CSE 332: Data Structures and Parallelism

Spring 2022

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Lecture 22: Parallel Algorithms

Announcements

- This week:
  - Monday: Parallel Sorting
  - Wednesday: Concurrency
  - Friday: Holiday
  - Monday: Concurrency

Data Parallel Programming

- Programming primitives for operating on Arrays
  - Reduce: Combine array elements with an operator, e.g., +.
  - Map: Apply an operation to every element, e.g., multiply by two
  - Prefix sum: Compute all partial sums
  - Pack: Shift all values satisfying a predicate to start of the array

Parallel Prefix

- Prefix-sum:

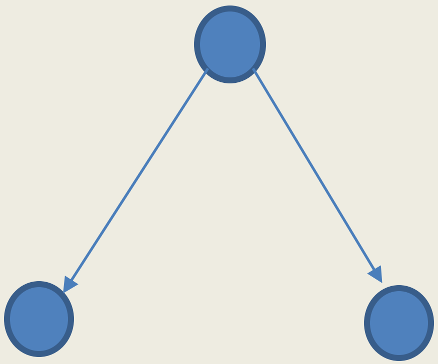
`input`	6	3	11	10	8	2	7	8	
`output`	0	6	9	20	30	38	40	47	55

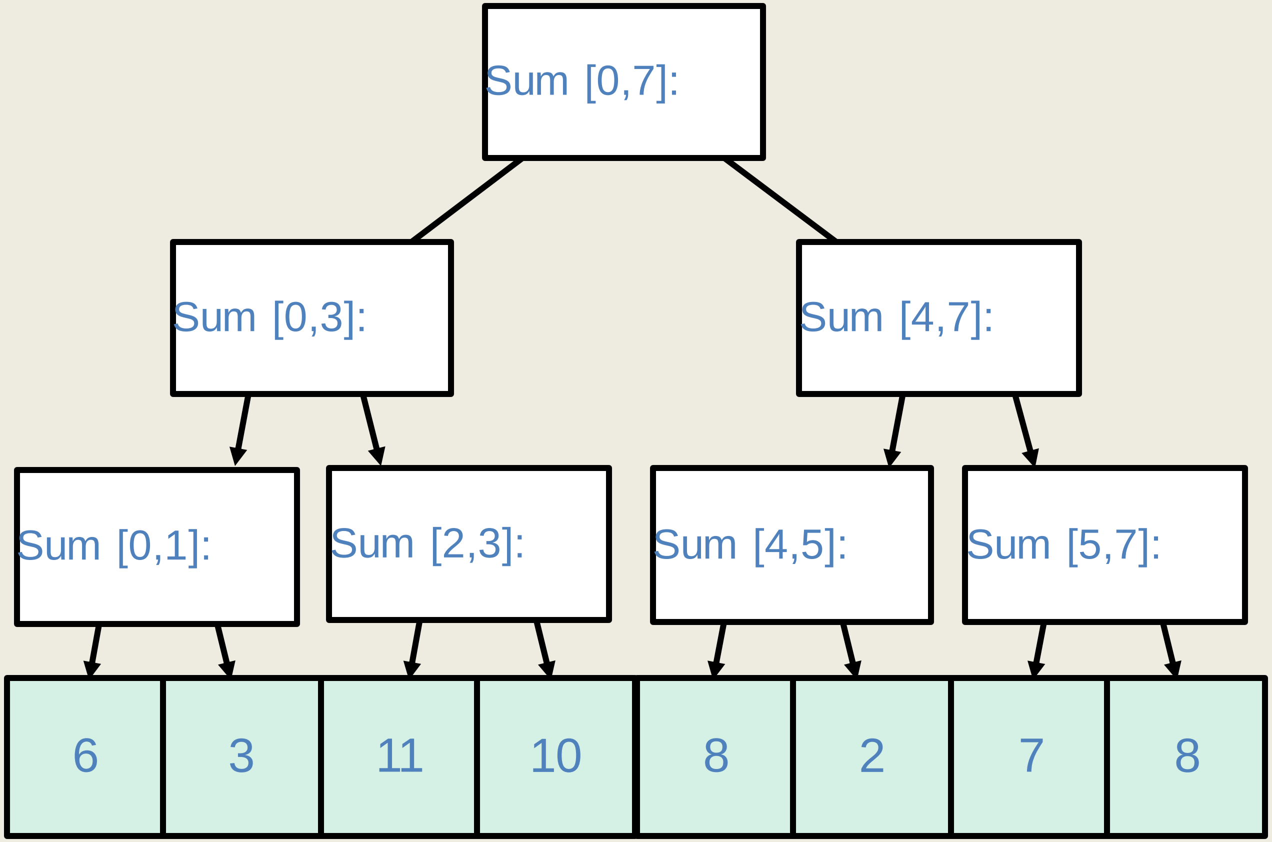
- $\text{output}[j+1] = \sum_{i=0}^{j} \text{input}[i]$

- Fork Join Implementation
- O(n) work, O(log n) span

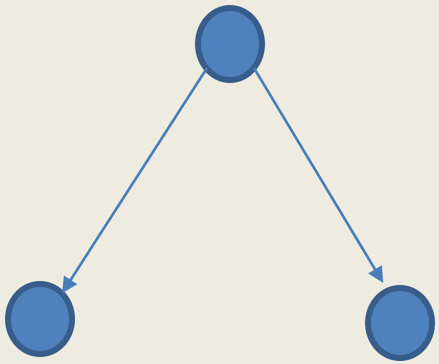
First Pass: Sum

sum = left.sum + right.sum



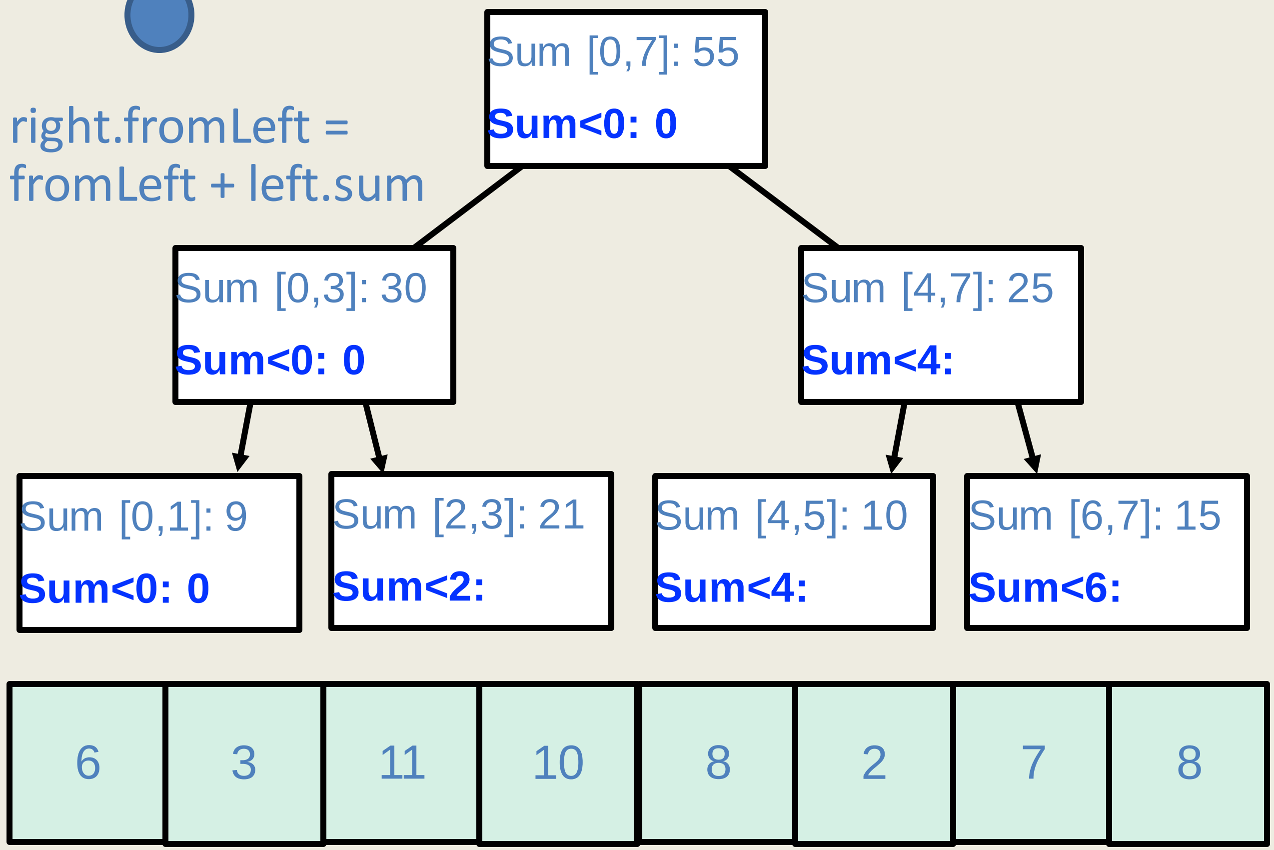


2nd Pass: Use Sum for Prefix-Sum



left.fromLeft = fromLeft

right.fromLeft = fromLeft + left.sum



Parallel Prefix, Generalized

- Parallel Prefix can be generalized to many operators
- Example: Sum by group

6	3	*	10	8	2	*	8	6	3	*	*	8	2	*	8
6	9	*	10	18	20	*	8	14	17	*	*	8	10	*	8

$A \bullet B = A + B$

$A \bullet * = *$

$* \bullet B = \underline{B}$

$* \bullet * = *$

$\underline{A} \bullet B = \underline{A + B}$

$A \bullet \underline{B} = \underline{B}$

$\underline{A} \bullet \underline{B} = \underline{B}$

$\underline{A} \bullet * = *$

$* \bullet \underline{B} = \underline{B}$

Parallel Pack

1. map test input, output [0,1] bit vector IsPrime(x) ?

input	6	3	11	10	8	2	7	8	13	4	23	5
test	0	1	1	0	0	1	1	0	1	0	1	1

2. prefix-sum on bit vector

pos	0	1	2	2	2	3	4	4	5	5	6	7

3. map input to corresponding positions in output

output	3	11	2	7	13	23	5					

– `if (test[i] == 1) output[pos[i]] = input[i]`

Parallel Algorithms

- $\mathbf{T_P}$ is the running time on **P** processors
- Work: How long it would take one processor: $\mathbf{T_1}$
- Span: How long it would take with infinite processors: $\mathbf{T_\infty}$
- Goal: parallel algorithm with $T_P \approx T_1/P$
- Assume $P << n$

Parallel Sorting

- Goal: O(n log n) work, O(log n) span
  - We will achieve O(n log n) work, O ($\log^2$ n) span
- Look at parallel versions of Quicksort and Mergesort

```
QS(S)
    if |S| < SeqCutoff
        return Sort(S)
    x = Pivot(S)
    S1, S2 = Partition(S, x)
    return  QS(S1), QS(S2)
```

```
MS
    if |S| < SeqCutoff
        return Sort(S)
    S1, S2 = Split(S)
    S1 = MS(S1);  S2 = MS(S2)
    return  Merge(S1, S2)
```

Sequential Quicksort

- Quicksort (review):
  1. Pick a pivot — $O(1)$
  2. Partition into two sub-arrays — $O(n)$
     A. values less than pivot
     B. values greater than pivot
  3. Recursively sort A and B — $2T(n/2)$ Sort of

- Complexity
  - $T(n) = n + 2T(n/2)$ $T(0) = T(1) = 1$
  - $O(n \log n)$

- How to parallelize?

Avoiding bad cases for quicksort

- Quicksort can be $\Omega(n^2)$ with bad pivot choices
- If input is random then Quicksort is O(n log n) with high probability
- If pivots are random then Quicksort is O(n log n) with high probability
- Pick 5 elements at random, choose the middle as a pivot

Parallel Quicksort

- Quicksort
  1. Pick a pivot — $O(1)$
  2. Partition into two sub-arrays — $O(n)$
     A. values less than pivot
     B. values greater than pivot
  3. Recursively sort A and B in parallel — $T(n/2)$, sort of

- Complexity (avg case)
  – $T(n) = n + T(n/2)$ $T(0) = T(1) = 1$
  – Span: $O(\quad)$
  – Parallelism (work/span) = $O(\quad)$

Parallel Partition

- Partition into sub-arrays
  A. values less than pivot
  B. values greater than pivot

- What parallel operation can we use for this?

Parallel Partition

- Pick pivot

8	1	4	9	0	3	5	2	7	6

- Pack (test: **<6**)

1	4	0	3	5	2				

- Right pack (test: **>=6**)

1	4	0	3	5	2	6	8	9	7

Parallel Quicksort

- Quicksort
  1. Pick a pivot O(1)
  2. Partition into two sub-arrays O(   ) span
     A. values less than pivot
     B. values greater than pivot
  3. Recursively sort A and B in parallel T(n/2), avg

- Complexity (avg case)
  – T(n) = O(   ) + T(n/2)   T(0) = T(1) = 1
  – Span: O(   )
  – Parallelism (work/span) = O(   )

Implementation

- Recommend random selection of pivot
- Choose sequential cutoff
  - Change over to sequential quick sort
- Constant factors in partitioning are higher for the parallel version

Sequential Mergesort

- Mergesort (review):
  1. Sort left and right halves 2T(n/2)
  2. Merge results O(n)

- Complexity (worst case)
  - T(n) = n + 2T(n/2) T(0) = T(1) = 1
  - O(n logn)

- How to parallelize?
  - Do left + right in parallel, improves to O(n)
  - To do better, we need to…

Parallel Mergesort

- MergeSort(Arr, lo, hi)
  - Threads to compute MS(Arr, lo, mid), MS(Arr, mid, hi)
  - Merge Arr[lo,mid] and Arr[mid,hi] into Arr[lo,hi]

- Can stop at a sequential cut off

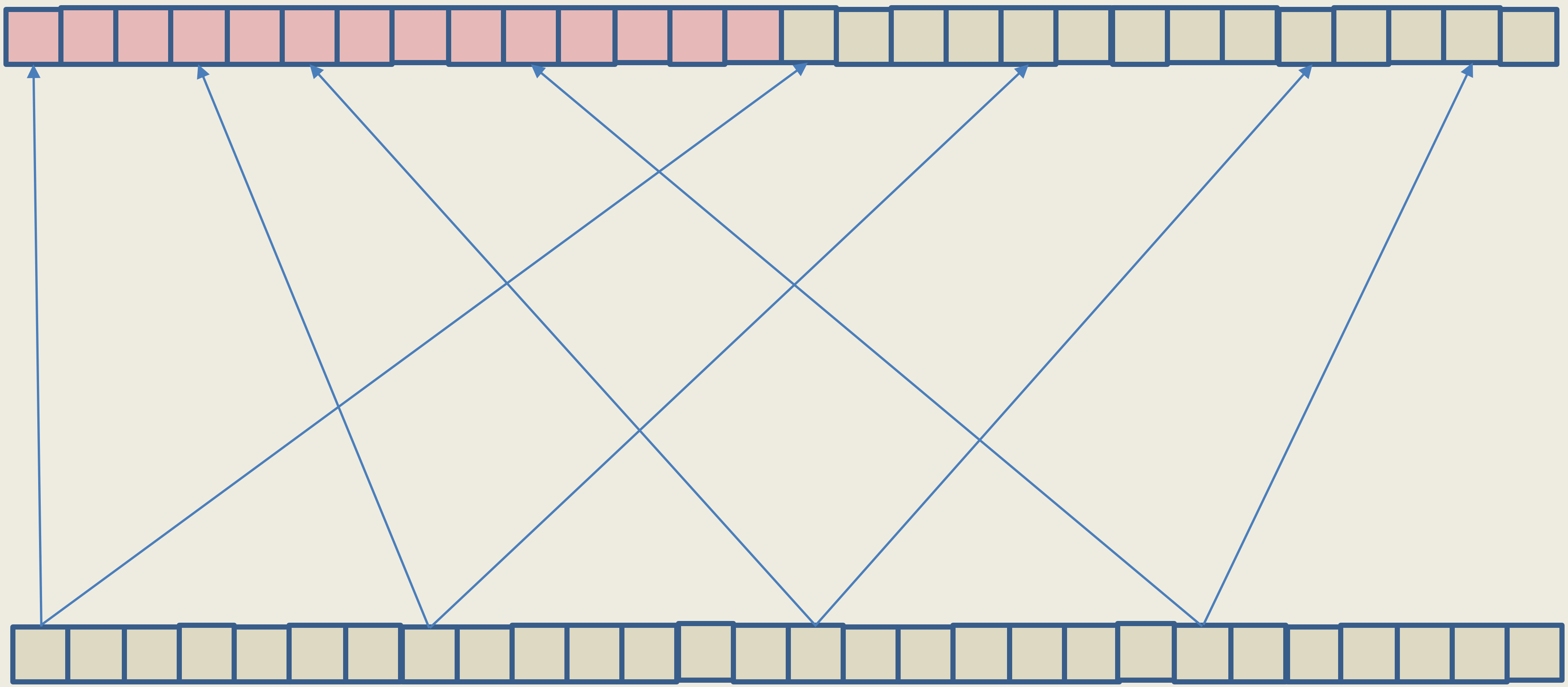
Parallel Merge

0	4	6	8	9

1	2	3	5	7

- How to merge two sorted lists in parallel?

Parallel Merge



Parallel Merge: n items with p threads

- Each thread needs to know where to start in the two arrays being merged
- If starting points are given, select the next n/p items

- Finding the starting points can be done in O(log n) time using a modified binary search

Finding the starting point

- Given two sorted arrays A, B, find the item of rank k in the combined arrays
- Compare A[k/2] and B[k/2]
  - If A[k/2] < B[k/2] discard first k/2 items of A, otherwise discard first k/2 items of B
- Look for item of rank k/2 in remaining items
- Logarithmic process

Parallel Quicksort and Mergesort

- Both algorithms can be implemented as efficient parallel algorithms
- With p processors, a speedup of p is achievable provided p << n
- Speedup comes from:
  - Doing much of the work on sorting items below the sequential cutoff
  - Taking advantage of parallelism in the combine steps to avoid a sequential bottleneck,