CSE 332 : 22Su Midterm Solutions

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Instructions

- The allotted time is **60** minutes. Please do not turn the page until the staff says so.
- This is a closed-book and closed-notes exam. You are not permitted to access electronic devices.
- Read directions carefully, especially for problems that require you to show work or provide an explanation.
- We can only give partial credit for work that you've written down.
- Unless otherwise noted, every time we ask for an O, Ω, or Θ bound, it must be simplified and tight.
- For answers that involve bubbling ◯ or □, make sure to fill in the shape completely: ● or ■.
- If you run out of room on a page, indicate that the answer continues on the back of that page. Try to avoid writing on the very edges of the pages: we scan your exams and edges often get cropped off.
- Make sure you also get a copy of the formula sheet.

Advice

- If you feel like you're stuck on a problem, you may want to skip it and come back at the end if you have time.
- Look at the question titles on the cover page to see if you want to start somewhere other than problem 1.
- Remember to take deep breaths.

Question	**Max points**
1. Big-O	16
2. Code Analysis	20
3. $\mathcal{O}, \Omega$, and Θ, oh, my!	12
4. Write a recurrence	9
5. Solve a recurrence	10
6. AVL	8
7. Heaps	15
8. B-Tree	8
Total	**98**

1. Big-O [16 points]

For questions asking you about runtime, give a simplified, tight big-$\mathcal{O}$ bound. This means that, for example, $\mathcal{O}\left(2^{n!}\right)$ or $\mathcal{O}\left(5n^2 + 7n + 3\right)$ are unlikely to get points. Unless otherwise specified, all logs are base 2. For short sentence answers, points may be deducted for answers exceeding the sentence limit. You may also leave you answer as an unsimplified formula like $7 \cdot 10^3$ when appropriate.

(a) Best-case runtime finding an element in a sorted array of size `n` **Solution:**

$O(1)$

(b) Given a binary search tree of `n` integers, worst-case runtime to convert the binary search tree into a sorted array. **Solution:**

$O(n)$

(c) Runtime constructing a binary min heap using Floyd's build heap, given an array of values n, n - 1, n - 2 ... 1, in descending order. **Solution:**

$O(n)$

(d) Given a B-tree of height 3, with `M=5` and `L=4` what is the **MINIMUM** number of data items in the tree? **Solution:**

36

(e) Given a B-tree of height 3, with `M=5` and `L=4`, what is the **MAXIMUM** number of data items in the tree? **Solution:**

500

(f) The TAs propose a new data data structure - MinThousandHeap, where it's a min heap, but each node has 1000 children.
Claim: MinThousandHeap is always more efficient than a binary heap, since the height of the heap is $\log_{1000} n$ instead of $\log_2 n$, so the runtime for operations such as `insert` and `deleteMin` will run much faster.
Do you agree with this claim, why or why not? Explain in 1 sentence. **Solution:**

Disagree - even though the height of the tree is much shorter, we need to compare 999 items in the worst case on each level time when calling `deleteMin`, rather than only once in the case of binary heap, making the operation much slower - constant matters!

(g) Name one benefit of using a B-tree over an AVL tree. **Solution:**

B-Trees work a lot better when we have a lot of data since it reduces expensive disk accesses.

(h) When designing a B-Tree, what is a possible downside of storing both the key AND value in the internal nodes

for the `find` operation? **Solution:**

Space per internal node must remain the same so since we are now storing key and value as well as pointers, the number of pointers per internal node will decrease which means that the value of M will decrease which in turn would increase the height of the B-Tree.

2. Code Analysis [20 points]

Describe the worst-case running time for the following pseudocode functions in Big-$\mathcal{O}$ notation in terms of the variable `n`. Your answer must be tight and simplified. **You do not have to show work or justify your answers for this problem.**

(a)
```
int kilimanjaro(int n, int steps) {
    steps = 0;

    for (i = 0; i < 332; i++) {
        steps += n;
    }
    return steps;
}
```

(a) _____

Solution:

$O(1)$

(b)
```
int everest(int n, int steps) {
    if (n < 5) {
        System.out.println("Keep climbing!");
        return 0;
    } else if (n < 100) {
        return everest(n / 3, steps);
    } else {
        for (int i = 0; i < n * n * n; i++) {
            steps++;
        }

        return everest(n - 2, steps);
    }
}
```

(b) _____ **Solution:**

$O(n^4)$

(c)
```
int matterhorn(int n, int steps) {
    int elevation = n;
```

```
    while (elevation > 0) {
        elevation  = elevation/2;
    }

    return steps;
}
```

(c)

Solution:

$O(n \log n)$

(d)
```
MinFourHeap rainierAscent(int n) {
    MinFourHeap mountain = new MinFourHeap();

    if (n < 50) {
        return mountain;
    } else {
        for (int i = 0; i < n; i++) {
            mountain.insert(i);
        }
    }

    return mountain;
}
```

(d)

Solution:

$O(n)$.

(e)
```
MinFourHeap rainierDescent(int n) {
    MinFourHeap mountain = new MinFourHeap();

    if (n < 50) {
        return mountain;
    } else {
        for (int i = n; i >= 0; i--) {
            mountain.insert(i);
        }
    }

    return mountain;
}
```

(e) _____

Solution:

$O(nlogn)$.

3. $\mathcal{O}, \Omega,$ and Θ, oh my! [12 points]

For each of the following statements, indicate whether it is always true, sometimes true, or never true. You do not need to include an explanation. Assume that the domain and codomain of all functions in this problem are natural numbers(1, 2, 3 ...).

(a) If $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(h(n))$ then $f(n)$ is $O(g(n) \cdot h(n))$.

Always True **Sometimes True** **Never True**

Solution:

Always True

(b) If $f(n)$ is $O(g(n))$ and $g(n)$ is $\Omega(h(n))$, then $f(n)$ is $\Theta(h(n))$.

Always True **Sometimes True** **Never True**

Solution:

Sometimes True

(c) Let $f(n)$ be the best-case running time of a call to `deleteMin` in a binary min heap with n distinct elements. $f(n)$ is $\Omega(1)$.

Always True **Sometimes True** **Never True**

Solution:

Always true

4. Write a recurrence [9 points]

Give a base case and a recurrence for the runtime of the following function. Use variables appropriately for constants (e.g. c1, c2, etc.) in your recurrence (you do not need to attempt to count the exact number of operations). **YOU DO NOT NEED TO SOLVE this recurrence**

```
int pirate(int n) {
    if (n < 3) {
        for (int i = 0; i < 10; i++) {
            print("Aye, Aye");
        }
        return 332;
    }
    for (int i = 1; i < pow(2, n); i *= 2) {
        print("Ahoy, matey!");
    }
    return n * pirate(n / 5) + 10 * pirate(n - 2);
}
```

$$T(n) = \begin{cases} \underline{\qquad\qquad} & \text{for } n < 3 \\ \underline{\qquad\qquad} & \text{for } n \geq 3 \end{cases}$$

Yay!! You do NOT need to solve *this* recurrence...

Solution:

$$T(n) = \begin{cases} c_0 & \text{for } n < 3 \\ T(\frac{n}{5}) + T(n-2) + c_1 \cdot n + c_2 & \text{for } n \geq 3 \end{cases}$$

Notes: The loop of "Ahoy, matey" runs n times.

5. Solve a Recurrence [10 points]

Suppose the running time of an algorithm satisfies the recurrence given below. Find the closed form for $T(N)$. **You may assume N is a large power of 2.** Your answer should *not* be in Big-$\mathcal{O}$ notation. Show the exact constants and bases of logarithms in your answer (e.g. do NOT use c_1, c_2 in your answer).
Your final answer must not have any summation symbols or recursion – you may find the list of summations and logarithm identities on the last page of the exam to be useful.
You must show your work to receive *any* credit.

$$T(n) = \begin{cases} 1 & \text{if n = 1} \\ 4T\left(\frac{n}{2}\right) + n & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + n \\ &= 4\left[4T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)\right] + n \\ &= 4^2 T\left(\frac{n}{2^2}\right) + 2n + n \\ &= 4^2\left[4T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)\right] + 2n + n \\ &= 4^3 T\left(\frac{n}{2^3}\right) + 2^2 n + 2n + n \\ &\cdots \\ &= 4^i T\left(\frac{n}{2^i}\right) + \sum_{j=0}^{i-1} 2^j n \end{aligned}$$

We will need the i that hits the base case, in this case that's when $n = 1$, so we solve for:

$$\frac{n}{2^i} = 1 \implies n = 2^i \implies i = \log_2(n)$$

Plugging in, we get:

$$\begin{aligned} 4^{\log_2(n)} T(1) + \sum_{j=0}^{\log_2(n)-1} 2^j n &= n^{\log_2(4)} + n \sum_{j=0}^{\log_2(n)-1} 2^j \\ &= n^2 + n \cdot \frac{2^{\log_2(n)} - 1}{2 - 1} \\ &= n^2 + n \cdot (n - 1) \\ &= 2n^2 - n \end{aligned}$$

6. Adelson-Velskii and Landis [8 points]

(a) (2 points) What is the balance condition for AVL trees? **Answer in one sentence**
Solution:

The invariant followed by an AVL Tree is that for any given node, the height of the left and right subtrees of that node cannot differ by more than 1.

(b) (3 points) If we have an AVL Tree with 11 nodes, what is the maximum height of the AVL Tree? Show your work. **Solution:**

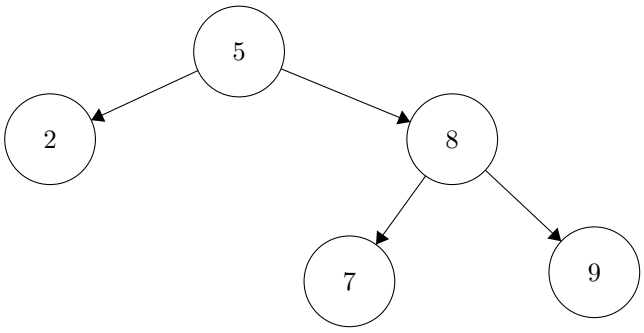
The maximum height is 3.

We can calculate this by using the formula $S(h) = S(h-1) + S(h-2)$.

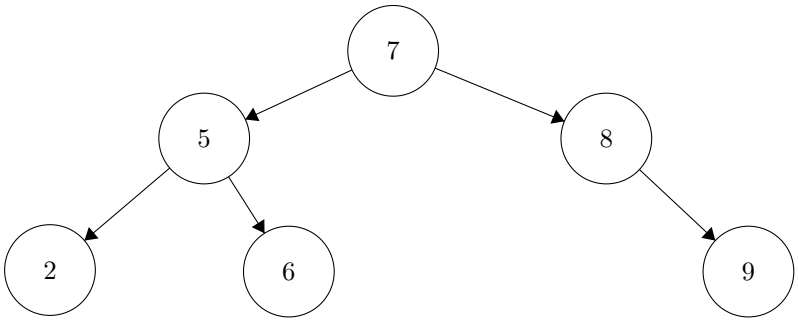
$$S(0) = 1 S(1) \quad = 2 S(2) = 2 + 1 + 1 = 4 S(3) \quad = 4 + 2 + 1 = 7 S(4) = 7 + 4 + 1 = 12$$

So, we know that we need to have at least 12 nodes for a height of 4 and thus 11 can only have a maximum height of 3.

(c) (3 points) Suppose we have the following AVL Tree:



Then, we **insert the value 6** into the AVL Tree. What does the AVL Tree look like **AFTER the AVL Tree is balanced**? Make sure to show work for possible partial credit. **Solution:**



7. Heaps [15 points]

(a) You are busy at your new job writing a binary min-heap implementation. However, your users often get confused on whether low priority numbers mean ”urgent” or ”not urgent”.

So, you decide to write a new operation:

`reverse(): given a binary min-heap (with root at index 0), convert it to a binary max-heap stored in the same array (with root at index 0)`

Your idea to implement this is to just reverse the order of the elements in the array.

ie: If the `array` backing the heap was [1, 2, 3, 4, 5], it would then become [5, 4, 3, 2, 1]. (Assume the `array` is exactly the `size` of the heap).

(5 points) Does this algorithm work? If it does, informally argue why (formal proof is not needed). If it doesn't, provide a counter-example (with explanation of why it is a counter-example)

Solution:

No, if we had a min-heap with the values [1, 3, 2], reversing this array would give us [2, 3, 1] which is not a valid max-heap. The 2 should not be at the root since the 3 is larger than it.

(b) (2 points) What are the two properties that define a min-heap. (Not necessarily binary!)

Solution:

(i) Every node is less than all of its children (ii) The tree is complete

For the next two questions, we are given a BINARY min-heap that contains the elements 1,2,3,...,100.

(c) (4 points) Which depth(s) of the heap can the value 4 be found in? Recall the root is at depth 0. For any credit, you must briefly explain your answer.

Solution:

The value 4 can occur at depths 1, 2, and 3. The value 1 must be at the root, depth 0. At depth 1, we can have 2 and 4 be the children and build the rest of the heap around that. For depth 2, we can have 2 and 3 be the children of 1 and then place 4 at depth 2. For depth 3, this is the deepest depth that 4 can exist at since at depth 2, we must place the value 3 and at depth 1, we must place the value 2.

(d) (4 points) Can the largest value, 100, be found in a non-leaf spot? Why or why not?

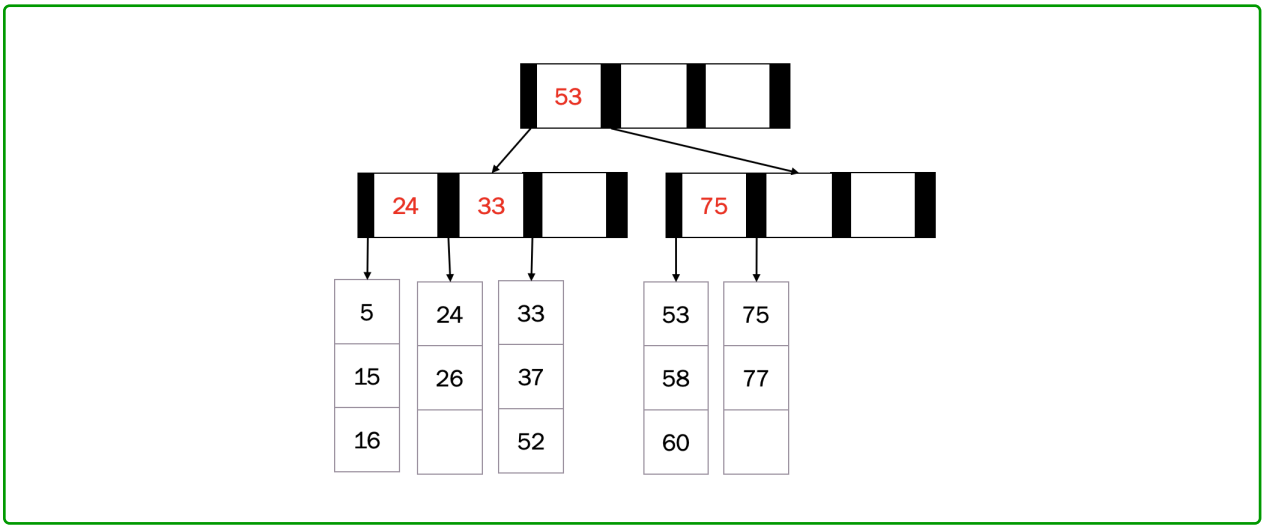
Solution:

No, the largest value must always be a leaf since if there were any values beneath it, those values would have to be larger than it.

8. B-Tree [8 points]

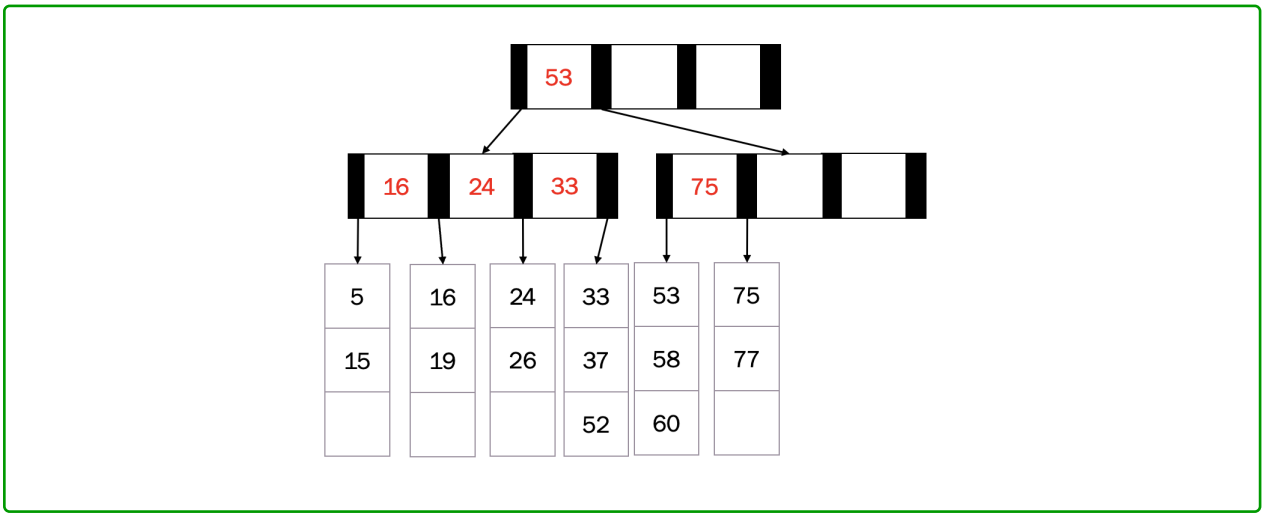
(a) (2 points) In this ORIGINAL B-Tree shown, **write in the correct values for the internal nodes**.

Original B-Tree: **Solution:**



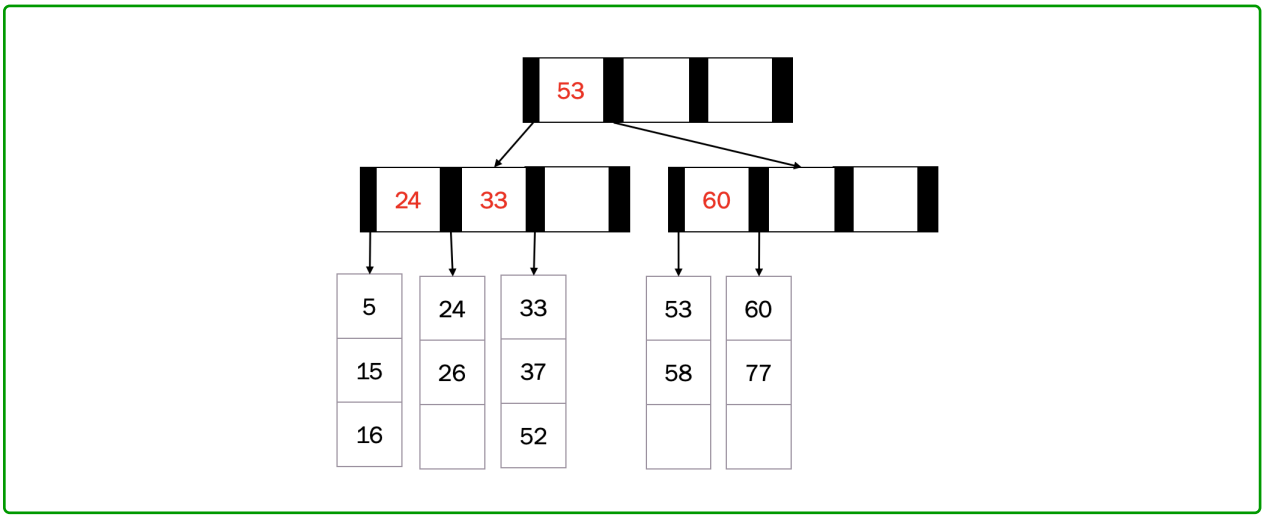
(b) (3 points) In the space below, **starting with the ORIGINAL B-Tree**, draw the tree resulting after ***inserting the value 19*** (including values for internal nodes). Use the method of insertion described in lecture and in the book.

Solution:



(c) (3 points) In the space below, **starting with the ORIGINAL B-Tree**, draw the tree resulting after ***deleting the value 77*** (including values for internal nodes). Use the method of deletion described in lecture and in the book.

Solution:


53
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Extra piece of paper for scratch work

Reference Sheet

Geometric series identities

$$\sum_{i=0}^{k} c^i = \frac{c^{k+1}-1}{c-1} \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c} \text{ if } |c| < 1$$

Sums of polynomials

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Log identities

$$b^{\log_b(a)} = a \qquad \log_b(x^y) = y \cdot \log_b(x) \qquad a^{\log_b(c)} = c^{\log_b(a)} \qquad \log_b(a) = \frac{\log_d(a)}{\log_d(b)}$$

Exponent properties

$$(a^m)^n = a^{m \cdot n} = (a^n)^m$$