CSE 332: Data Structures and Parallelism

Spring 2022

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Lecture 21: Graph Theory

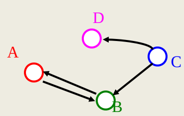
Announcements

- Upcoming lectures
  - Intro to graphs
  - Topological Sort
  - Graph Algorithms (3 lectures)
  - Union-Find Data Structure
  - Theory of NP-Completeness (2 lectures)

Graphs

A formalism for representing relationships between objects

- Graph **G = (V,E)**
- *Set of vertices*:
  $V = \{v_1, v_2, \ldots, v_n\}$
- *Set of edges*:
  $E = \{e_1, e_2, \ldots, e_m\}$
  where each e_i connects one
- vertex to another (v_j, v_k)

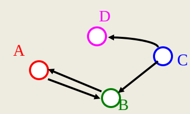


V = {A, B, C, D}
E = {(C, B),
(A, B),
(B, A)
(C, D)}

For *directed edges*, (v_j, v_k) and (v_k, v_j) are distinct.

Graphs

Notation
- |V| = number of vertices
- |E| = number of edges



v is *adjacent* to **u** if **(u,v) ∈ E**
- *neighbor* of = adjacent to
- Order matters for directed edges

It is possible to have an edge **(v,v)**, called a *loop*.
- We will assume graphs without loops.

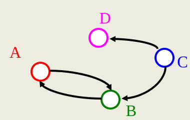
V = {A, B, C, D}
E = {(C, B),
(A, B),
(B, A)
(C, D)}

Examples of Graphs

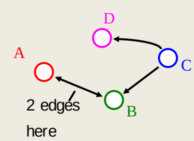
- For each, what are the vertices and edges?
- The web
- Facebook
- Highway map
- Airline routes
- Call graph of a program
- …

Directed Graphs

In *directed* graphs (a.k.a., *digraphs*), edges have a direction:



or

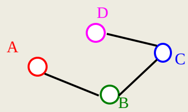


Thus, **(u,v) ∈ E** does *not* imply **(v,u) ∈ E**.
I.e., **v** adjacent to **u** does *not* imply **u** adjacent to **v**.

In-degree of a vertex: number of inbound edges.
Out-degree of a vertex : number of outbound edges.

Undirected Graphs

In *undirected* graphs, edges have no specific direction (edges are always two-way):

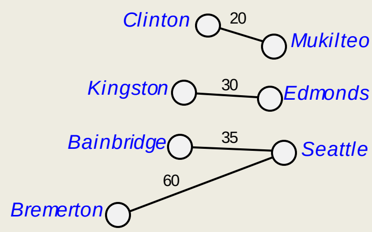


Thus, `(u,v)` ∈ `E` *does* imply `(v,u)` ∈ `E`. Only one of these edges needs to be in the set; the other is implicit.

Degree of a vertex: number of edges containing that vertex. (Same as number of adjacent vertices.)

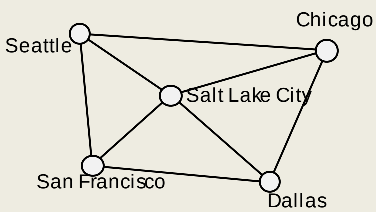
Weighted Graphs

Each edge has an associated weight or cost.



Paths and Cycles

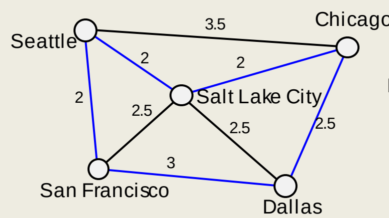
- A *path* is a list of vertices $\{w_1, w_2, \ldots, w_q\}$ such that $(w_i, w_{i+1}) \in E$ for all $1 \leq i < q$
- A *cycle* is a path that begins and ends at the same node



P = {Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle}

Path Length and Cost

- *Path length*: the number of edges in the path
- *Path cost*: the sum of the costs of each edge



For path P:
length(P) = 5
cost(P) = 11.5

How would you ensure that length(p)=cost(p) for all p?

Simple Paths and Cycles

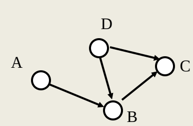
A *simple path* repeats no vertices (except that the first can also be the last):
- P = {Seattle, Salt Lake City, San Francisco, Dallas}
- P = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}

A *cycle* is a path that starts and ends at the same node:
- P = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}
- P = {Seattle, Salt Lake City, Seattle, San Francisco, Seattle}

A *simple cycle* is a cycle that is also a simple path (in undirected graphs, no edge can be repeated).

Paths/Cycles in Directed Graphs

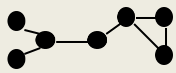
Consider this directed graph:



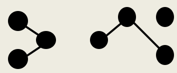
Is there a path from A to D?
Does the graph contain any cycles?

Undirected Graph Connectivity

- Undirected graphs are *connected* if there is a path between any two vertices:

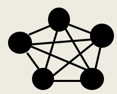


Connected graph



Disconnected graph

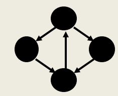
- A *complete undirected* graph has an edge between every pair of vertices:



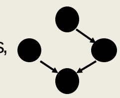
- (Complete = *fully connected*)

Directed Graph Connectivity

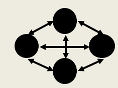
Directed graphs are *strongly connected* if there is a path from any one vertex to any other.



Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction.*



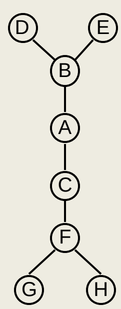
A *complete directed* graph has a directed edge between every pair of vertices. (Again, complete = *fully connected.*)



Trees as Graphs

A tree is a graph that is:

- *undirected*
- *acyclic*
- *connected*



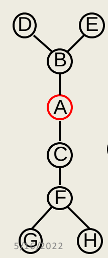
Rooted Trees

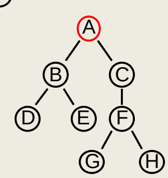
We are more accustomed to:

Rooted trees (a tree node that is "special")

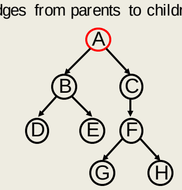
Directed edges from parents to children (parent closer to root).

A rooted tree (root indicated in red) drawn two ways





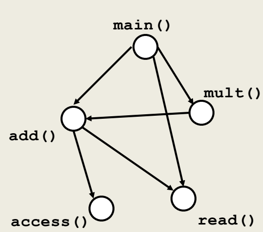
Rooted tree with directed edges from parents to children.



Characteristics of this one?

Directed Acyclic Graphs (DAGs)

- DAGs are directed graphs with no (directed) cycles.

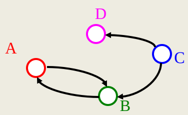


What's the data structure?

Common query: which edges are adjacent to a vertex

Representation 2: Adjacency List

A list (array) of length `|V|` in which each entry stores a list (linked list) of all adjacent vertices



A	
B	
C	
D	

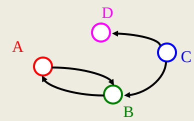
Runtimes:
Iterate over vertices?
Iterate over edges?
Iterate edges adj. to vertex?
Existence of edge?

Space requirements?

Best for what kinds of graphs?

Representation 1: Adjacency Matrix

A `|V| x |V|` matrix `M` in which an element `M[u,v]` is true if and only if there is an edge from `u` to `v`



	A	B	C	D
A				
B				
C				
D				

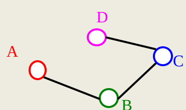
Runtimes:
Iterate over vertices?
Iterate over edges?
Iterate edges adj. to vertex?
Existence of edge?

Space requirements?

Best for what kinds of graphs?

Representing Undirected Graphs

What do these reps look like for an undirected graph?



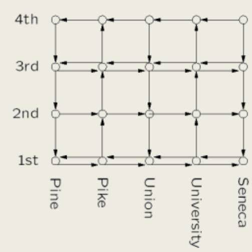
Adjacency matrix:

	A	B	C	D
A				
B				
C				
D				

Adjacency list:

A	
B	
C	
D	

Some Applications: Bus Routes in Downtown Seattle



If we're at 3[rd] and Pine, how can we get to 1[st] and University using Metro?

How about 4[th] and Seneca?