CSE 332: Data Structures & Parallelism
Lecture 4: Priority Queues and Heaps

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Announcements

- Checkpoint 1 due last night
- EX02 due Friday

Today – Priority Queues and Heaps

- Priority Queue ADT
- Binary Min-Heap Datastructure

- (More recurrences on Friday)

Scenario

What is the difference between waiting for service at a pharmacy versus an ER?

Pharmacies usually follow the rule First Come, First Served

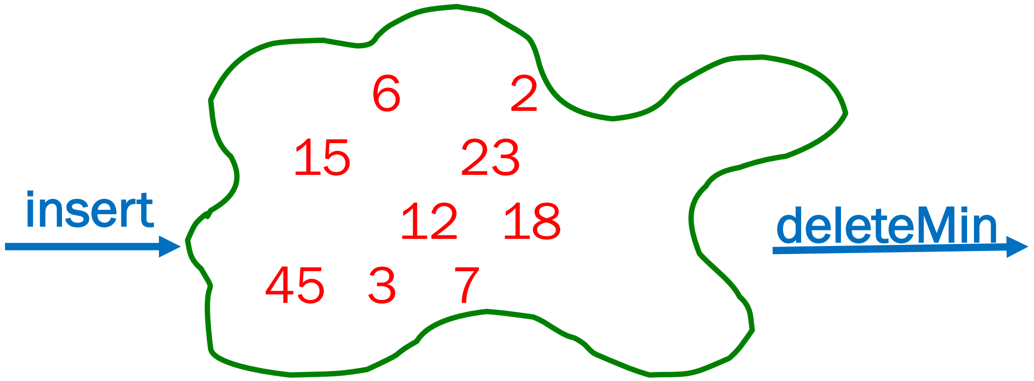
Queue

Emergency Rooms assign priorities based on each individual's need

Priority Queue

Priority Queue ADT

Priority Queue ADT
State: • Set of comparable elements   • Order based on "priority" Operations: • **insert(element)** • deleteMin() – returns the element with the smallest priority, removes it from the collection • **findMin()**



- Assume each item has a "priority"
  - The *lesser* item is the one with the *greater* priority
  - So "priority 1" is more important than "priority 4"
  - Just a convention, could also do a maximum priority

Aside: We will use ints as data <u>and</u> priority

For simplicity in lecture, we'll often suppose items are just **`int`**s and the **`int`** is also the priority

- So an operation sequence could be

```
insert 6
insert 5
x = deleteMin  // Now x = 5.
```

– **`int`** priorities are common, but really just need comparable

- Not having "other data" is very rare
  - Example: print job has a priority *and* the file to print is the data

Applications

Like all good ADTs, the priority queue arises often

- Sometimes "directly", sometimes less obvious

- Run multiple programs in the operating system
  - "critical" before "interactive" before "compute-intensive"
  - Maybe let users set priority level
- Treat hospital patients in order of severity (or triage)
- Select print jobs in order of decreasing length?
- Forward network packets in order of urgency
- Select most frequent symbols for data compression (peep CSE143)
- Sort: **`insert`** all, then repeatedly **`deleteMin`**

More applications

- “Greedy” algorithms
  - Select the ‘best-looking’ choice at the moment
  - Will see an example when we study graphs in a few weeks
- Discrete event simulation (system modeling, virtual worlds, …)
  - Simulate how state changes when events fire
  - Each event *e* happens at some time t and generates new events *e1*, …, *en* at times *t+t1*, …, *t+tn*
  - Naïve approach: advance “clock” by 1 unit at a time and process any events that happen then
  - Better:
    - *Pending events* in a priority queue (priority = time happens)
    - Repeatedly: **`deleteMin`** and then **`insert`** new events
    - Effectively, “set clock ahead to next event”

<u>Preliminary</u> Implementations of Priority Queue ADT

	insert	deleteMin
Unsorted Array		
Unsorted Linked-List		
Sorted Circular Array		
Sorted Linked-List		
Binary Search Tree (BST)		

Notes: Worst case, Assume arrays have enough space

Need a good data structure!

- Next we will show an efficient, non-obvious data structure for this ADT
  - But first let's analyze some "obvious" ideas for *n* data items
  - All times worst-case; assume arrays "have room"

data	*insert algorithm / time*		*deleteMin algorithm / time*	
unsorted array	add at end	$O(1)$	search	$O(n)$
unsorted linked list	add at front	$O(1)$	search	$O(n)$
sorted circular array	search / shift	$O(n)$	move front	$O(1)$
sorted linked list	put in right place	$O(n)$	remove at front	$O(1)$
binary search tree	put in right place	$O(n)$	leftmost	$O(n)$

Aside: More on possibilities

- Note: If priorities are inserted in random order, binary search tree will likely do better than $O(n)$
  - $O(\texttt{log}\ n)$ **`insert`** and $O(\texttt{log}\ n)$ **`deleteMin`** on average
  - Could get same performance from a *balanced* binary search tree (e.g. AVL tree we will study later)

- One more idea: if priorities are 0, 1, …, k can use array of lists
  - **`insert`**: add to front of list at **`arr[priority]`**, $O(1)$
  - **`deleteMin`**: remove from lowest non-empty list $O(k)$

Our Data Structure: The Heap

The Heap:

- Worst case: O(log n) for **insert**
  - If items arrive in random order, then the average-case of insert is O(1) !!
- Worst case: O(log n) for **deleteMin**
- Very good constant factors

Key idea: Only pay for functionality needed

- We need something better than scanning unsorted items
- But we do not need to maintain a full sorted list

- Do “log”s remind you of anything? 🌲🌲 We will ***visualize*** our heap as a tree

Q: Reviewing Some Tree Terminology

root(**T**):

leaves(**T**):

children(B):

parent(H):

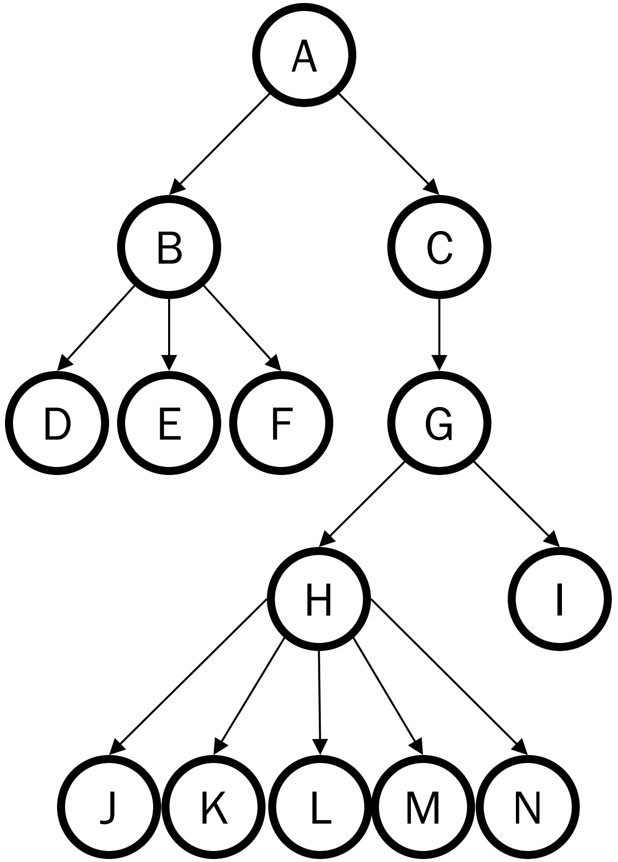
siblings(E):

ancestors(F):

descendents(G):

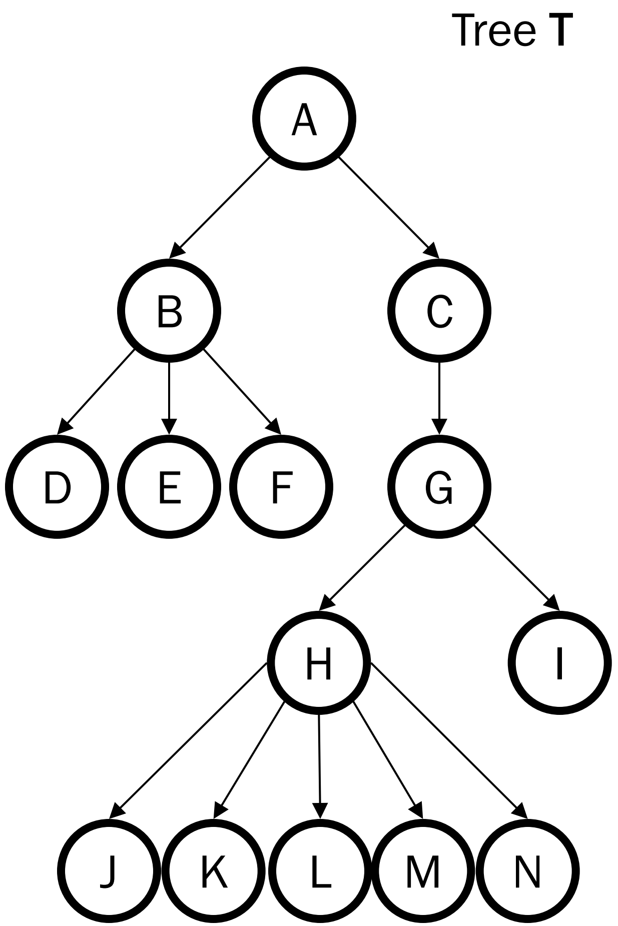
subtree(G):

Tree **T**



A: Reviewing Some Tree Terminology

root(**T**):	A
leaves(**T**):	D-F, I, J-N
children(B):	D, E, F
parent(H):	G
siblings(E):	D, F
ancestors(F):	B, A
descendents(G):	H, I, J-N
subtree(G):	G and its children



Q: Some More Tree Terminology

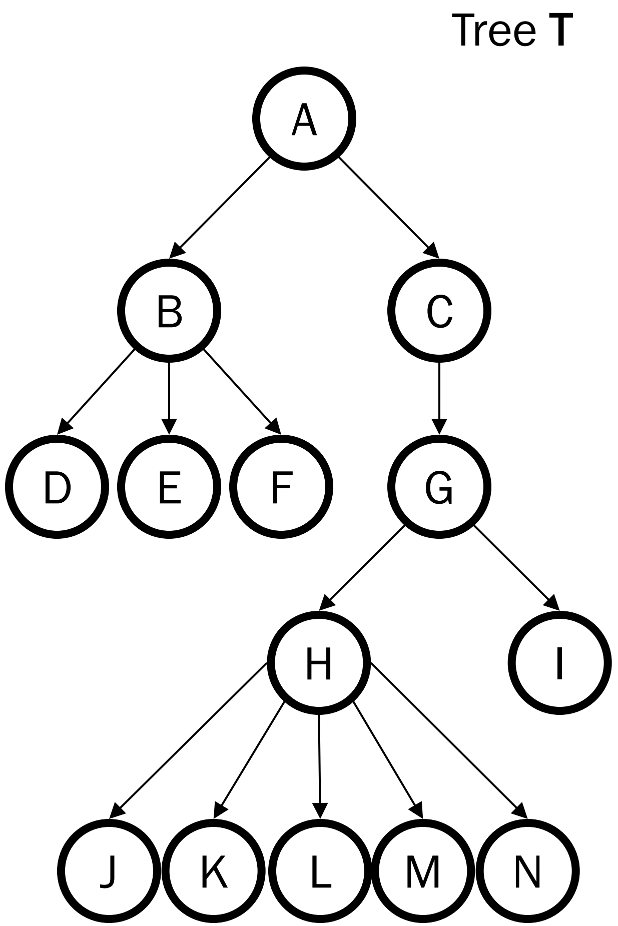
depth(B):

height(G):

height(**T**):

degree(B):

branching factor(**T**):



A: Some More Tree Terminology

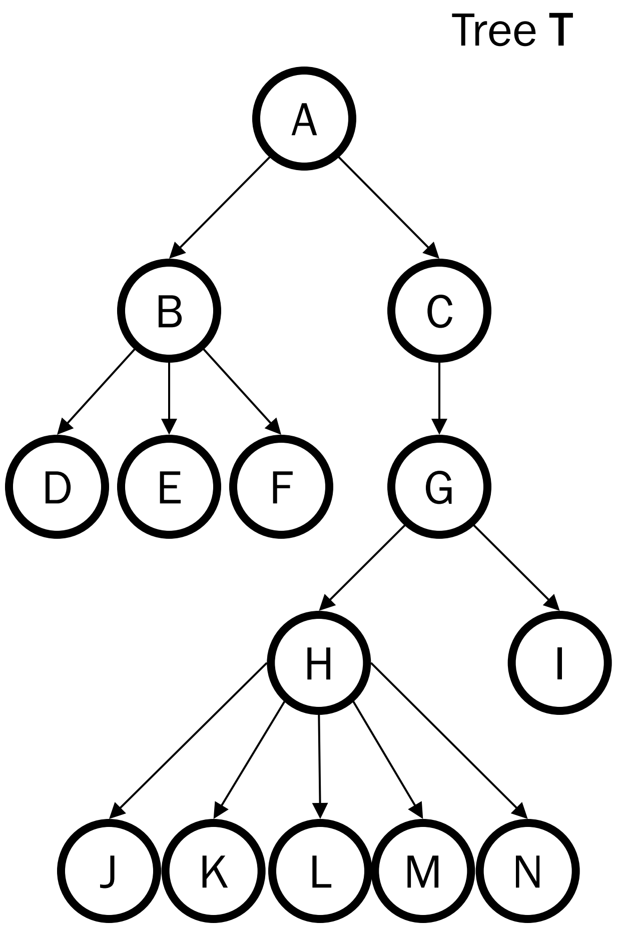
depth(B): *1*

height(G): *2*

height(**T**): *4*

degree(B): *3*

branching factor(**T**): *0-5*



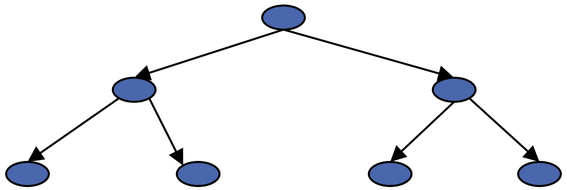
Types of Trees

Binary tree: Every node has ≤2 children

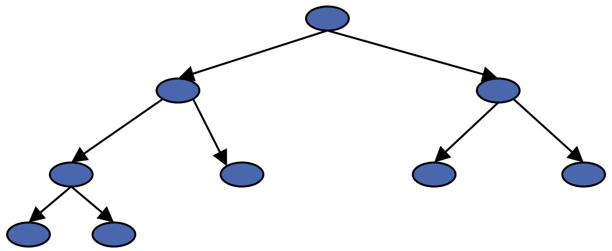
n-ary tree: Every node has ≤n children

Perfect tree: Every row is completely full

Complete tree: All rows except possibly the bottom are completely full, and it is filled from left to right



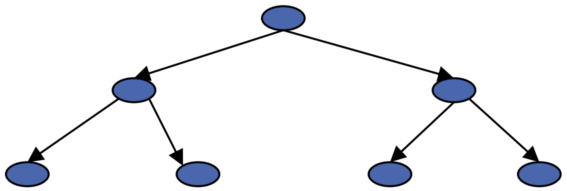
Perfect Tree



Complete Tree

More on Perfect Trees

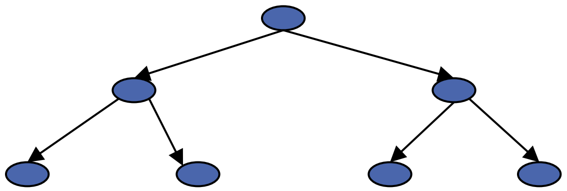
Perfect tree: Every row is completely full



Perfect Tree

More on Perfect Trees

Perfect tree: Every row is completely full



Perfect Tree

height	# of nodes	# of leaves
0	1	1
1	3	2
2	7	4
3	15	8
h	$2^{h+1} - 1$	2^h

Some Basic Tree Properties

Nodes in a perfect binary tree of height h?

$2^{h+1}-1$

Leaf nodes in a perfect binary tree of height h?

2^h

Height of a perfect binary tree with n nodes?

$\lfloor \log_2 n \rfloor$

Height of a ***complete*** binary tree with n nodes?

$\lfloor \log_2 n \rfloor$

Now Formalizing: Binary Min-Heap Datastructure

More commonly known as a binary heap or simply a heap

- Structure Property:
  A complete [binary] tree
- Heap-Order Property:
  Every non-root node has a priority value larger than (or possibly equal to) the priority of its parent

Now Formalizing: Binary Min-Heap Datastructure

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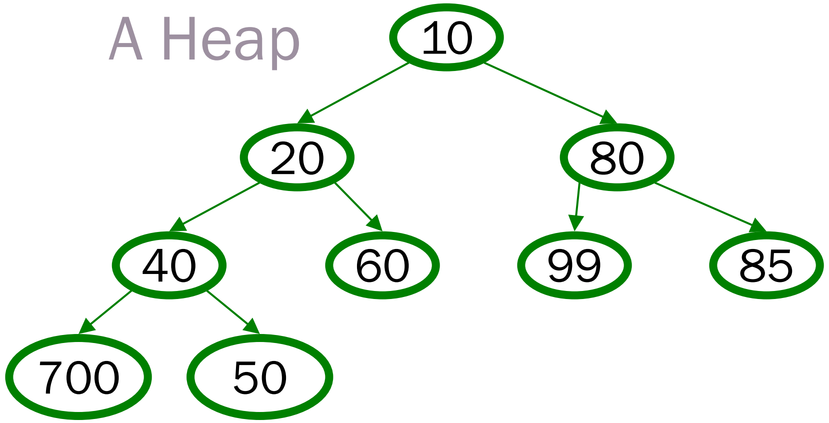
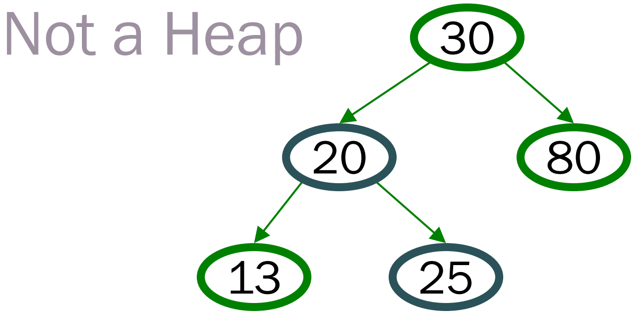
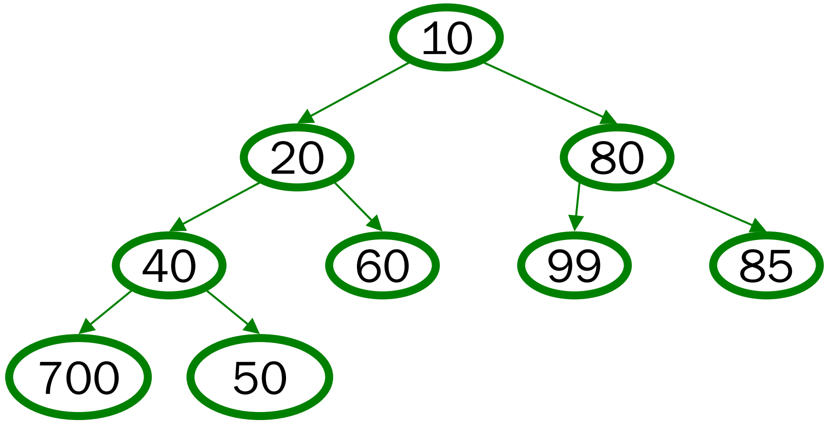




Properties of Binary Min-Heap

- Where is the minimum priority item?

- What is the height of a heap with n items?

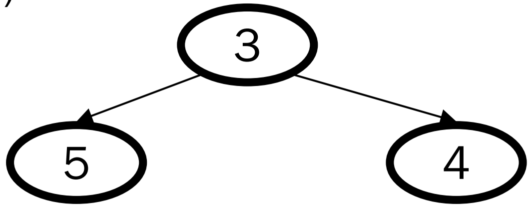


Poll Everywhere

pollev.com/artliu

Are these valid binary heaps?

1)



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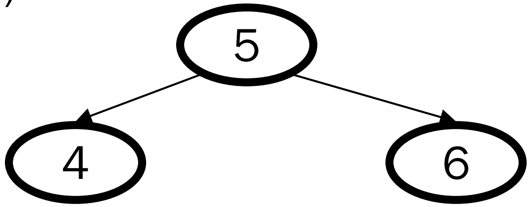
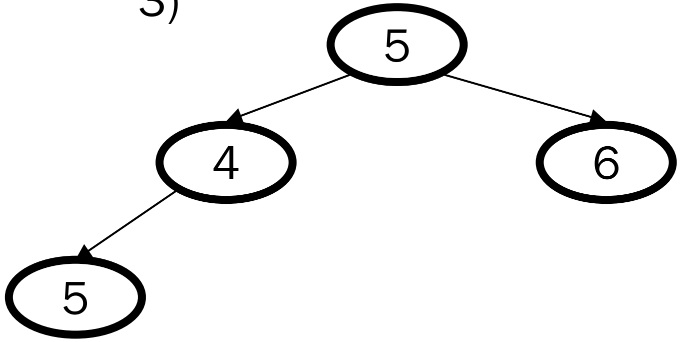


3)



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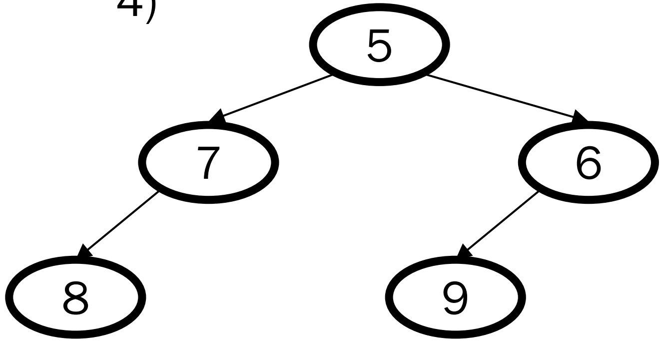
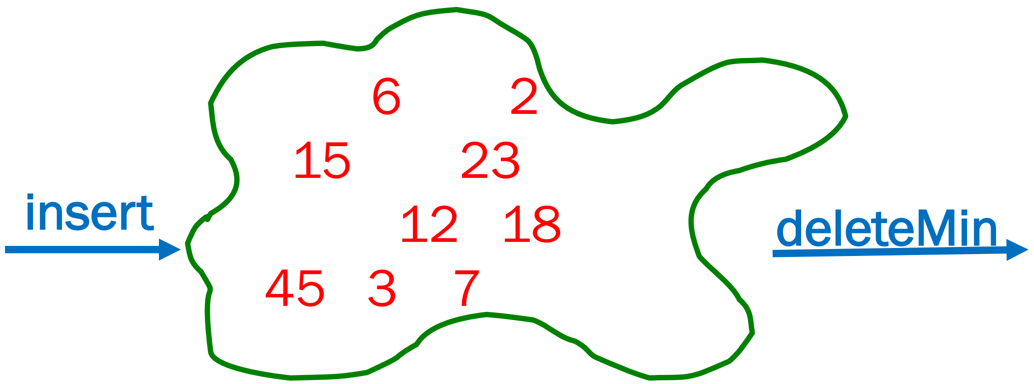


Implementing Priority Queue ADT

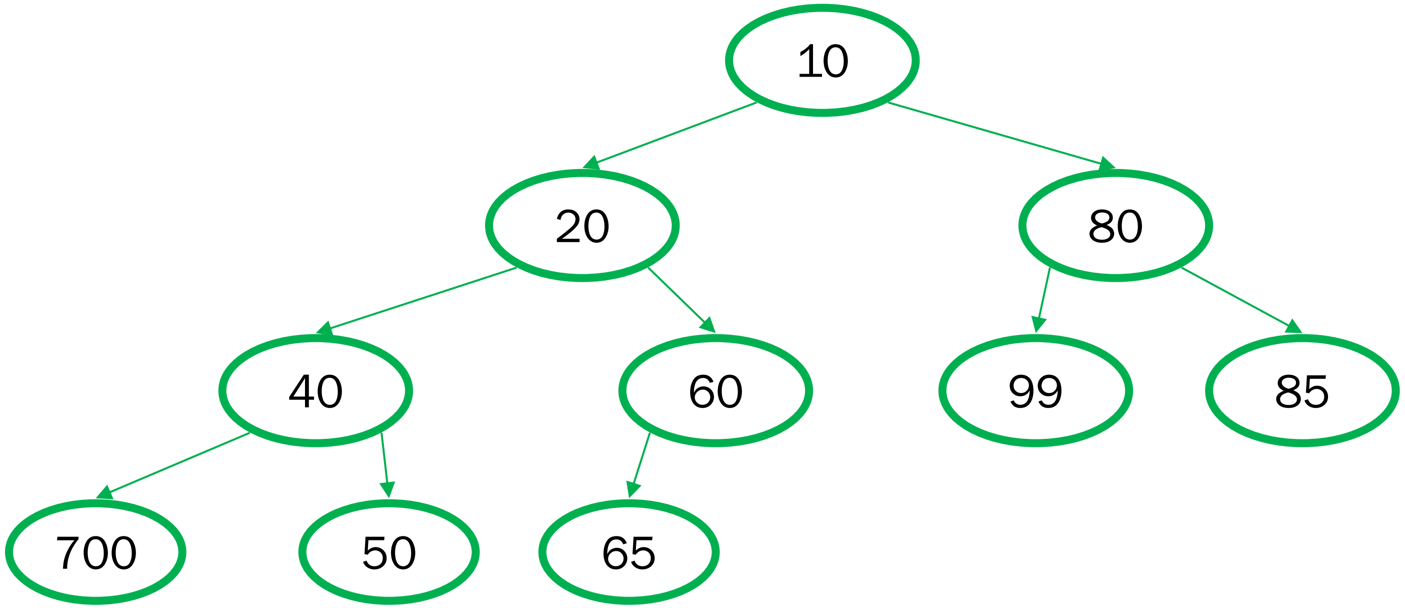
Reminder :)

Priority Queue ADT
State: • Set of comparable elements   • Order based on "priority" Operations: • **insert(element)** • deleteMin() – returns the element with the smallest priority, removes it from the collection • **findMin()**



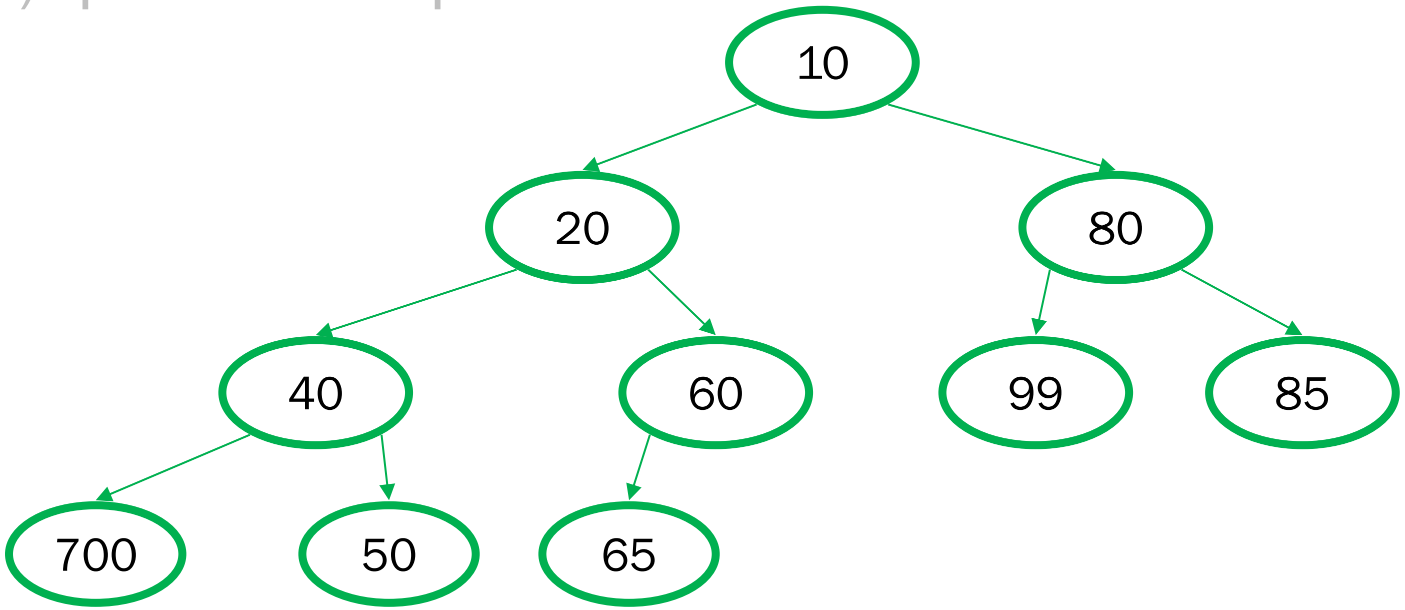
Heap Operations

- insert(val): percolate up



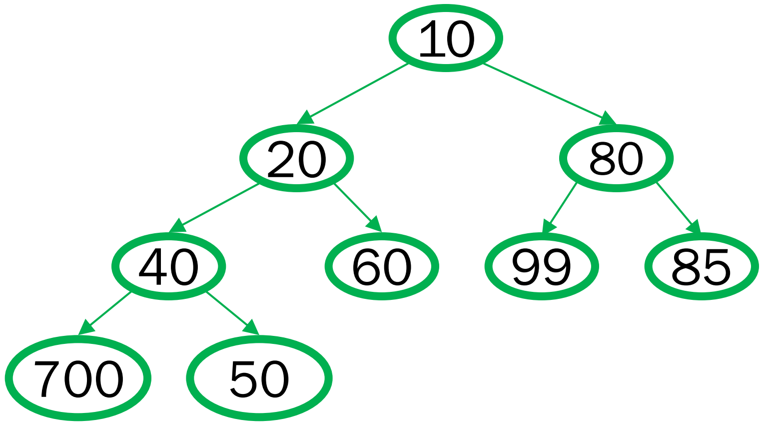
Heap Operations

- findMin:
- deleteMin: percolate down
- insert(val): percolate up



Operations: basic idea

- **`findMin`**:
  return **`root.data`**
- **`deleteMin`**:
  1. **`answer = root.data`**
  2. Move right-most node in last row to root to restore structure property
  3. "Percolate down" to restore heap order property
- **`insert:`**
  1. Put new node in next position on bottom row to restore structure property
  2. "Percolate up" to restore heap order property



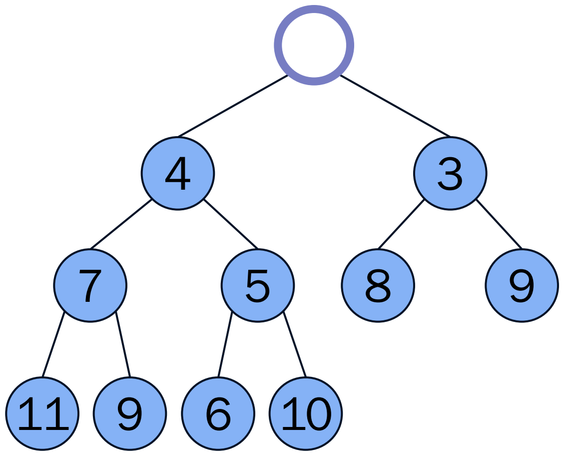
Overall strategy:

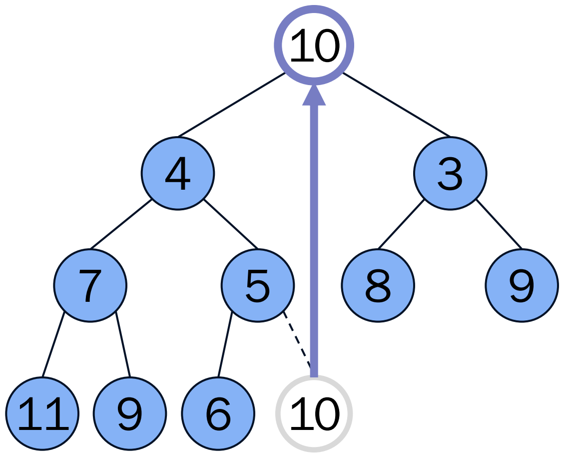
- *Preserve complete tree structure property*
- *This may break heap order property*
- *Percolate to restore heap order property*

DeleteMin Implementation

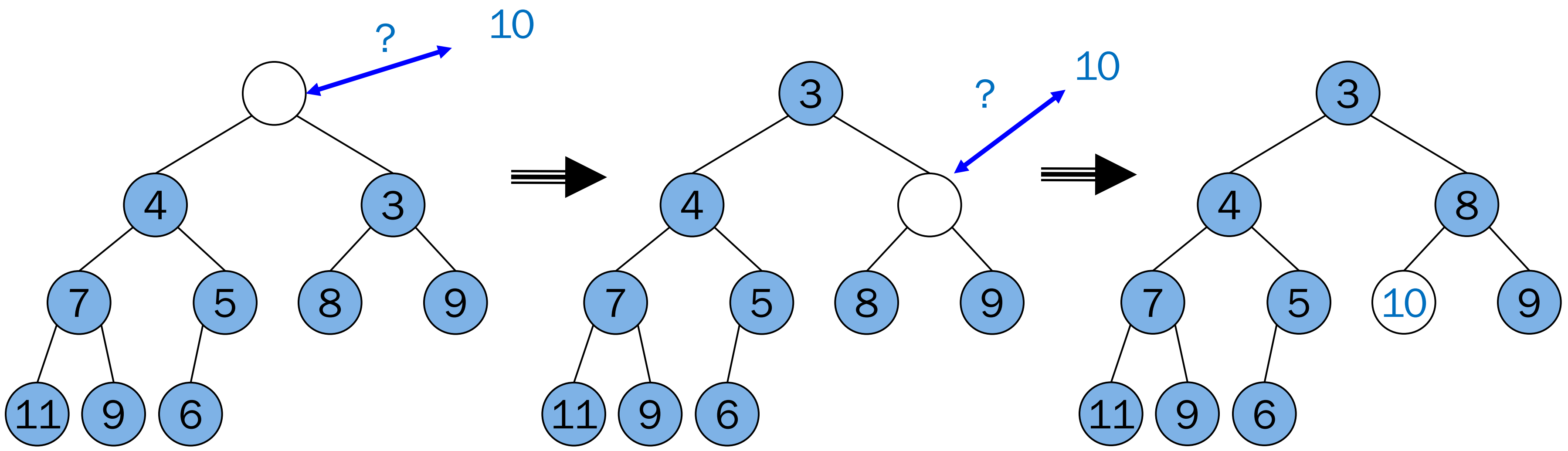
1. Delete value at root node (and store it for later return)
2. There is now a "hole" at the root. We must "fill" the hole with another value, must have a tree with one less node, and it must still be a complete tree
3. The "last" node is the obvious choice, but now the heap order property is violated
4. We percolate down to fix the heap order:

```
While greater than either child
    Swap with smaller child
```





Percolate Down



Percolate down:
- Keep comparing with both children
- Move smaller child up and go down one level
- Done if both children are ≥ item or reached a leaf node
- Why does this work? What is the run time?

DeleteMin: Run Time Analysis

- Run time is *O*(height of heap)

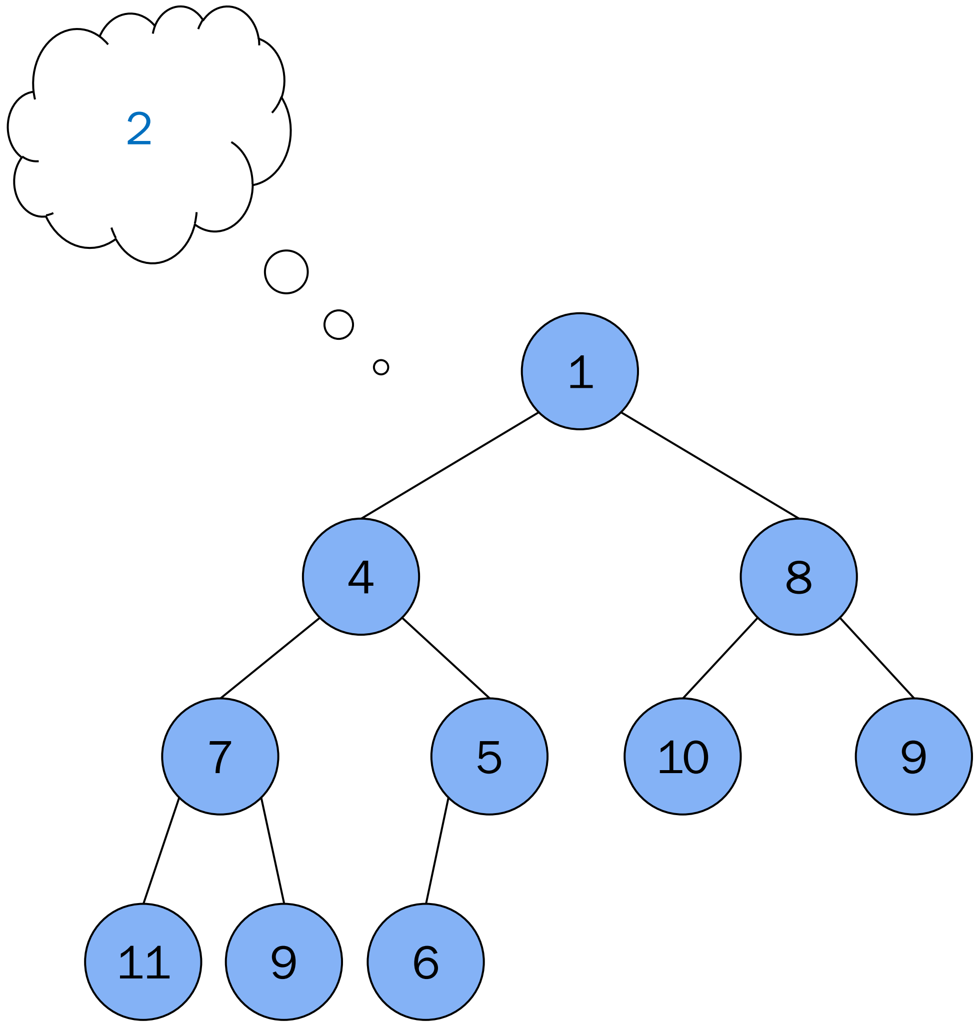
- A heap is a complete binary tree

- Height of a complete binary tree of *n* nodes?
  height = $\lfloor \mathbf{log}_2(n) \rfloor$

- Run time of **`deleteMin`** is $O(\mathbf{log}\ n)$

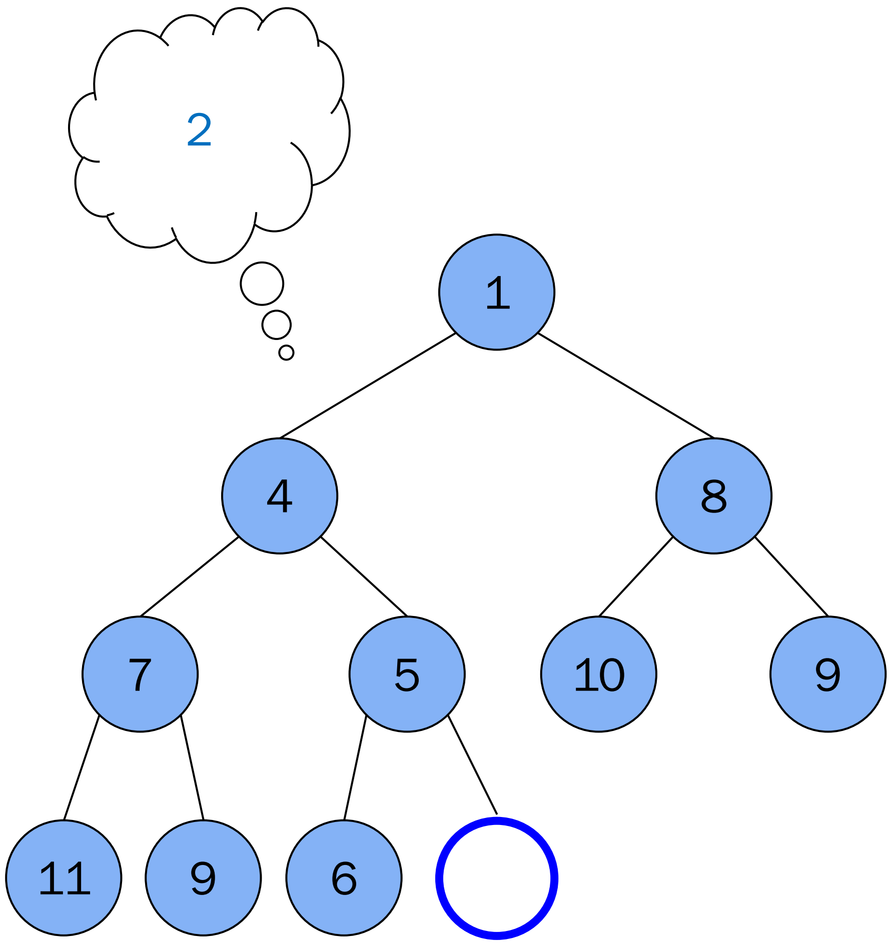
Insert

- Add a value to the tree
- Structure and heap order properties must still be correct afterwards

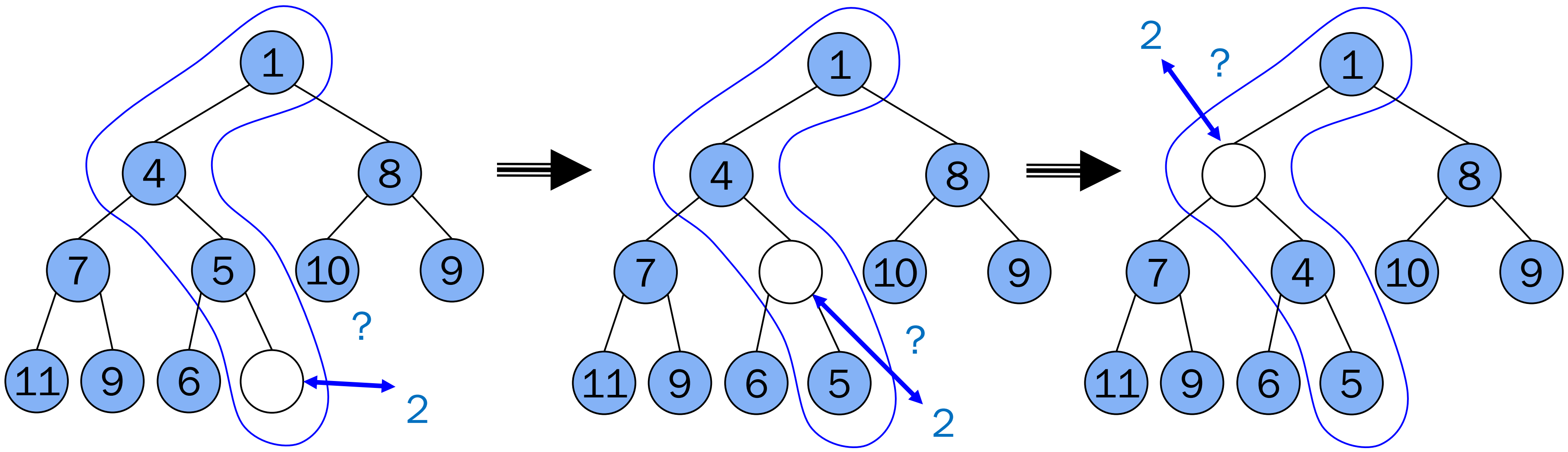


Insert: Maintain the ***Structure*** Property

- There is only **one** valid tree shape after we add one more node!
- So put our new data there and then focus on restoring the heap order property



Insert: Maintain the ***Heap Order*** property



Percolate up:

- Put new data in new location
- If parent larger, swap with parent, and continue
- Done if parent ≤ item or reached root
- Why does this work? What is the run time?

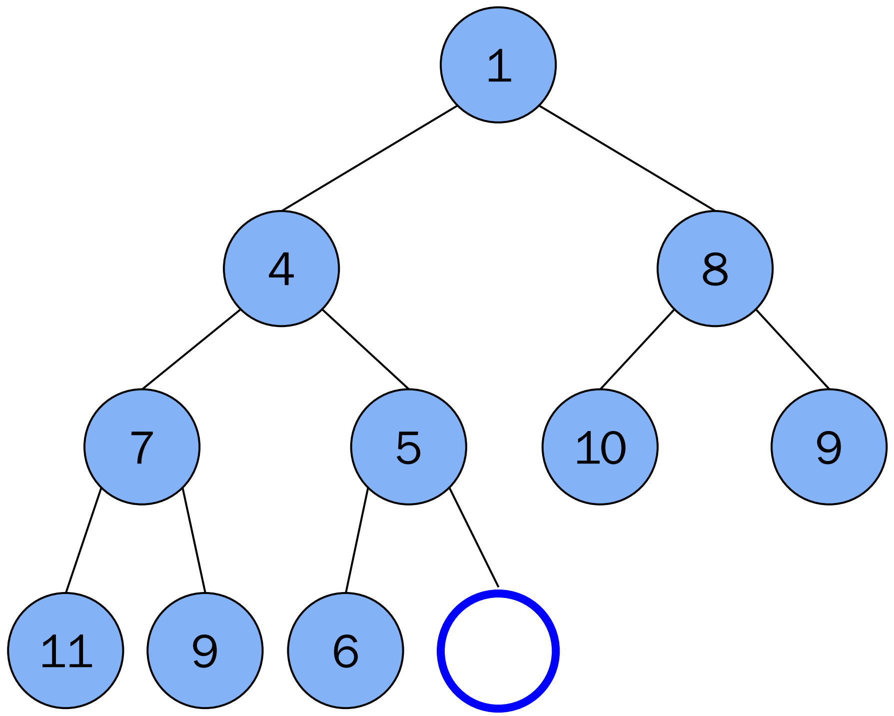
Clever trick for storing the heap…

Need to have access to "next to use" position in the tree. Requires at minimum log(n)…

How could we get O(1) average-case insertion?

Hint: why did we insist the tree be complete?

- All complete trees have the same edges, so we don't need to explicitly represent edges



Note: Exercises and P2 start counting from 0

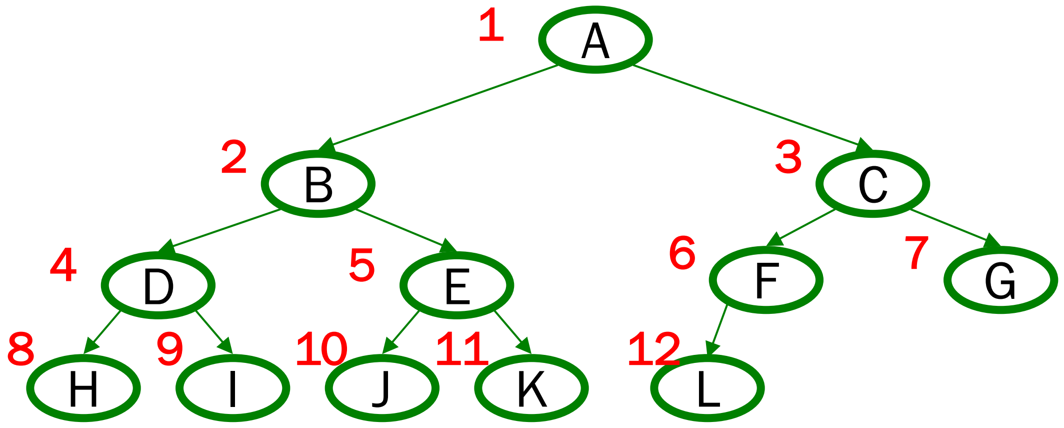
Array Representation of a Binary Heap

From node i:

left child:

right child:

parent:



	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

- We skip index 0 to make the math simpler
- Actually, it can be a good place to store the current size of the heap

Note: Exercises and P2 start counting from 0

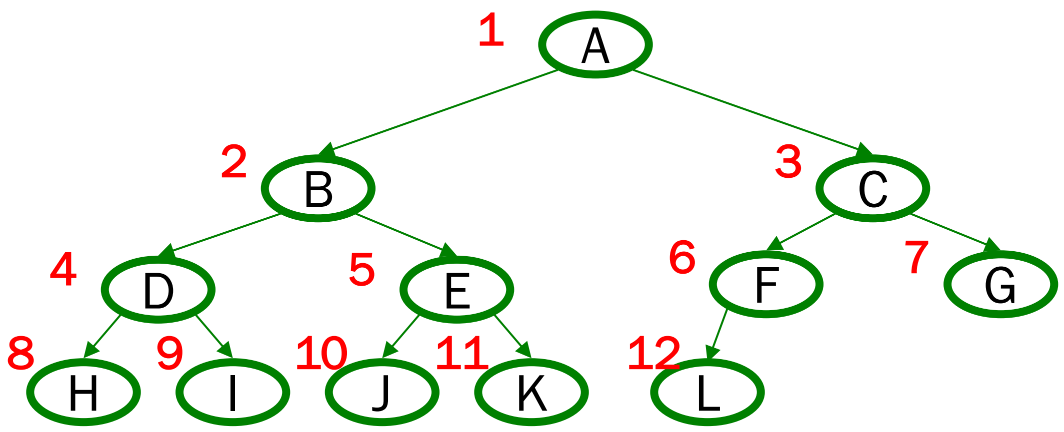
Array Representation of a Binary Heap

From node i:

left child: 2i

right child: 2i+1

parent: floor(i / 2)



	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

- We skip index 0 to make the math simpler
- Actually, it can be a good place to store the current size of the heap

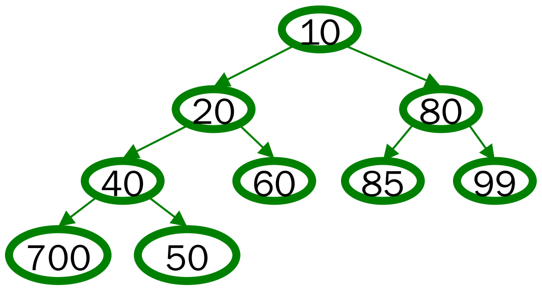
Note: Exercises and P2 start counting from 0

Pseudocode: insert

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
void insert(int val) {
   if(size==arr.length-1)
     resize();
   size++;
   i=percolateUp(size,val);
   arr[i] = val;
}
```

```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
          val < arr[hole/2]){
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

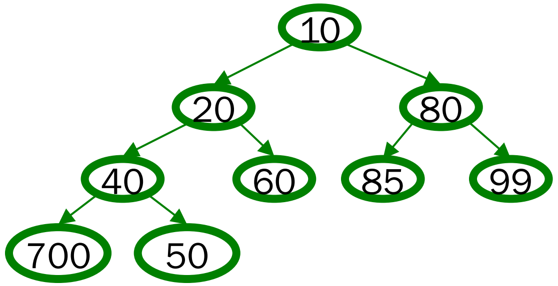
Note: Exercises and P2 start counting from 0

Pseudocode: deleteMin

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
int deleteMin() {
  if(isEmpty()) throw…
  ans = arr[1];
  hole = percolateDown
          (1,arr[size]);
  arr[hole] = arr[size];
  size--;
  return ans;
}
```

```
int percolateDown(int hole,
                  int val) {
 while(2*hole <= size) {
  left  = 2*hole;
  right = left + 1;
  if(arr[left] < arr[right]
     || right > size)
    target = left;
  else
    target = right;
  if(arr[target] < val) {
    arr[hole] = arr[target];
    hole = target;
  } else
      break;
 }
 return hole;
}
```



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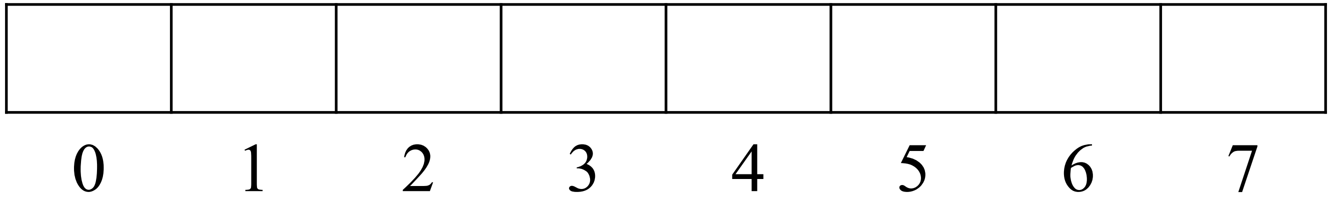
Note: Exercises and P2 start counting from 0

Example

1. insert: 16, 32, 4, 57, 80, 43, 2
2. deleteMin

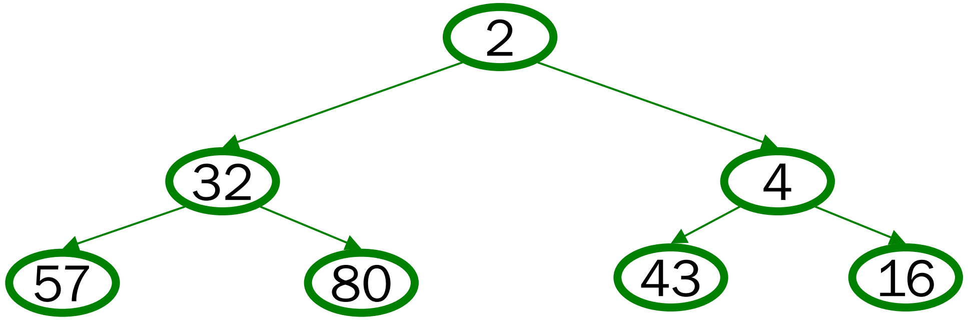


Note: Exercises and P2 start counting from 0

Example: After insertion

1. insert: 16, 32, 4, 57, 80, 43, 2
2. deleteMin

	2	**32**	**4**	**57**	**80**	**43**	**16**
0	1	2	3	4	5	6	7

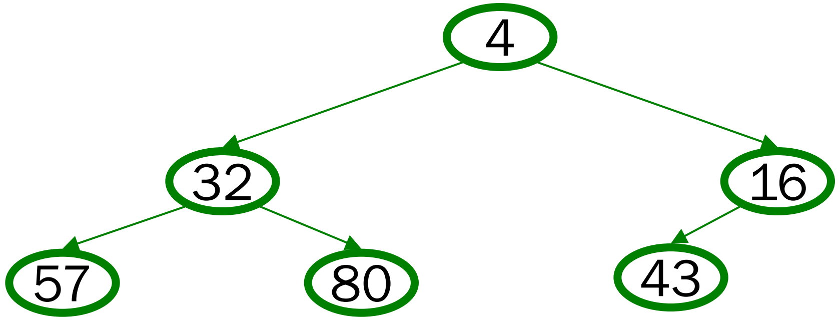


Note: Exercises and P2 start counting from 0

Example: After deletion

1. insert: 16, 32, 4, 57, 80, 43, 2
2. deleteMin

	4	**32**	**16**	**57**	**80**	**43**	
0	1	2	3	4	5	6	7



So why O(1) average-case insert?

- Yes, insert's **worst case** is O(log n)
- The trick is that it all depends on the order the items are inserted (What is the worst case order?)
- Experimental studies of randomly ordered inputs shows the following:
  - Average 2.607 comparisons per insert (# of percolation passes)
  - An element usually moves up 1.607 levels
- deleteMin is average O(log n)
  - Moving a leaf to the root usually requires re-percolating that value back to the bottom

Evaluating the Array Implementation...

Advantages:

Minimal amount of wasted space:

- Only index 0 and any unused space on right in the array
- No "holes" due to complete tree property
- No wasted space representing tree edges

Fast lookups:

- Benefit of array lookup speed
- Multiplying and dividing by 2 is extremely fast (can be done through bit shifting (see CSE 351)
- Last used position is easily found by using the PQueue's size for the index

Disdvantages:

- What if the array gets too full (or wastes space by being too empty)? Array will have to be resized.

Advantages outweigh Disadvantages: This is how it is done!

Other (specialized) operations

- **`decreaseKey`**: *given pointer* to object in priority queue (e.g., its array index), lower its priority value by p
  - Change priority and percolate up
- **`increaseKey`**: *given pointer* to object in priority queue (e.g., its array index), raise its priority value by p
  - Change priority and percolate down

- **`remove`**: *given pointer* to object in priority queue (e.g., its array index), remove it from the queue
  - **`decreaseKey`** with $p = \infty$, then **`deleteMin`**

Running time for all these operations?

Building a Heap

Suppose you have *n* items you want to put in a new priority queue

- A sequence of *n* **`insert`** operations works
- Runtime?

Can we do better?

- If we only have access to **`insert`** and **`deleteMin`** operations, then NO.
- There is a faster way - O(n), but that requires the ADT to have a specialized **`buildHeap`** operation

Floyd's `buildHeap` Method

Recall our general strategy for working with the heap:

- Preserve structure property
- Break and restore heap ordering property

Floyd's ***buildHeap***:

1. Create a complete tree by putting the n items in array indices 1, ... N

   (Requires having all the elements that we want to insert all at once!)

2. Treat the array as a heap and fix the heap-order property

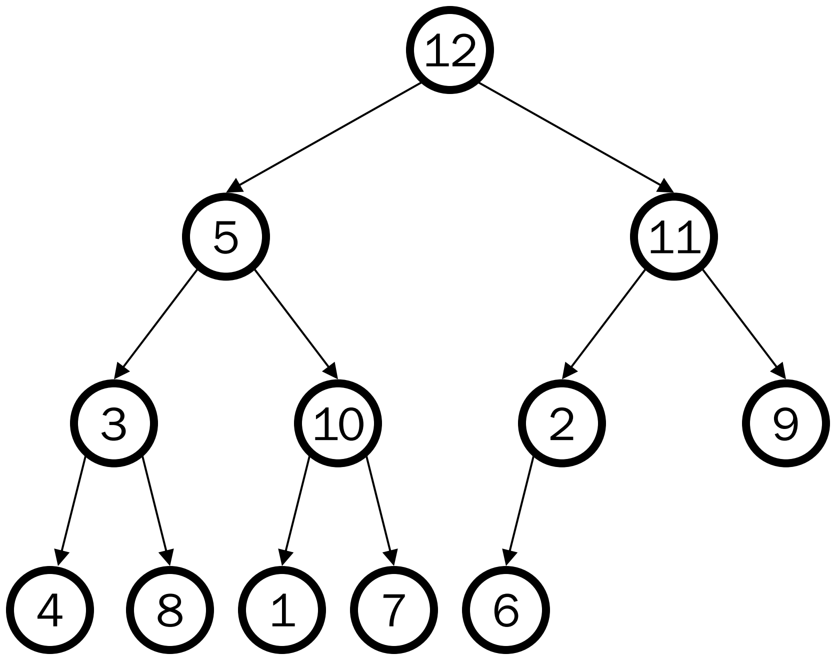
   Exactly how we do this is where we gain efficiency

Thinking about `buildHeap`

- Say we start with this array:

  [12,5,11,3,10,2,9,4,8,1,7,6]

- To “fix” the ordering should we use:
  - percolateUp?
  - percolateDown?



Note: Exercises and P2 start counting from 0

Floyd's `buildHeap` Method

percolateDown, **bottom-up**:

- Leaves are already in heap order
- Work up toward the root one level at a time

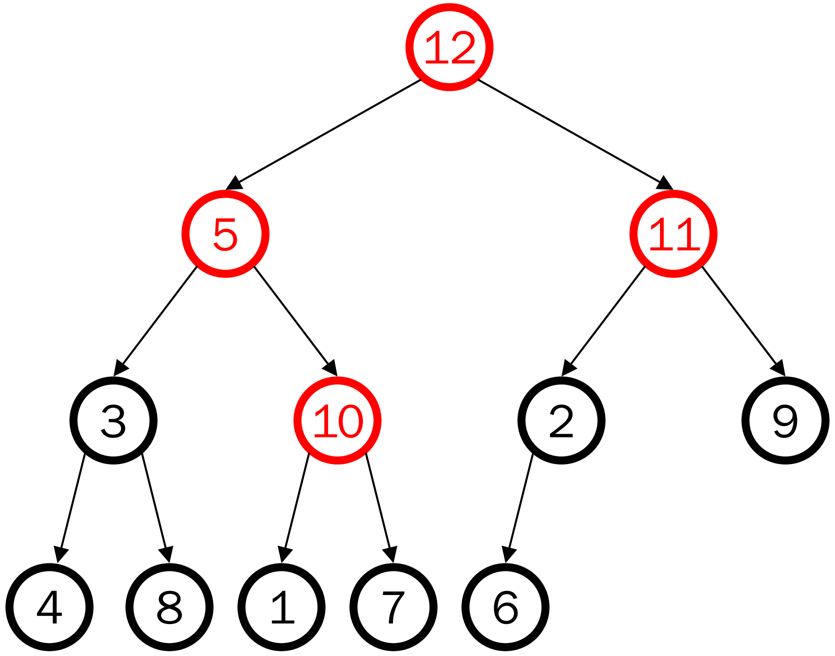
```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

`buildHeap` Example

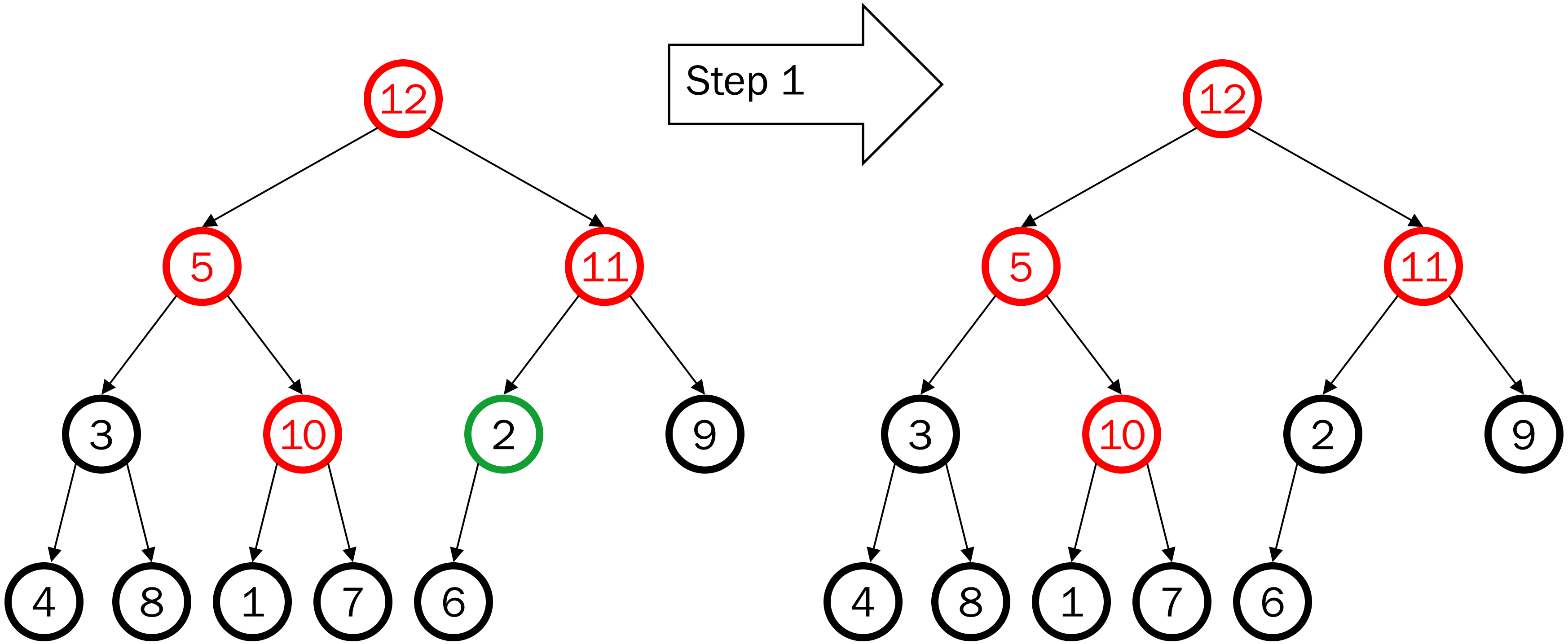
- Say we start with this array:

  [12,5,11,3,10,2,9,4,8,1,7,6]

- In tree form for readability
  - Red for node not less than descendants
    - heap-order problem
  - Notice no leaves are red
  - Check/fix each non-leaf bottom-up (6 steps here)

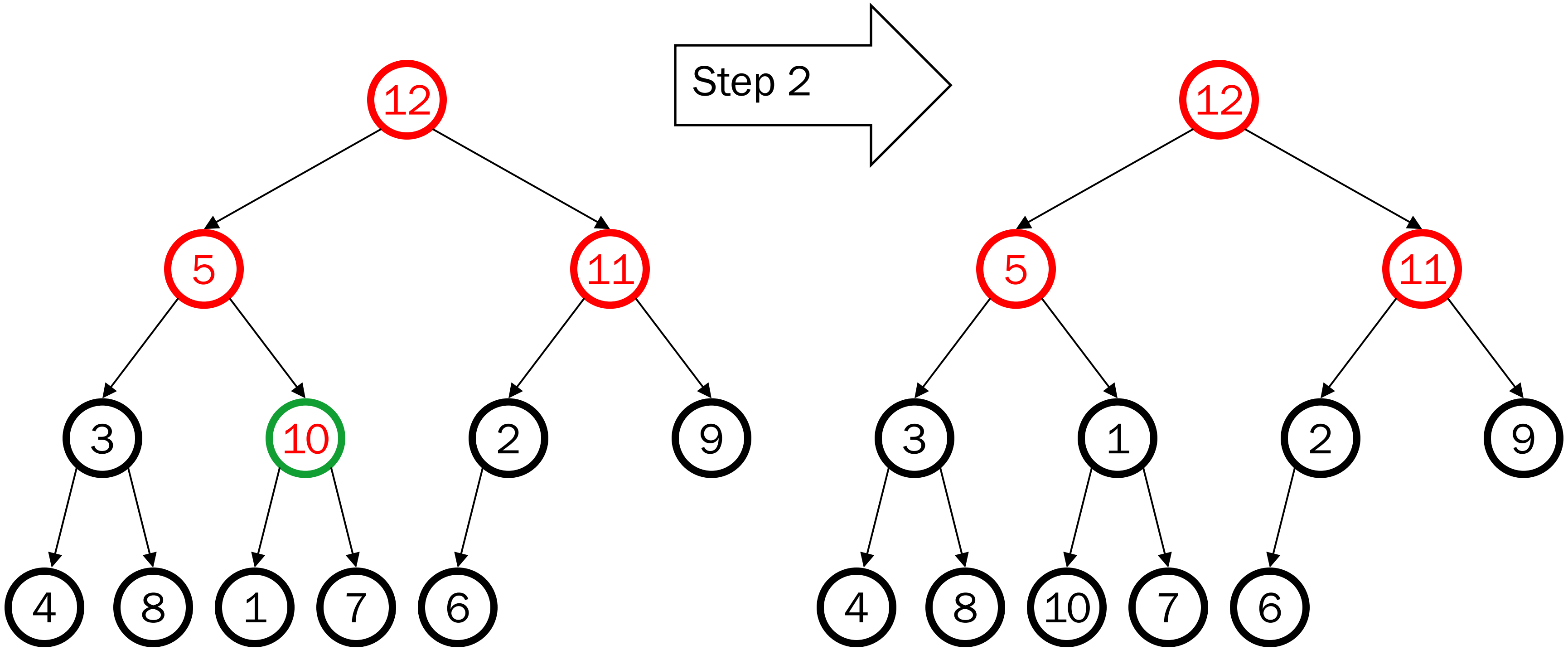


`buildHeap` Example



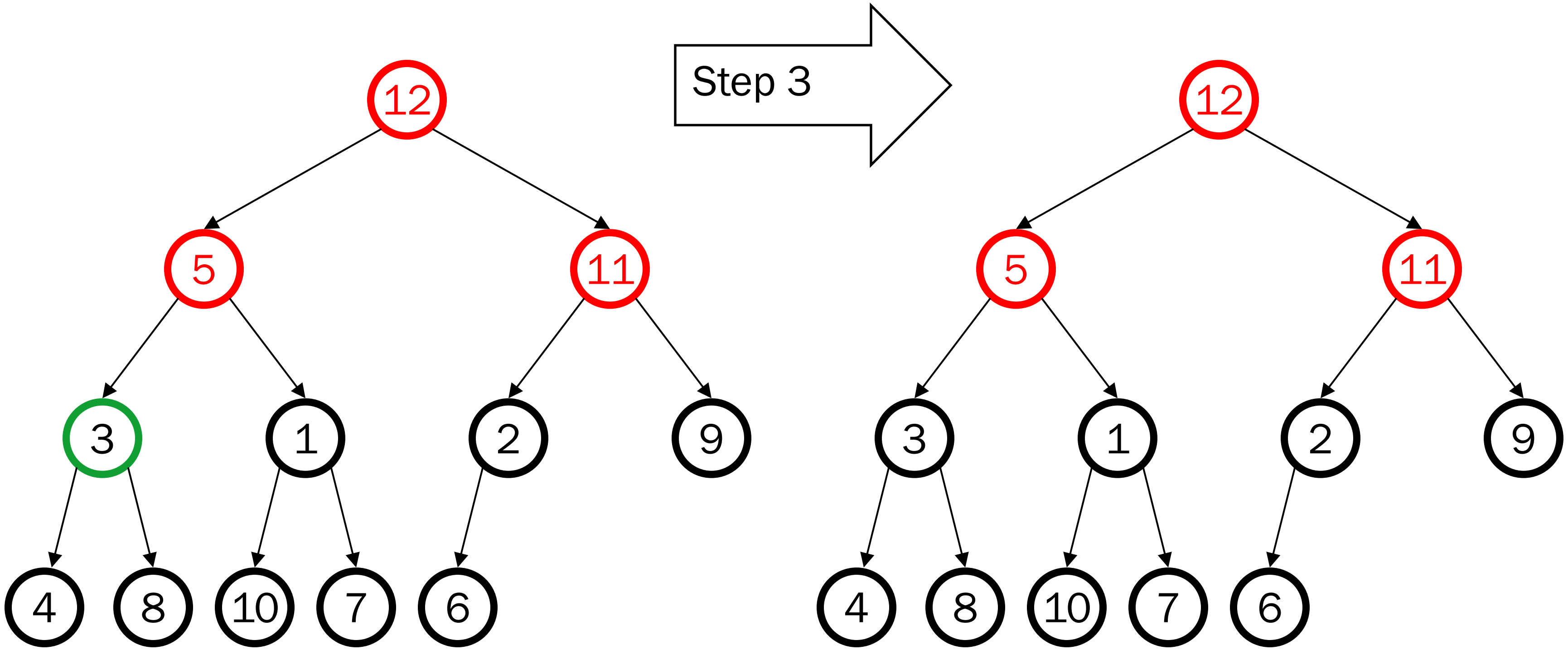
- Happens to already be less than child

`buildHeap` Example



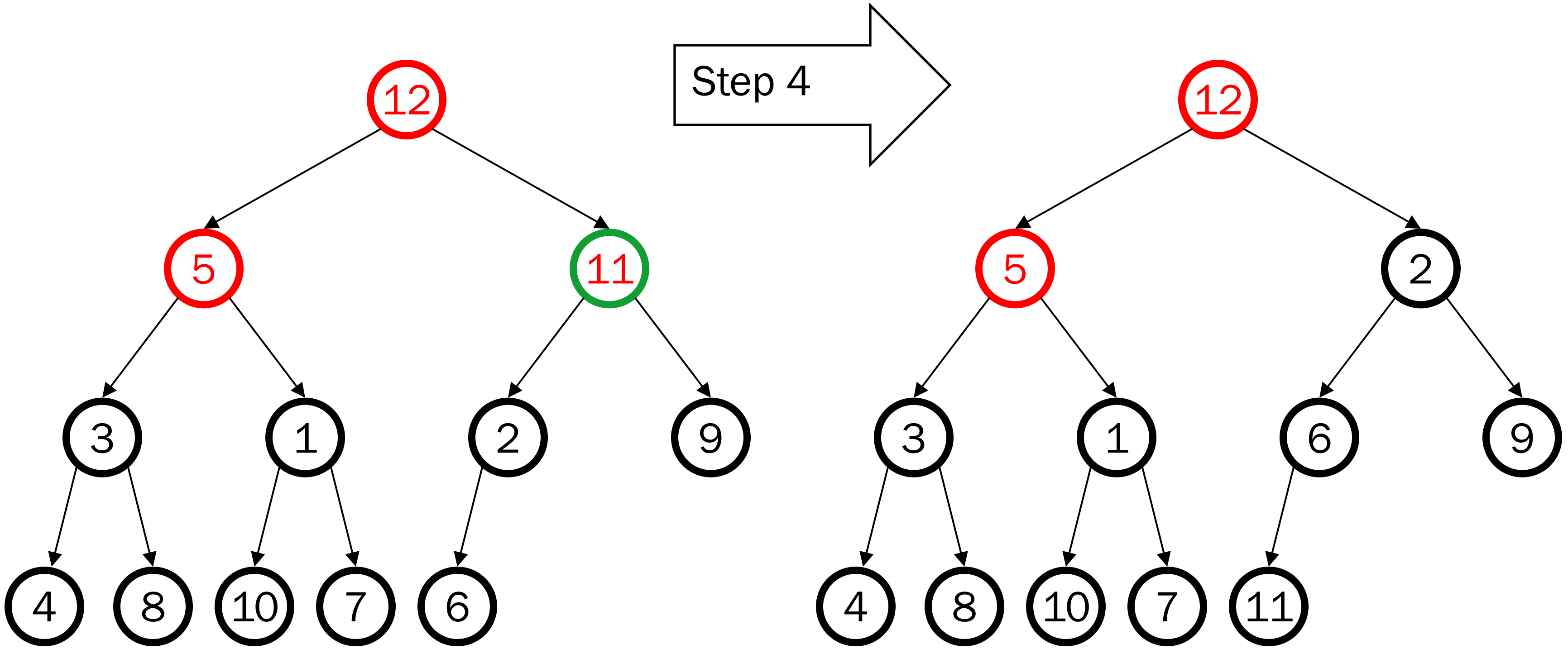
- Percolate down (notice that moves 1 up)

`buildHeap` Example



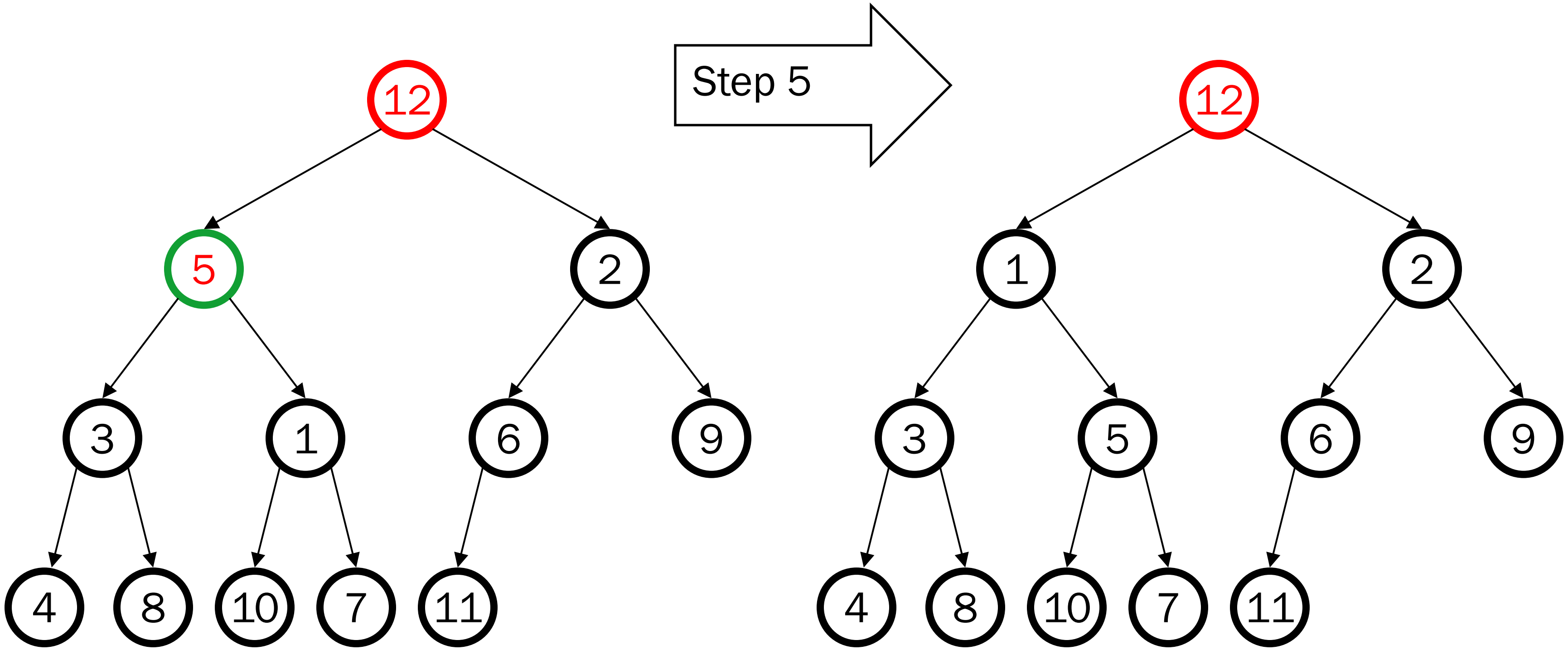
- Another nothing-to-do step

`buildHeap` Example

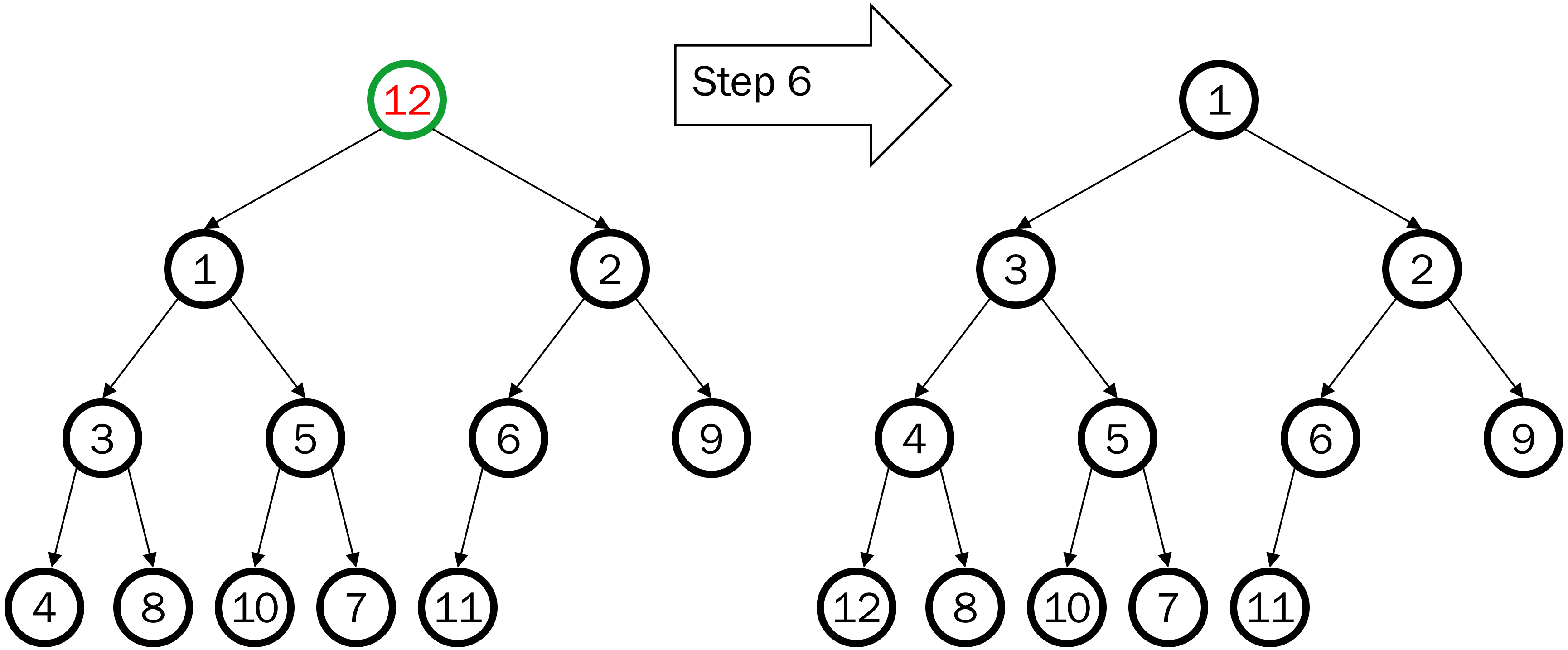


- Percolate down as necessary (steps 4a and 4b)

`buildHeap` Example



`buildHeap` Example



But is it right?

- "Seems to work"
  - Let's *prove* it restores the heap property (correctness)
  - Then let's *prove* its running time (efficiency)

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Correctness

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
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    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Loop Invariant: For all `j>i`, `arr[j]` is less than its children

- True initially: If `j > size/2`, then `j` is a leaf
  - Otherwise its left child would be at position > `size`
- True after one more iteration: loop body and `percolateDown` make `arr[i]` less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Loop Invariant:

For all `j>i`, `arr[j]` is less than its children

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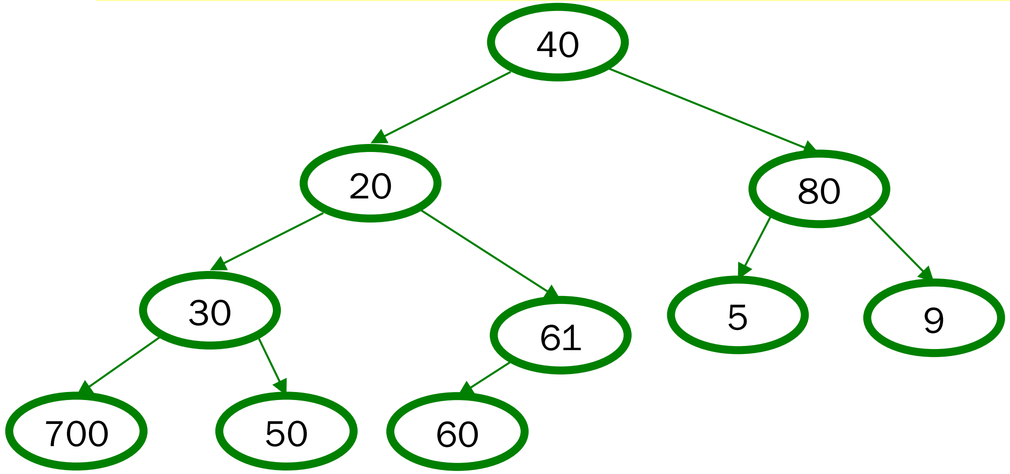

	40	20	80	30	61	5	9	700	50	60			
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
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  }
}
```

Easy argument: **`buildHeap`** is $O(n$ **`log`** $n)$ where n is **`size`**

- **`size/2`** loop iterations
- Each iteration does one **`percolateDown`**, each is $O($**`log`** $n)$

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
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Better argument: **`buildHeap`** is $O(n)$ where n is **`size`**

Efficiency

```
void buildHeap() {
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    arr[hole] = val;
  }
}
```

Better argument: **`buildHeap`** is $O(n)$ where n is **`size`**

- **`size/2`** total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps… etc.
- ((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + …) = 2 (page 4 of Weiss)
  - So at most **`2(size/2)`** *total* percolate steps: $O(n)$
  - Also see Weiss 6.3.4, sum of heights of nodes in a perfect tree

Lessons from `buildHeap`

- Without **`buildHeap`**, our ADT already let clients implement their own in $\theta(n \texttt{ log } n)$ worst case
  - Worst case is inserting lower priority values later
- By providing a specialized operation internally (with access to the data structure), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness: Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was $O(n \texttt{ log } n)$
    - A “tighter” analysis shows same algorithm is $O(n)$

More heaps (see Weiss if curious)

- *d*-heaps: have *d* children instead of 2 (Weiss 6.5)
  - Makes heaps shallower, useful for heaps too big for memory
  - How does this affect the asymptotic run-time (for small d's)?
- Leftist heaps, skew heaps, binomial queues (Weiss 6.6-6.8)
  - Different data structures for priority queues that support a logarithmic time **`merge`** operation (impossible with binary heaps)
  - **`merge:`** given two priority queues, make one priority queue
  - Insert & deleteMin defined in terms of merge

Aside: How might you merge *binary* heaps:

- If one heap is much smaller than the other?
- If both are about the same size?