CSE 332: Data Structures and Parallelism

Spring 2022

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Lecture 26: Dijkstra’s Algorithm

Announcements

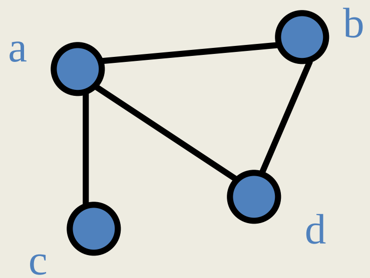
- Upcoming lectures
  - ~~Intro to graphs~~
  - ~~Topological Sort~~
  - Graph Algorithms
    - ~~Graph Traversal~~
    - Shortest Paths
    - Minimum Spanning Tree
  - Theory of NP-Completeness (2 lectures)

Graph Theory

- $G = (V, E)$
  - V: vertices, $|V| = n$
  - E: edges, $|E| = m$
- Undirected graphs
  - Edges sets of two vertices $\{u, v\}$
- Directed graphs
  - Edges ordered pairs (u, v)
- Many other flavors
  - Edge / vertices weights
  - Parallel edges
  - Self loops

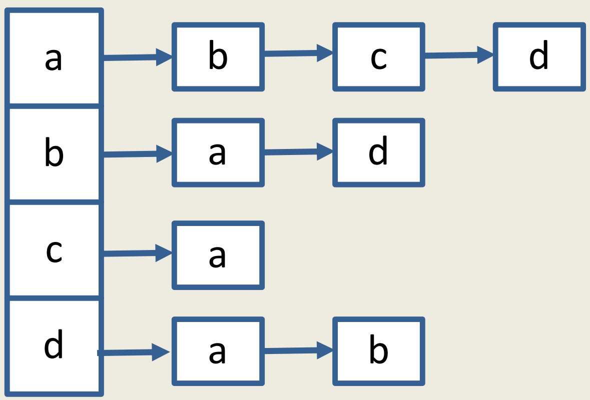
- Path: $v_1, v_2, \ldots, v_k$, with (v_i, v_{i+1}) in E
  - Simple Path
  - Cycle
  - Simple Cycle
- Neighborhood
  - $N(v)$
- Distance
- Connectivity
  - Undirected
  - Directed (strong connectivity)
- Trees
  - Rooted
  - Unrooted

Graph Representation



V = { a, b, c, d}

E = { {a, b}, {a, c}, {a, d}, {b, d} }



Adjacency List

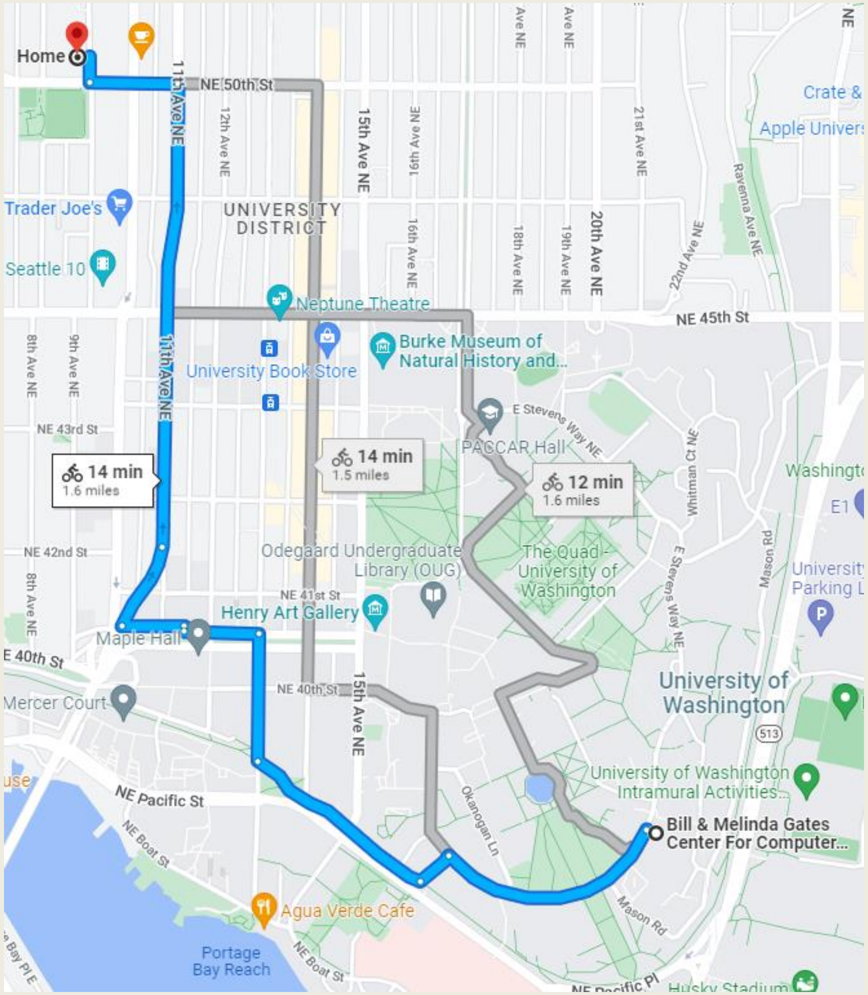
$O(n + m)$ space

	1	1	1
1		0	1
1	0		0
1	1	0	

Adjacency Matrix

$O(n^2)$ space

Find the shortest path



The Shortest Path Problem

Given a graph G, and vertices s and t in G, find the shortest path from s to t.

Two cases: weighted and unweighted.

For a path $p = v_0\ v_1\ v_2 \ldots v_k$

- *unweighted length* of path $p = k$ (*length*)
- *weighted length* of path $p = \sum_{i=0..k-1} c_{i,i+1}$ (*cost*)

We will assume the graph is directed

Single Source Shortest Paths (SSSP)

Given a graph *G* and vertex *s*, find the shortest paths from *s* to all vertices in G.

- How much harder is this than finding single shortest path from s to t?
  - Most algorithms will have to find the shortest path to every vertex in the graph in the worst case
    - Although may stop early in some cases

SSSP: Unweighted Version

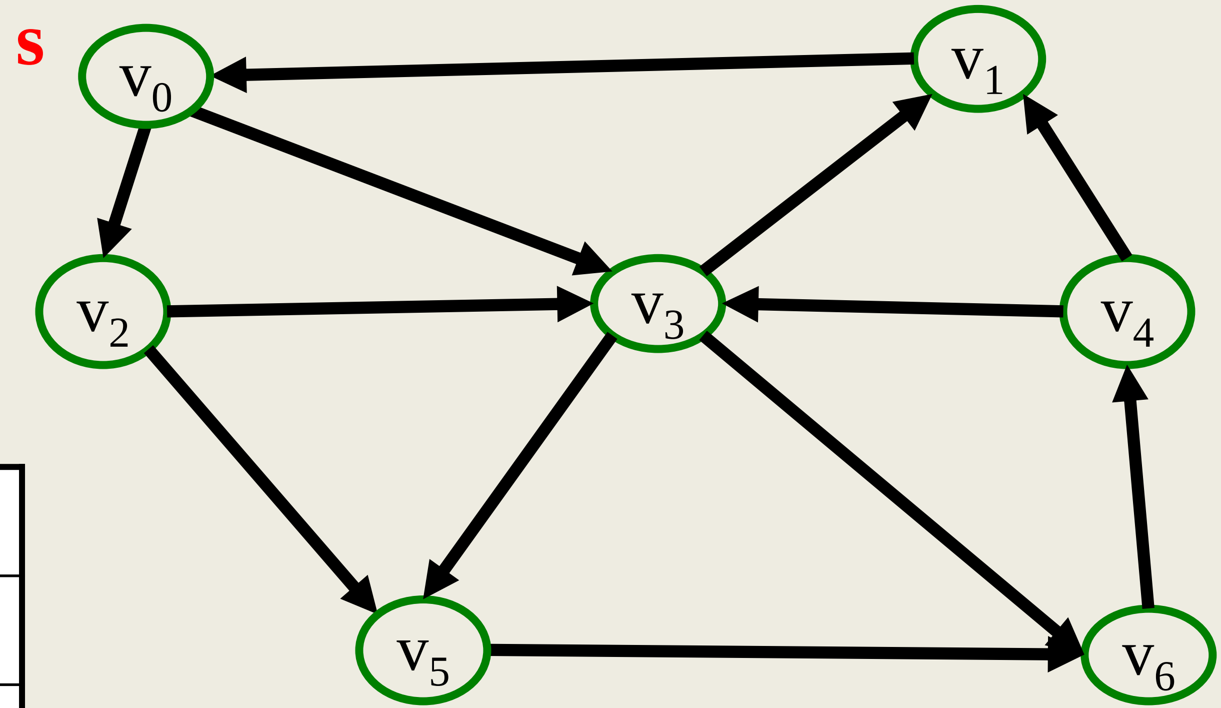
- This is just Breadth First Search
  - Build a breadth first search tree starting from s

```
void BFS(Vertex s){
  Queue q(NUM_VERTICES);
  Vertex v, w;
  for each w {
    w.dist = INFINITY;
    w.prev = -1;
  }
  s.dist = 0;
  q.enqueue(s);

  while (!q.isEmpty()){
    v = q.dequeue();
    for each w adjacent to v
      if (w.dist == INFINITY){
        w.dist = v.dist + 1;
        w.prev = v;
        q.enqueue(w);
      }
    }
  }
```

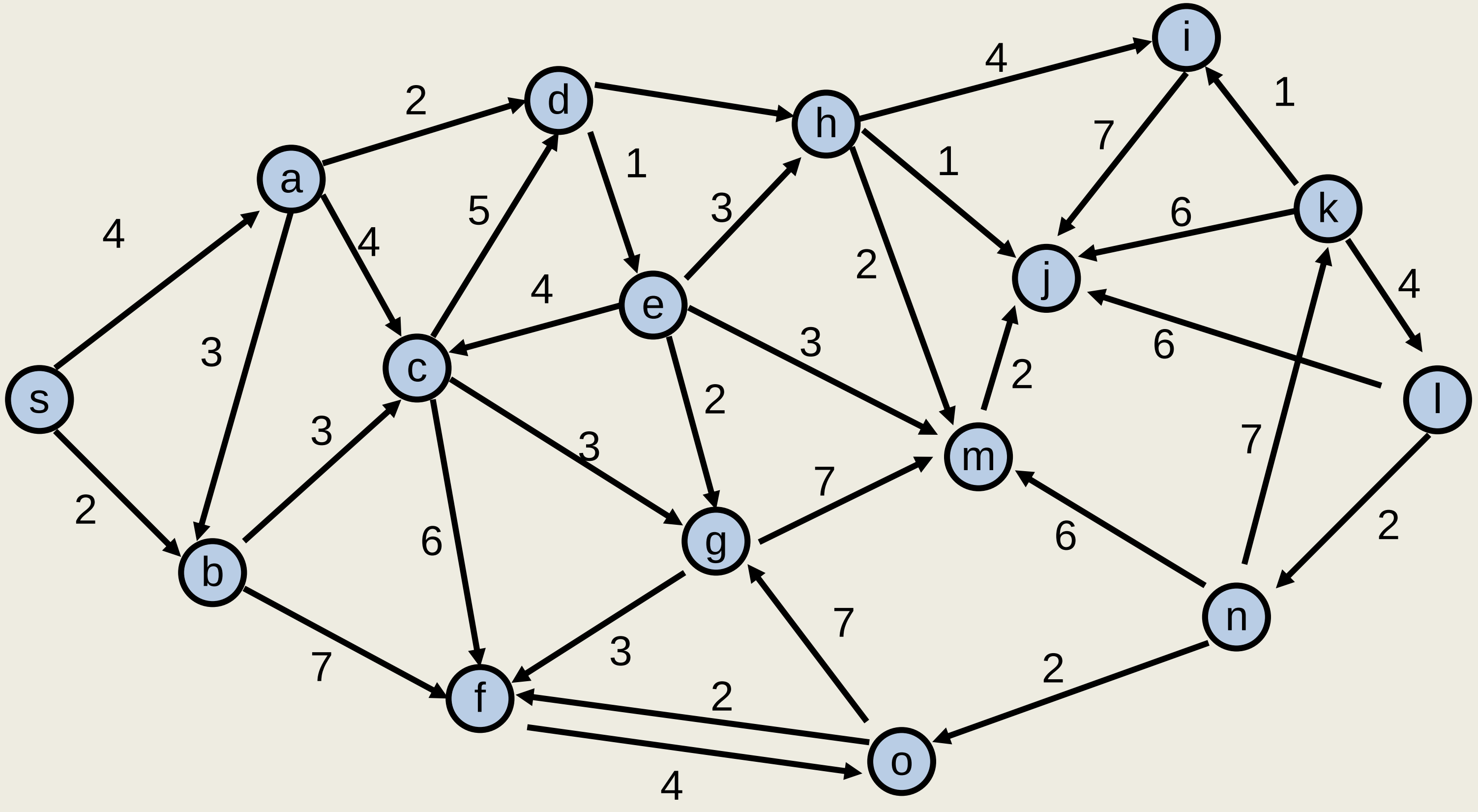
each edge examined at most once – if adjacency lists are used

each vertex enqueued at most once

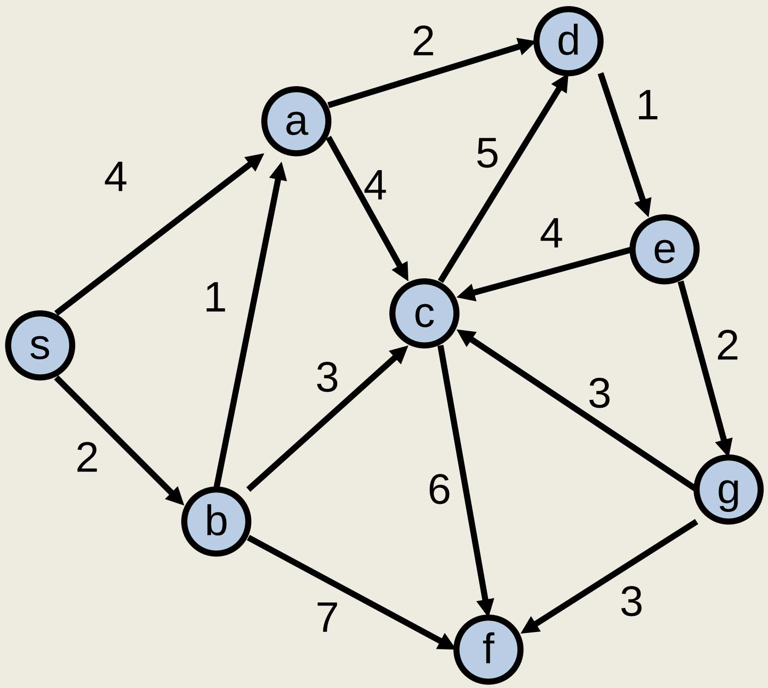


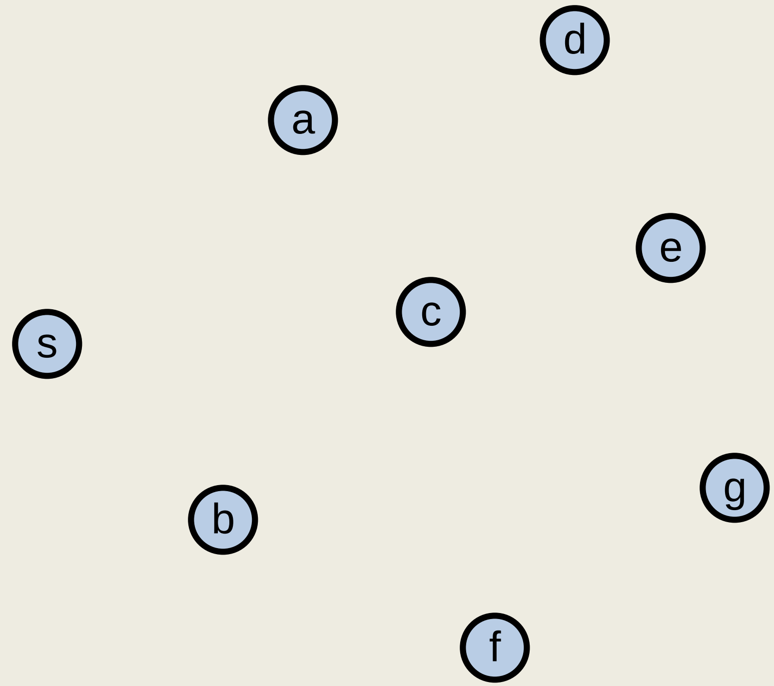
V	Dist	prev
v_0		
v_1		
v_2		
v_3		
v_4		
v_5		
v_6		

Weighted shortest paths problem



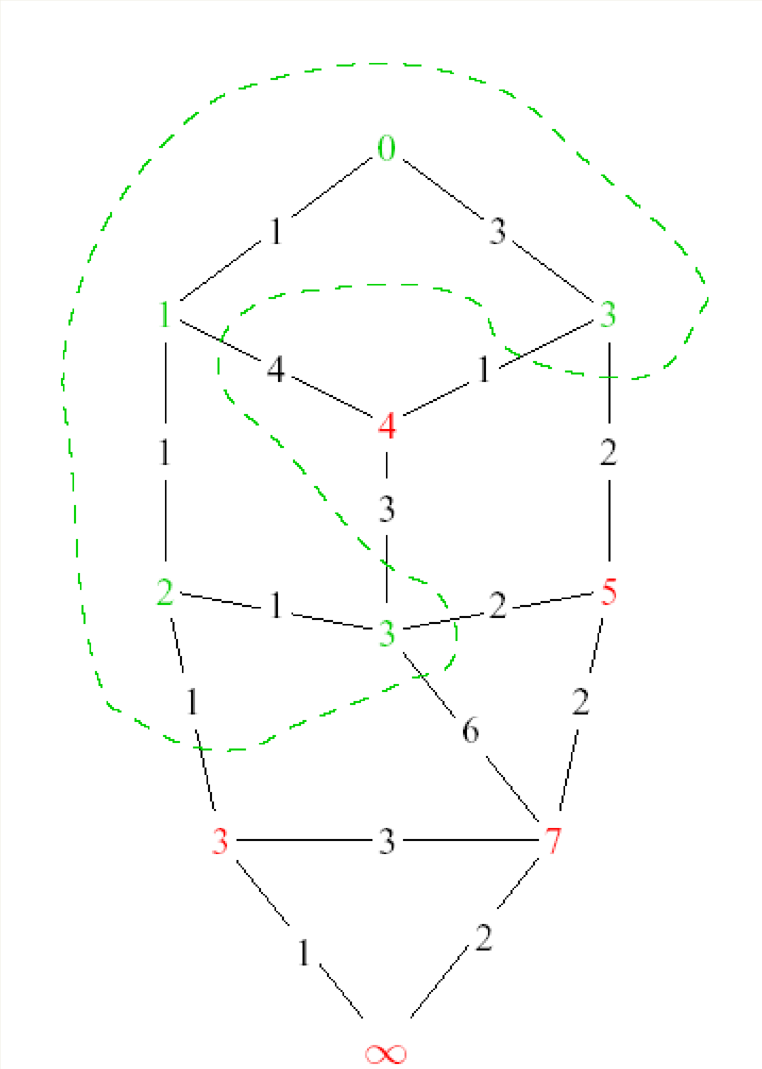
Construct Shortest Path Tree from s





Assume all edges have non-negative cost

Dijkstra's Algorithm: Idea

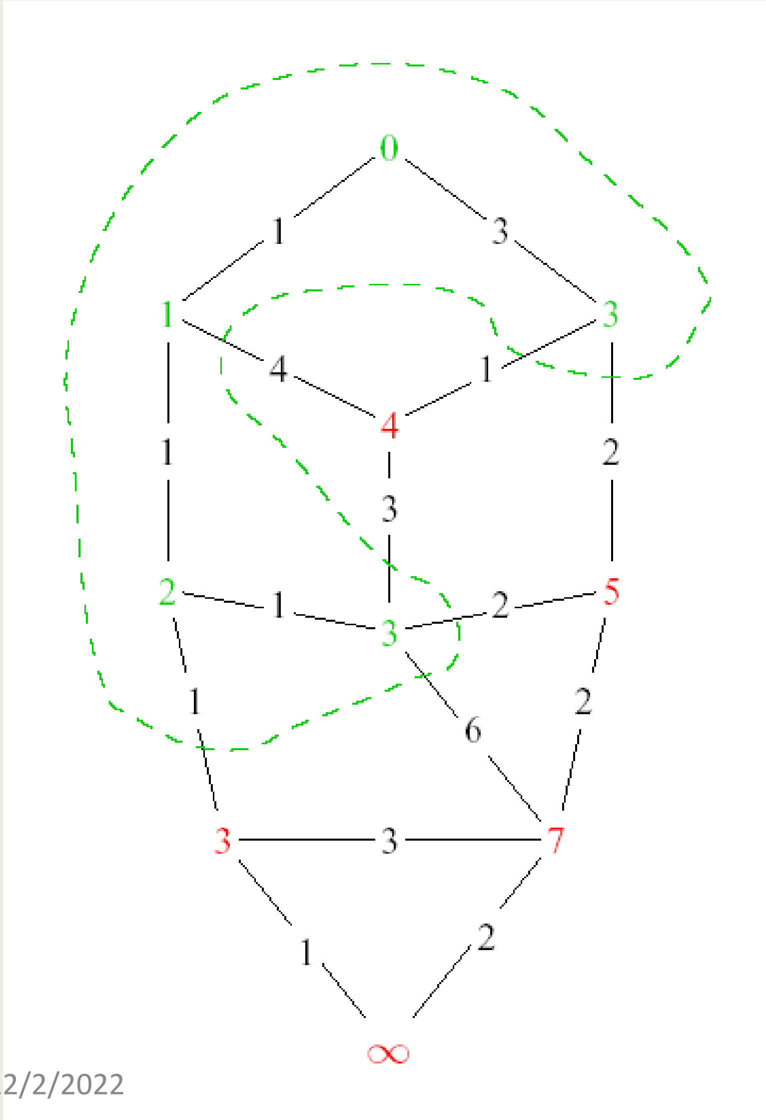


Adapt BFS to handle weighted graphs

Two kinds of vertices:

- Known
  - shortest distance is already known
- Unknown
  - Have tentative distance

Dijkstra's Algorithm: Idea



At each step:

1) Pick closest unknown vertex
2) Add it to known vertices
3) Update distances

Assume all edges have non-negative cost

Dijkstra's Algorithm

S = { }; d[s] = 0; d[v] = infinity for v != s

while S != V

Choose v in V-S with minimum d[v]

Add v to S

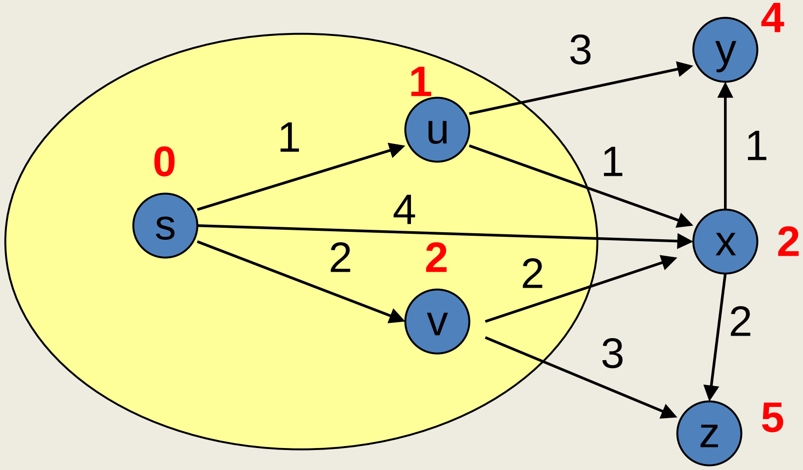
for each w in the neighborhood of v

newCost = d[v] + c(v, w)

if (newCost < d[w])

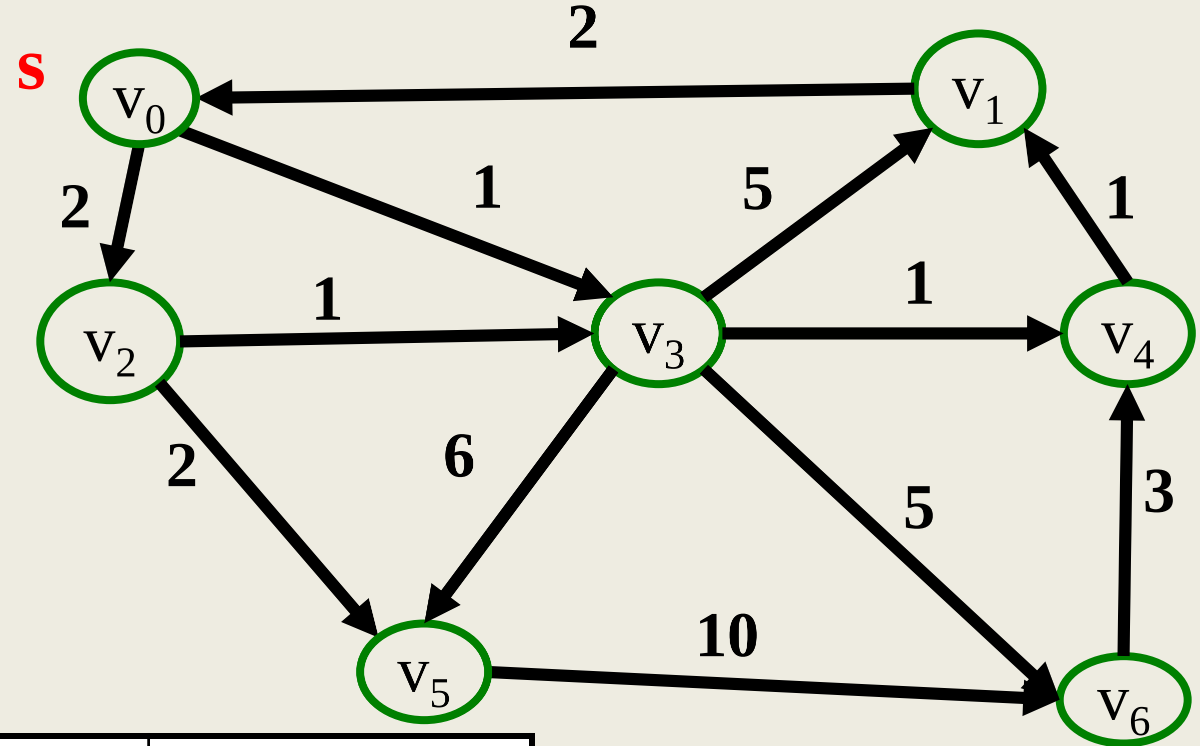
d[w] = newCost

prev[w] = v



Important Features

- Once a vertex is known (in S), the cost of the shortest path to that vertex is correct
- While a vertex is still unknown, another shorter path to it might still be found
- The shortest path can found by following the previous pointers stored at each vertex



V	Known?	Cost	Previous
v0			
v1			
v2			
v3			
v4			
v5			
v6			

Implementation

S = { }; d[s] = 0; d[v] = infinity for v != s

while S != V

  Choose v in V-S with minimum d[v]

  Add v to S

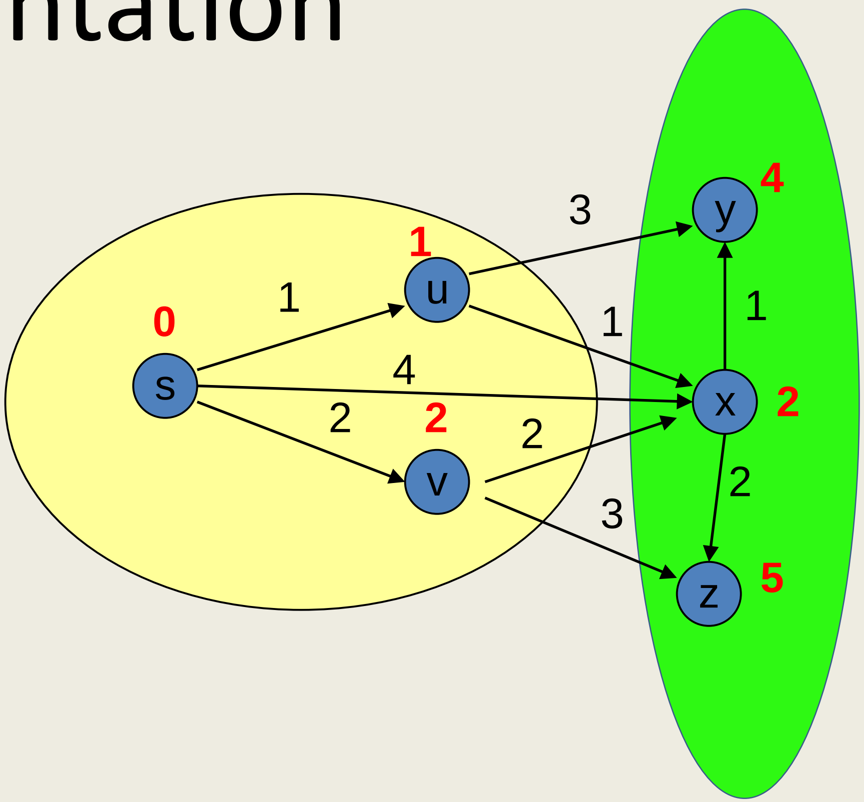
  for each w in the neighborhood of v

    newCost = d[v] + c(v, w)

    if (newCost < d[w])

      d[w] = newCost

      prev[w] = v



What are the heap operations?

How many heap operations?

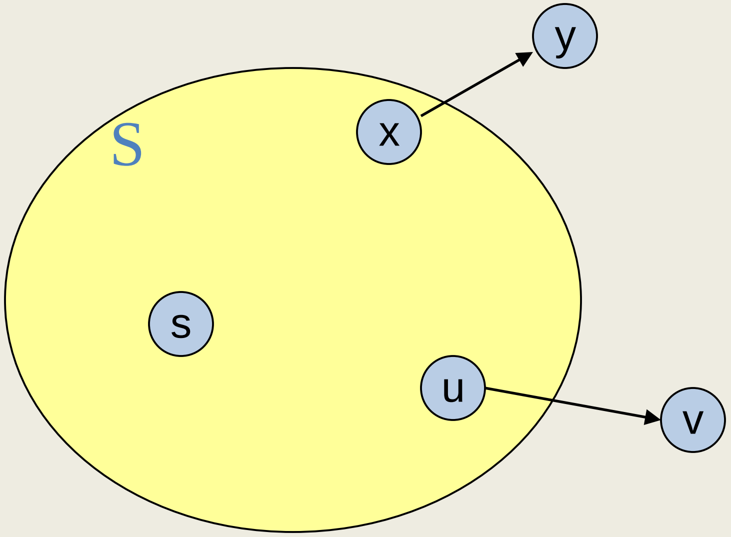
Dijkstra Algorithm

```
int dist[N], prev[N];
for (int i = 0; i < N; i++){
    dist[i] = INFINITY;
    prev[i] = -1;
}
dist[s] = 0;
Heap h = new Heap(dist);

while (!h.isEmpty()){
  v = h.DeleteMin();
  for each w adjacent to v {
    int newCost = dist[v] + cost(v,w);
    if (newCost < dist[w]){
       dist[w] = newCost;
       h.DecreaseKey(w, newCost);
       prev[w] = v;
    }
  }
}
```

Correctness Proof

- Elements in S have the correct label
- Induction: when v is added to S, it has the correct distance label
  - Dist(s, v) = d[v] when v added to S



D-Heaps (again)

- Heaps with branching factor D
- DeleteMin runtime $O(D\log_D N)$
- Decrease Key runtime $O(\log_D N)$

Dijkstra's Algorithm with D heaps

- n DeleteMin operations
- m DecreaseKey operations
- Runtime $O(n\ D\log_D n + m \log_D n)$
- What value for D?

Why do we worry about negative cost edges?