CSE 332: Data Structures and Parallelism

Spring 2022

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Lecture 21: Parallel Algorithms

Announcements

- Read parallel computing notes by Dan Grossman 3.5-5.4

Recap

- Last lectures
  - simple parallel programs
  - fork-join/thread programming
  - common patterns: reduce, map
  - analysis tools (task graph, work, span, parallelism)

- Now
  - Amdahl’s Law
  - useful building blocks: prefix, pack
  - parallel quicksort, merge sort

Analyzing Parallel Programs

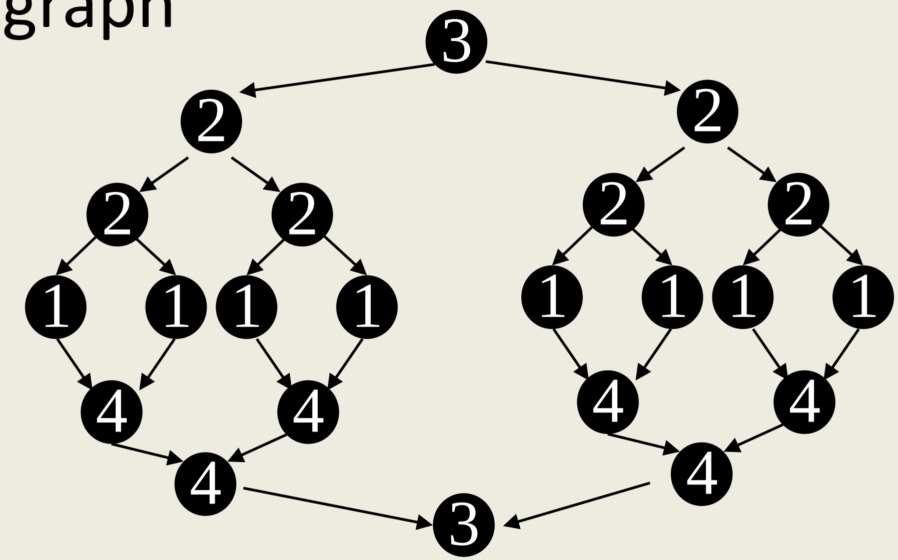
Let $\mathbf{T_P}$ be the running time on **P** processors

Two key measures of run-time:

- Work: How long it would take 1 processor = $\mathbf{T_1}$
- Span: How long it would take infinity processors = $\mathbf{T_\infty}$
  - The hypothetical ideal for parallelization
  - This is the longest “dependence chain” in the computation
  - Example: $O($`log` $n)$ for summing an array
  - Also called “critical path length” or “computational depth”

Task Graph

A parallel program can be modelled as a directed acyclic graph



Work – T_1, sum of times of all of the nodes

Span - T_∞, longest path

Parallel Speed-up

- Speed-up on **P** processors: $\mathbf{T_1 / T_P}$
- If speed-up is **P**, we call it perfect linear speed-up
  - e.g., doubling **P** halves running time
  - hard to achieve in practice
- Parallelism is the maximum possible speed-up: $\mathbf{T_1 / T_\infty}$
  - if you had infinite processors

Estimating T_p

- How to estimate T_P ?

- Lower bounds on T_P

  $T_p \geq T_\infty$

  $T_p \geq T_1 / P$

  – which one is the tighter (higher) lower bound?

- The ForkJoin Java Framework achieves the following asymptotic time bound:

$$T_P \text{ is } O(T_\infty + T_1 / P)$$

  – this bound is optimal

Amdahl’s Law

- Most programs have
  1. parts that parallelize well
  2. parts that don’t parallelize at all

- The latter become bottlenecks

Amdahl's Law

- Let T_1 = 1 unit of time
- Let S = proportion that can't be parallelized

$$1 = \mathbf{T_1 = S + (1 - S)}$$

- Suppose we get perfect linear speedup on the parallel portion:

$$\mathbf{T_P =}$$

- So the overall speed-up on P processors is (Amdahl's Law):

$$\mathbf{T_1 / T_P =}$$

$$\mathbf{T_1 / T_\infty =}$$

- If 1/3 of your program is parallelizable, max speedup is:

Take Aways

- Parallel algorithms can be a big win
- Many fit standard patterns that are easy to implement
- Can’t just rely on more processors to make things faster (Amdahl’s Law)

Parallelizable?

- Prefix-sum:

`input`	6	3	11	10	8	2	7	8
`output`								

- $\text{output}[j] = \sum_{i=0}^{j} \text{input}[i]$

Parallel prefix-sum

- The parallel-prefix algorithm does two passes
  - Each pass has $O(n)$ work and $O(\log n)$ span
  - So in total there is $O(n)$ work and $O(\log n)$ span
  - So like with array summing, parallelism is $n/\log n$
- First pass builds a tree bottom-up: the "up" pass
- Second pass traverses the tree top-down: the "down" pass

Parallel Prefix: The Up Pass

We build want to build a binary tree where

- Root has sum of the range [x,y)
- If a node has sum of [lo,hi) and hi>lo,
  - Left child has sum of [lo,middle)
  - Right child has sum of [middle,hi)
  - A leaf has sum of [i,i+1), which is simply input[i]

It is critical that we actually create the tree as we will need it for the down pass

- We do not need an actual linked structure
- We could use an array as we did with heaps

Analysis of first step: Work = Span =

The algorithm, part 1

1. Propagate 'sum' up: Build a binary tree where
   – Root has sum of **`input[0]..input[n-1]`**
   – Each node has sum of **`input[lo]..input[hi-1]`**
     • Build up from leaves; parent.sum=left.sum+right.sum
   – A leaf's sum is just it's value; **`input[i]`**

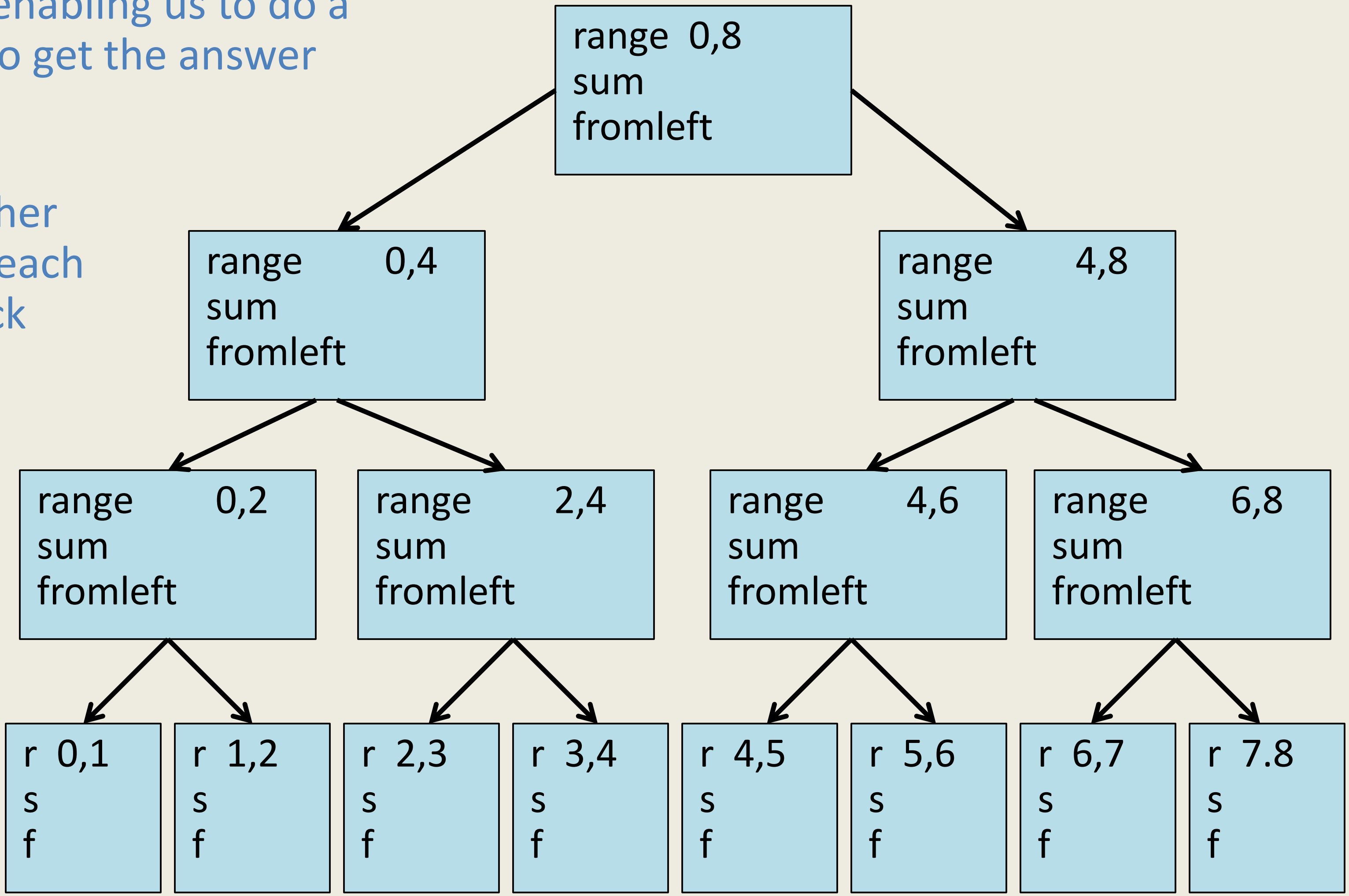
This is an easy fork-join computation: combine results by actually building a binary tree with all the sums of ranges

– Tree built bottom-up in parallel
– Could be more clever; ex. Use an array as tree representation like we did for heaps

Analysis of first step: $O(n)$ **work**, $O(\log n)$ **span**

Do an initial pass to gather information, enabling us to do a second pass to get the answer

First we'll gather the 'sum' for each recursive block



`input`	6	4	16	10	16	14	2	8
`output`								

The algorithm, part 2

2. Propagate 'fromleft' down:
   - Root given a **`fromLeft`** of **`0`**
   - Node takes its **`fromLeft`** value and
     - Passes its left child the same **`fromLeft`**
     - Passes its right child its **`fromLeft`** plus its left child's **`sum`** (as stored in part 1)
   - At the leaf for array position **`i`**, **`output[i]=fromLeft+input[i]`**

This is an easy fork-join computation: traverse the tree built in step 1 and produce no result (the leaves assign to **`output`**)

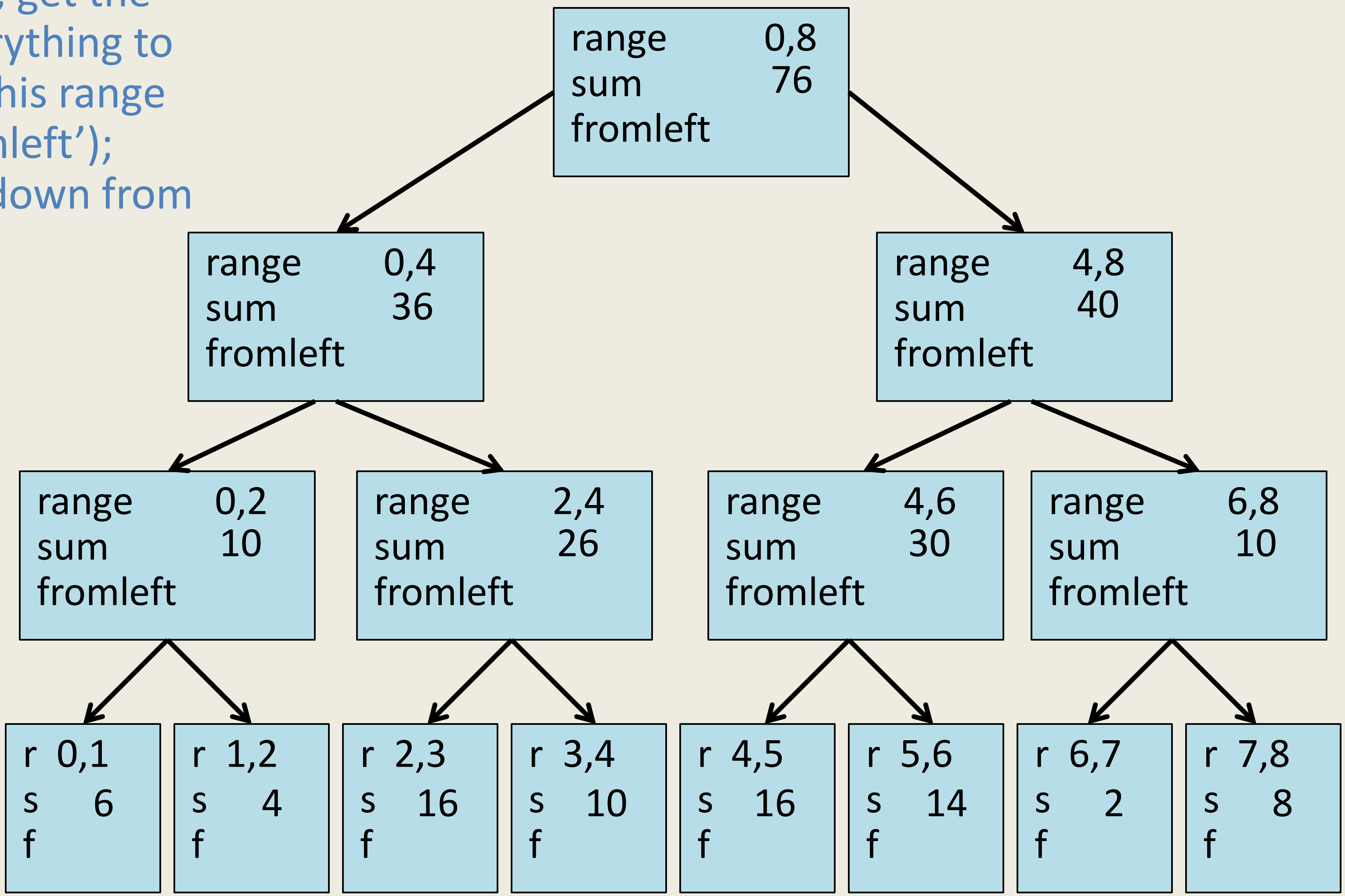
- Invariant: **`fromLeft`** is sum of elements left of the node's range

Analysis of first step: *O*(*n*) **work**, *O*(**`log`** *n*) **span**

Analysis of second step: *O*(*n*) **work**, *O*(**`log`** *n*) **span**

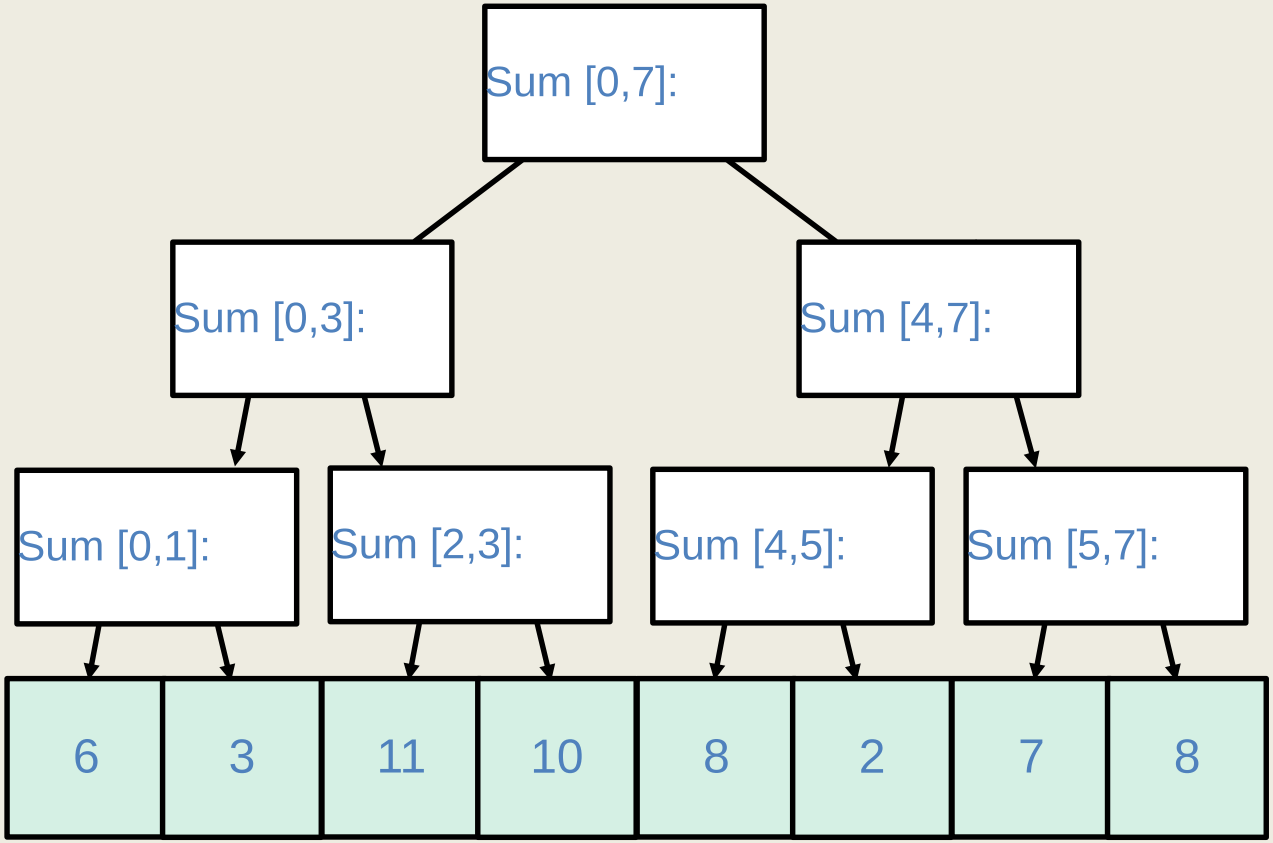
Total for algorithm: *O*(*n*) **work**, *O*(**`log`** *n*) **span**

Using ‘sum’, get the sum of everything to the left of this range (call it ‘fromleft’); propagate down from root

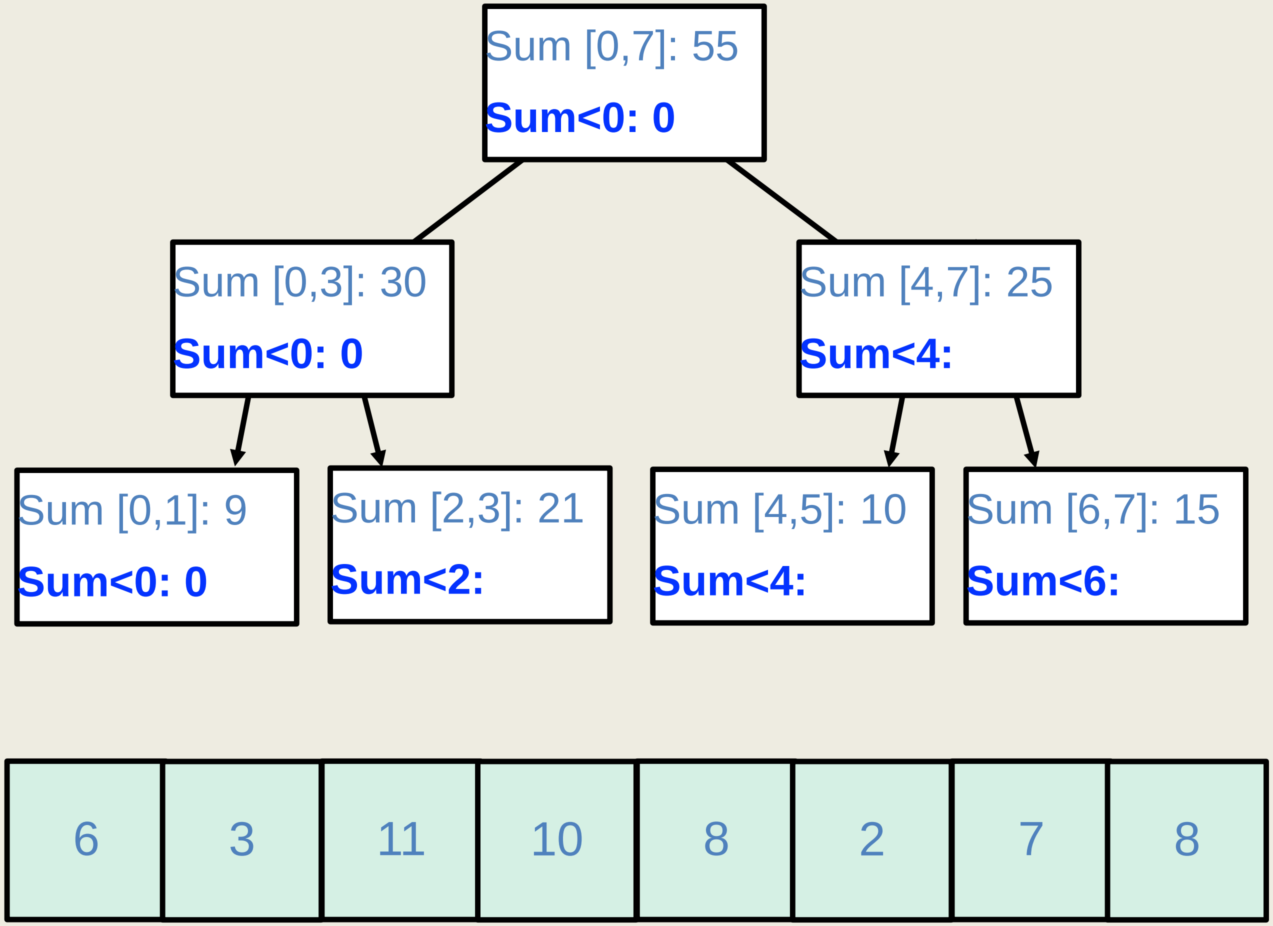


`input`	6	4	16	10	16	14	2	8
`output`								

First Pass: Sum



2nd Pass: Use Sum for Prefix-Sum



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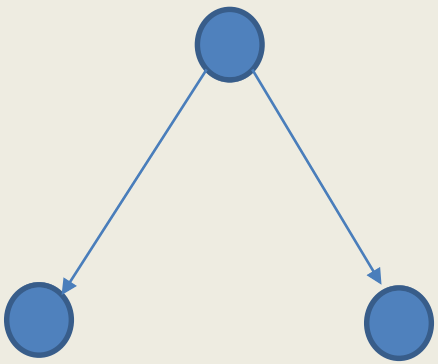
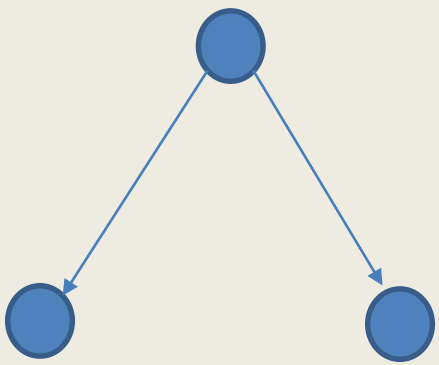
A nodes computation

sum = left.sum + right.sum





left.fromLeft = fromLeft

right.fromLeft = fromLeft + left.sum

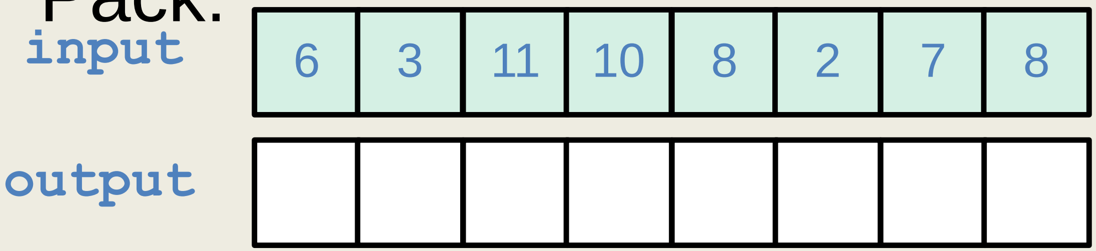
Parallel Prefix, Generalized

- Prefix-sum is another common pattern (prefix problems)
  - maximum element to the left of i
  - is there an element to the left of i satisfying some property?
  - count of elements to the left of i satisfying some property
  - …

- We can solve all of these problems in the same way

Pack

- Pack:



`test:  X < 8?`

- Output array of elements satisfying **`test`**, in original order

Parallel Pack?

- Pack

`input`	6	3	11	10	8	2	7	8
`output`	6	3	2	7				

`test:  X < 8?`

- Determining **which** elements to include is **easy**
- Determining **where** each element goes in output is **hard**
  - seems to depend on previous results

Parallel Pack

1. map test input, output [0,1] bit vector

input	6	3	11	10	8	2	7	8
test	1	1	0	0	0	1	1	0

`test: X < 8?`

Parallel Pack

1. map test input, output [0,1] bit vector

input	6	3	11	10	8	2	7	8
test	1	1	0	0	0	1	1	0

test: X < 8?

2. transform bit vector into array of indices into result array

pos	1	2				3	4	

Parallel Pack

1. map test input, output [0,1] bit vector

`test:  X < 8?`

input	6	3	11	10	8	2	7	8
test	1	1	0	0	0	1	1	0

2. prefix-sum on bit vector

pos	**1**	**2**	2	2	2	**3**	**4**	4

3. map input to corresponding positions in output

output	6	3	2	7				

- `if (test[i] == 1) output[pos[i]] = input[i]`

Parallel Pack Analysis

- Parallel Pack
  1. map: O( ) span
  2. sum-prefix: O( ) span
  3. map: O( ) span

- Total: O( ) span