, 4, 57, 80, 43, 2
2. deleteMin

	4	**32**	**16**	**57**	**80**	**43**	
0	1	2	3	4	5	6	7



So why O(1) average-case insert?

- Yes, insert's **worst case** is O(log n)
- The trick is that it all depends on the order the items are inserted (What is the worst case order?) reverse 5,4,3,2,1
- Experimental studies of randomly ordered inputs shows the following:
  - Average 2.607 comparisons per insert (# of percolation passes)
  - An element usually moves up 1.607 levels



- deleteMin is average O(log n)
  - Moving a leaf to the root usually requires re-percolating that value back to the bottom

Evaluating the Array Implementation…

Advantages:

Minimal amount of wasted space:

- Only index 0 and any unused space on right in the array
- No "holes" due to complete tree property
- No wasted space representing tree edges

Fast lookups:

- Benefit of array lookup speed
- Multiplying and dividing by 2 is extremely fast (can be done through bit shifting (see CSE 351)
- Last used position is easily found by using the PQueue's size for the index

Disdvantages:

- What if the array gets too full (or wastes space by being too empty)? Array will have to be resized.

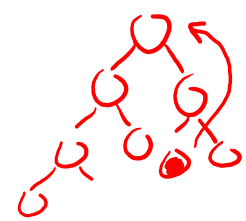
Advantages outweigh Disadvantages: This is how it is done!

Other (specialized) operations

- **`decreaseKey`**: *<u>given pointer</u>* to object in priority queue (e.g., its array index), lower its priority value by *p*   O(log n)
  - Change priority and percolate up
- **`increaseKey`**: *<u>given pointer</u>* to object in priority queue (e.g., its array index), raise its priority value by *p*   O(log n)
  - Change priority and percolate down

- **`remove`**: *<u>given pointer</u>* to object in priority queue (e.g., its array index), remove it from the queue
  - **`decreaseKey`** with $p = \infty$, then **`deleteMin`**

O(log n) + O(log n) = O(log n)



Running time for all these operations?

Building a Heap

Suppose you have *n* items you want to put in a new priority queue

- A sequence of *n* **`insert`** operations works
- Runtime? $n \cdot O(\log n) = O(n \log n)$

Can we do better?

- If we only have access to **`insert`** and **`deleteMin`** operations, then NO.
- There is a faster way - O(n), but that requires the ADT to have a specialized **`buildHeap`** operation

Floyd's `buildHeap` Method

Recall our general strategy for working with the heap:

- Preserve structure property
- Break and restore heap ordering property

Floyd's ***`buildHeap`***:

1. Create a complete tree by putting the n items in array indices 1, ... N

   (Requires having all the elements that we want to insert all at once!)

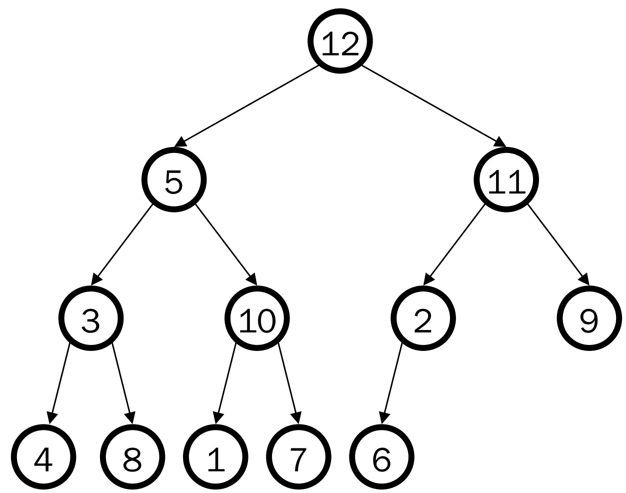
2. Treat the array as a heap and fix the heap-order property

   Exactly how we do this is where we gain efficiency

Thinking about `buildHeap`

- Say we start with this array:
  [12,5,11,3,10,2,9,4,8,1,7,6]

- To "fix" the ordering should we use:
  - percolateUp?
  - percolateDown?



Note: Exercises and P2 start counting from 0

Floyd's `buildHeap` Method

percolateDown, **bottom-up**:

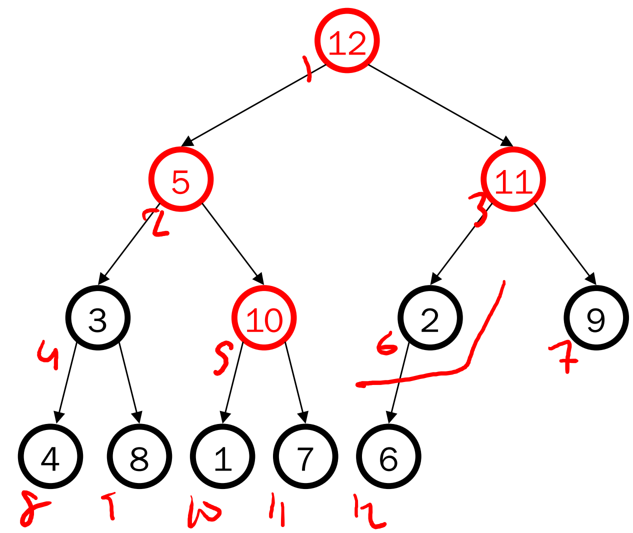
- Leaves are already in heap order
- Work up toward the root one level at a time

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

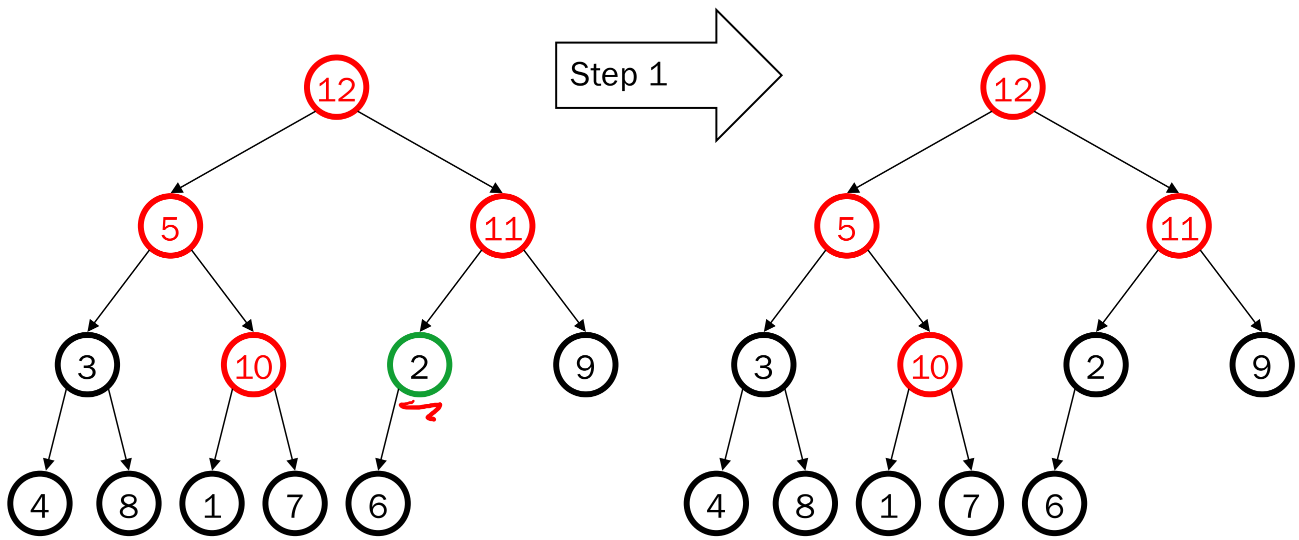
`buildHeap` Example

- Say we start with this array:
  [12,5,11,3,10,2,9,4,8,1,7,6]

- In tree form for readability
  - Red for node not less than descendants
    - heap-order problem
  - Notice no leaves are red
  - Check/fix each non-leaf bottom-up (6 steps here)

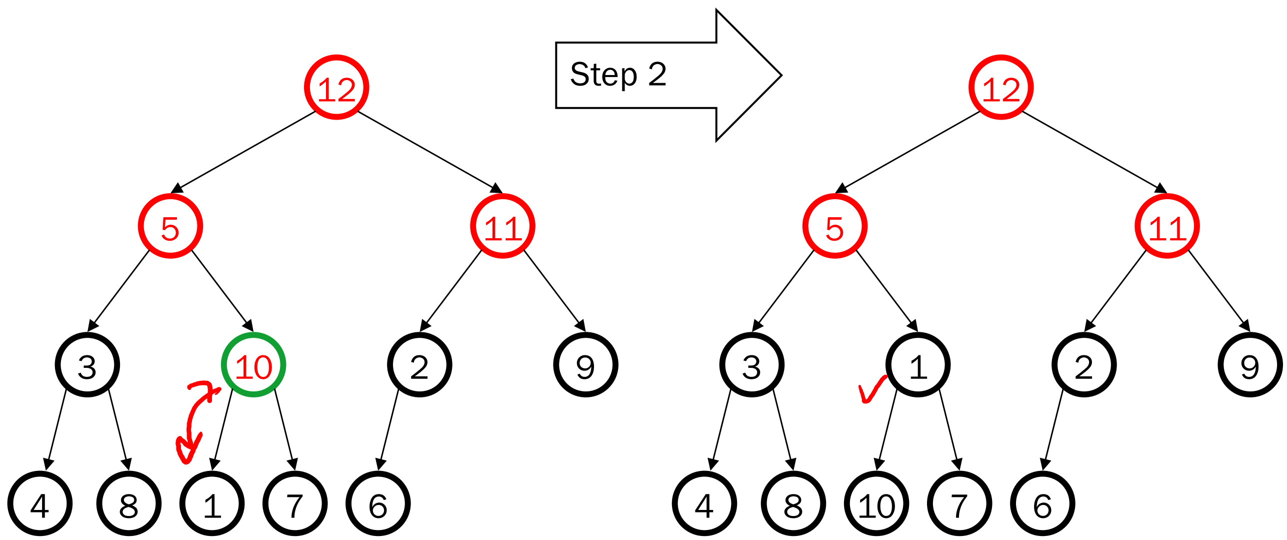


buildHeap Example



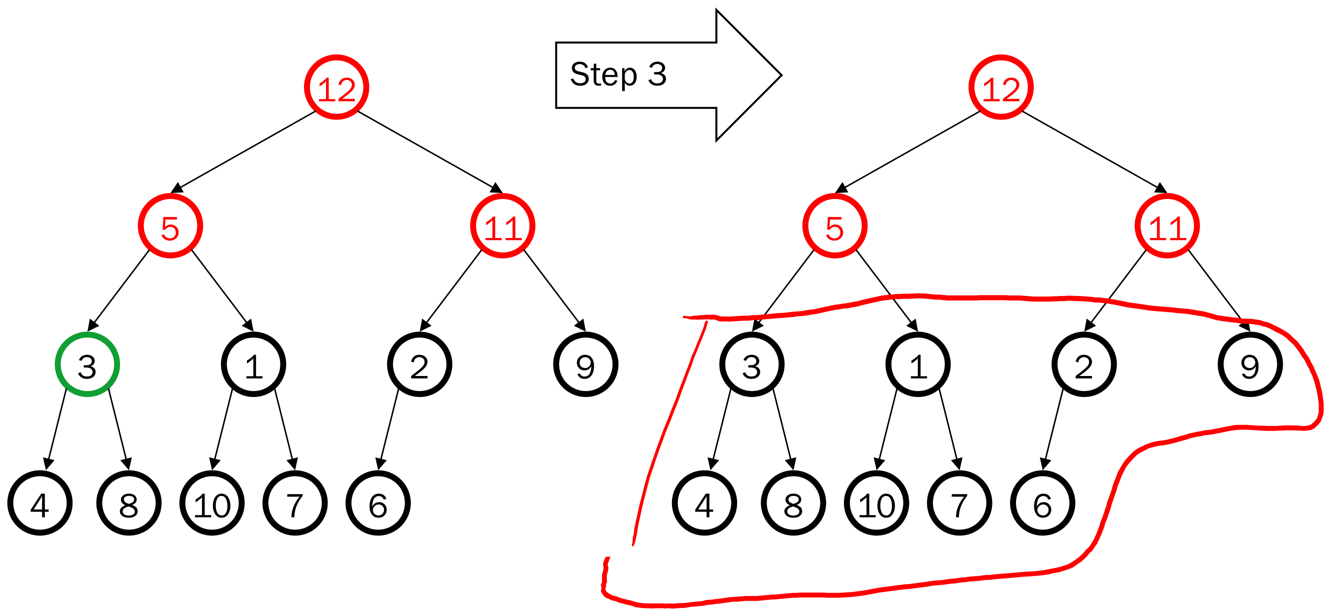
- Happens to already be less than child

`buildHeap` Example



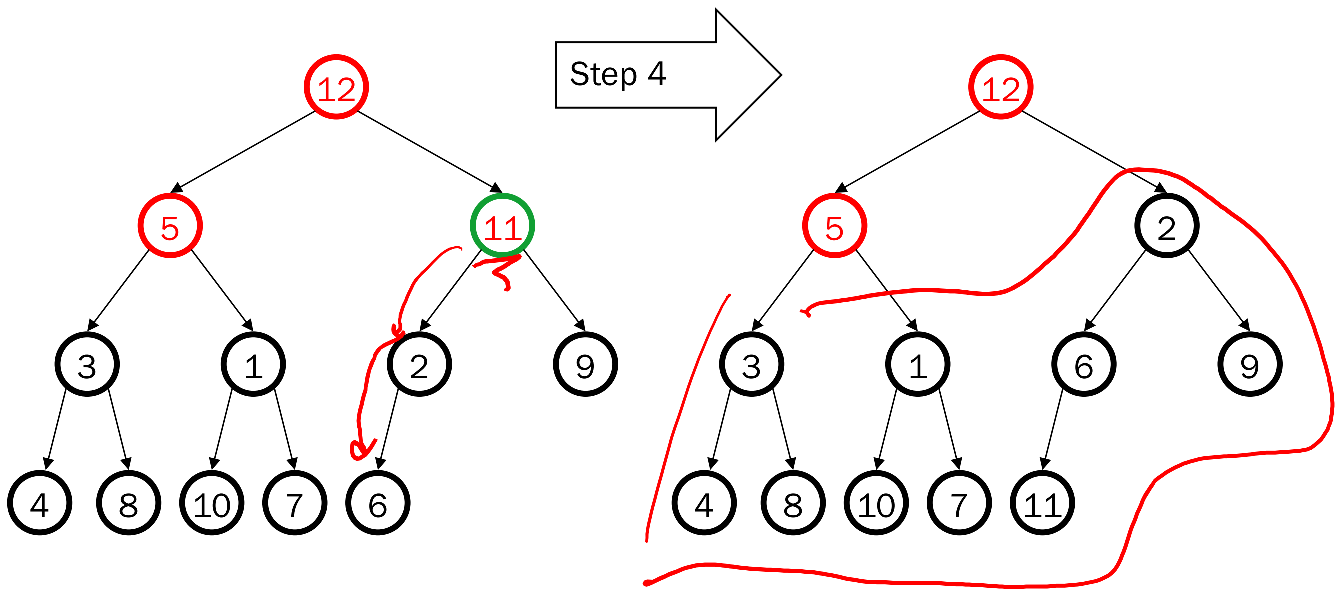
- Percolate down (notice that moves 1 up)

`buildHeap` Example



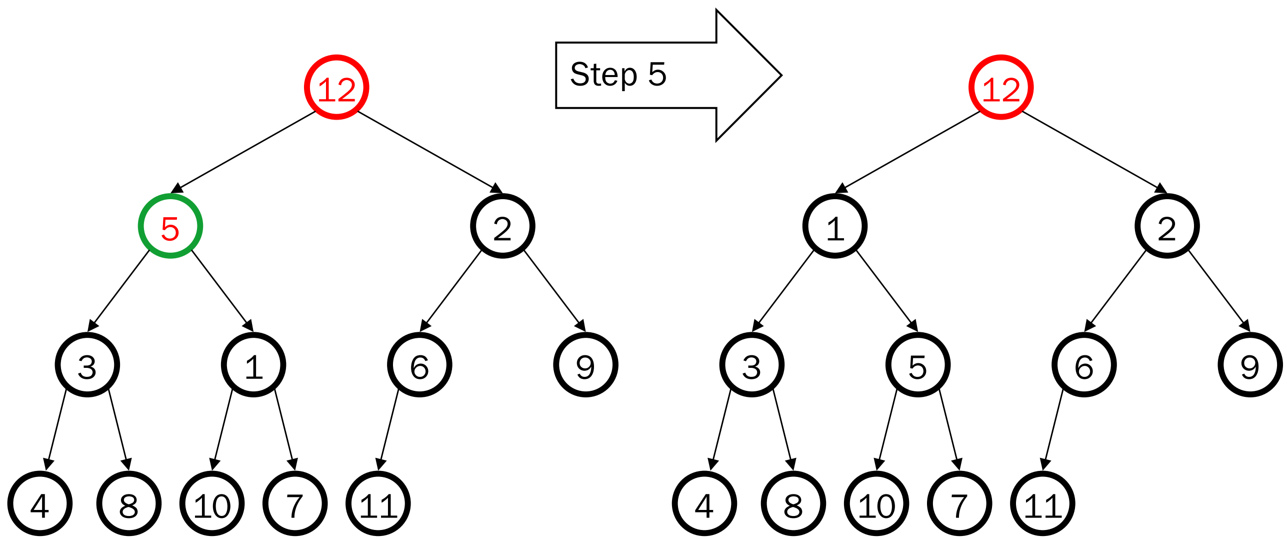
- Another nothing-to-do step

`buildHeap` Example

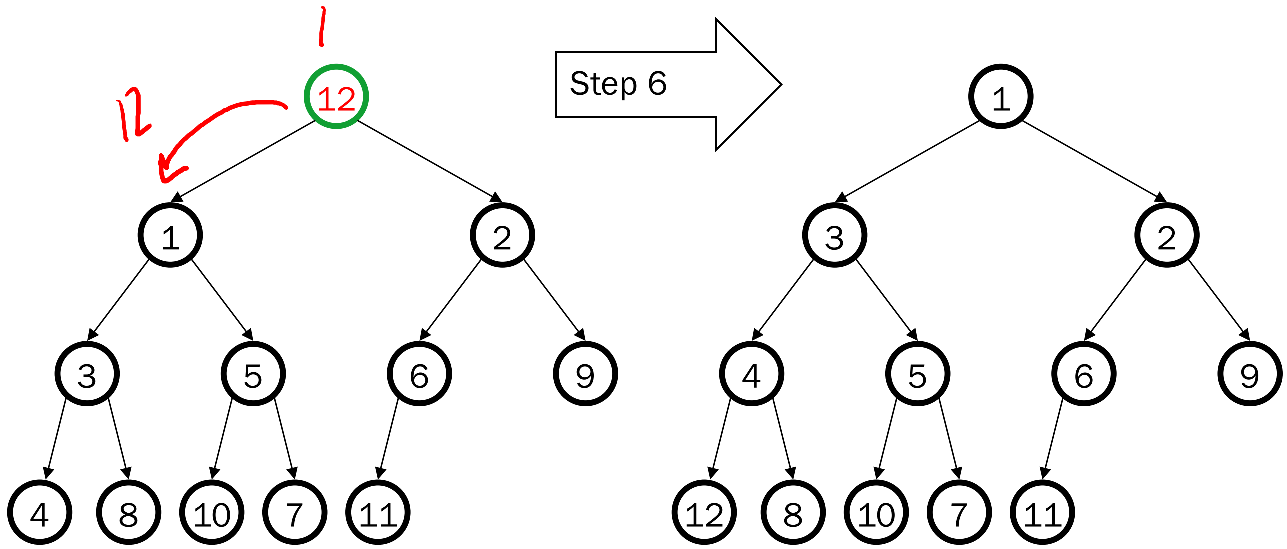


- Percolate down as necessary (steps 4a and 4b)

`buildHeap` Example



`buildHeap` Example



But is it right?

- "Seems to work"
  - Let's *prove* it restores the heap property (correctness)
  - Then let's *prove* its running time (efficiency)

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Correctness

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Loop Invariant: For all **`j>i`**, **`arr[j]`** is less than its children

- True initially: If **`j > size/2`**, then **`j`** is a leaf
  - Otherwise its left child would be at position > **`size`**
- True after one more iteration: loop body and **`percolateDown`** make **`arr[i]`** less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Loop Invariant:

For all `j>i`, `arr[j]` is less than its children

- True initially:
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So after the loop finishes,
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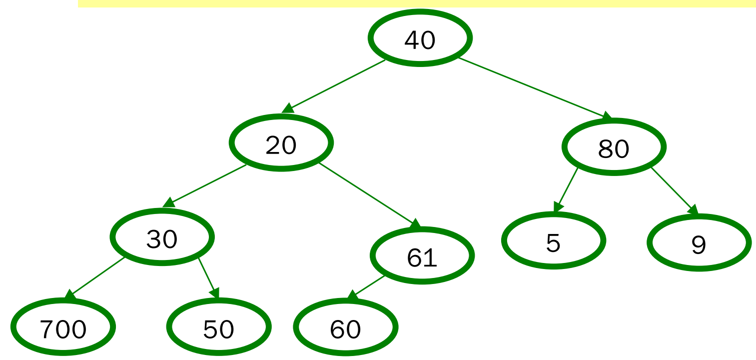


	40	**20**	**80**	**30**	**61**	**5**	**9**	**700**	**50**	**60**			
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Efficiency

$O(n^2)$ ✓

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

$O(\log n)$

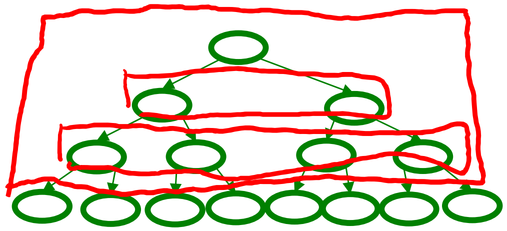
$\frac{n}{2} \cdot O(n \log n)$

Easy argument: **buildHeap** is $O(n$ **log** $n)$ where n is **size**

- **size/2** loop iterations
- Each iteration does one **percolateDown**, each is $O($**log** $n)$

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

Efficiency



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    arr[hole] = val;
  }
}
```

Better argument: **buildHeap** is $O(n)$ where n is **size**

$$\frac{n}{2}\left[\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8} 3 \cdots \frac{1}{2^k}\cdot k\right]$$

$$= \frac{n}{2}\sum_{i=1}^{k}\frac{i}{2^i} < \frac{n}{2}\underbrace{\sum_{i=1}^{\infty}\frac{i}{2^i}}_{2} = \frac{n}{2}\cdot 2 = O(n)$$

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Better argument: **`buildHeap`** is $O(n)$ where n is **`size`**

- **`size/2`** total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps... etc.
- ((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + ...) = 2 (page 4 of Weiss)
  - So at most **`2(size/2)`** *total* percolate steps: $O(n)$
  - Also see Weiss 6.3.4, sum of heights of nodes in a perfect tree

Lessons from `buildHeap`

- Without **`buildHeap`**, our ADT already let clients implement their own in $\theta(n$ **`log`** $n)$ worst case
  - Worst case is inserting lower priority values later
- By providing a specialized operation internally (with access to the data structure), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness: Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was $O(n$ **`log`** $n)$
    - A "tighter" analysis shows same algorithm is $O(n)$

More heaps (see Weiss if curious)

- *d*-heaps: have *d* children instead of 2 (Weiss 6.5)
  - Makes heaps shallower, useful for heaps too big for memory
  - How does this affect the asymptotic run-time (for small d's)?
- Leftist heaps, skew heaps, binomial queues (Weiss 6.6-6.8)
  - Different data structures for priority queues that support a logarithmic time **`merge`** operation (impossible with binary heaps)
  - **`merge:`** given two priority queues, make one priority queue
  - Insert & deleteMin defined in terms of merge

Aside: How might you merge *binary* heaps:

- If one heap is much smaller than the other?
- If both are about the same size?