

CSE 332: Data Structures and Parallelism

Spring 2022

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Lecture 21: Graph Theory

Announcements

- Upcoming lectures
 - ~~Intro to graphs~~
 - Topological Sort
 - Graph Algorithms (3 lectures)
 - Union-Find Data Structure
 - Theory of NP-Completeness (2 lectures)

Graphs

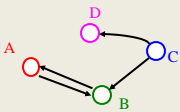
A formalism for representing binary relationships between objects

– Graph $G = (V, E)$

– Set of vertices: $V = \{v_1, v_2, \dots, v_n\}$

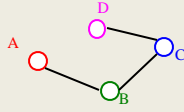
– Set of edges: $E = \{e_1, e_2, \dots, e_m\}$

Directed



$V = \{A, B, C, D\}$
 $E = \{(A, B), (A, B), (B, A), (C, D)\}$

Undirected



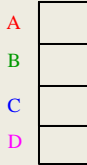
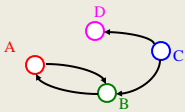
$V = \{A, B, C, D\}$
 $E = \{(C, B), (A, B), (A, B), (C, D)\}$

What's the data structure?

Common query: which edges are adjacent to a vertex

Representation 2: Adjacency List

A list (array) of length $|V|$ in which each entry stores a list (linked list) of all adjacent vertices

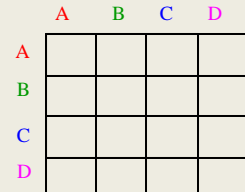
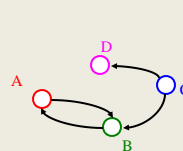


Runtimes:
 Iterate over vertices?
 Iterate over edges?
 Iterate edges adj. to vertex?
 Existence of edge?

Space requirements?
 Best for what kinds of graphs?

Representation 1: Adjacency Matrix

A $|V| \times |V|$ matrix M in which an element $M[u, v]$ is true if and only if there is an edge from u to v

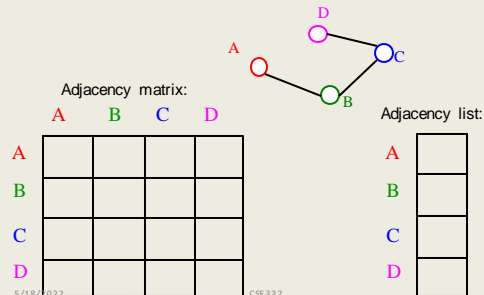


Runtimes:
 Iterate over vertices?
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Space requirements?
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Representing Undirected Graphs

What do these reps look like for an undirected graph?

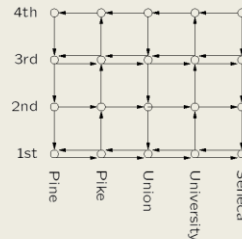


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Some Applications: Bus Routes in Downtown Seattle



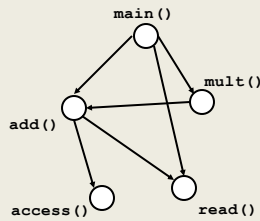
If we're at 3rd and Pine, how can we get to 1st and University using Metro?
How about 4th and Seneca?

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Directed Acyclic Graphs (DAGs)

- DAGs are directed graphs with no (directed) cycles.



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$|E|$ and $|V|$

- How many edges $|E|$ in a graph with $|V|$ vertices?
- What if the graph is directed?
- What if it is undirected and connected?
- Some (semi-standard) terminology:
 - A graph is *sparse* if it has $O(|V|)$ edges (upper bound).
 - A graph is *dense* if it has $\Theta(|V|^2)$ edges.

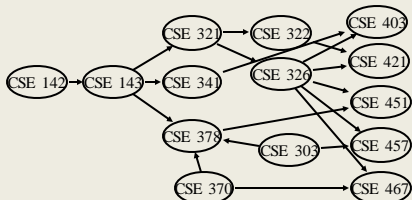
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Application: Topological Sort

- Given a graph, $G = (V, E)$, output all the vertices in V sorted so that no vertex is output before any other vertex with an edge to it.



What kind of input graph is allowed?

Is the output unique?

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Topological Sort: Take One

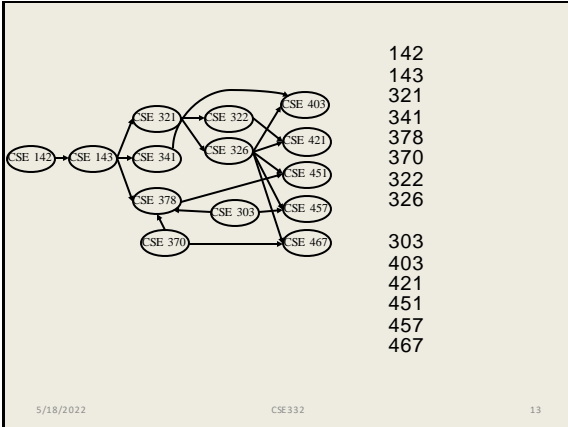
- Label each vertex with its *in-degree* (# inbound edges)
- While** there are vertices remaining:
 - Choose a vertex v of *in-degree zero*; output v
 - Reduce the in-degree of all vertices adjacent to v
 - Remove v from the list of vertices

Runtime:

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```
void topsort(){
    Vertex v, w;

    labelEachVertexWithItsInDegree();

    for (int counter=0; counter < NUM_VERTICES; counter++){
        v = findNewVertexOfDegreeZero();

        v.topologicalNum = counter;
        for each w adjacent to v
            w.indegree--;
    }
}
```

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Topological Sort: Take Two

1. Label each vertex with its in-degree
2. Initialize a queue Q to contain all in-degree zero vertices
3. While Q not empty
 - a. $v = Q.dequeue()$; output v
 - b. Reduce the in-degree of all vertices adjacent to v
 - c. If new in-degree of any such vertex u is zero $Q.enqueue(u)$

Runtime:

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```
topsort(){
    Queue q(NUM_VERTICES);
    Vertex v, w;

    labelEachVertexWithItsIn-degree();

    q.makeEmpty();
    for each vertex v
        if (v.indegree == 0)
            q.enqueue(v);

    while (!q.isEmpty()){
        v = q.dequeue();
        v.topologicalNum = ++counter;
        for each w adjacent to v
            if (--w.indegree == 0)
                q.enqueue(w);
    }
}
```

initialize the queue

get a vertex with indegree 0

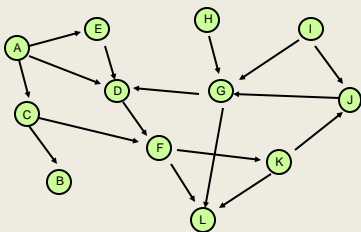
insert new eligible vertices

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Find a topological order for the following graph



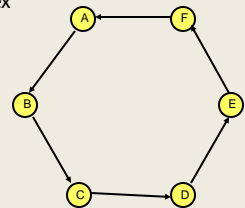
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If a graph has a cycle, there is no topological sort

- Consider the first vertex on the cycle in the topological sort
- It must have an incoming edge



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Lemma: If a graph is acyclic, it has a vertex with in degree 0

- Proof:
 - Pick a vertex v_1 , if it has in-degree 0 then done
 - If not, let (v_2, v_1) be an edge, if v_2 has in-degree 0 then done
 - If not, let (v_3, v_2) be an edge . . .
 - If this process continues for more than n steps, we have a repeated vertex, so we have a cycle