CSE 332: Data Structures and Parallelism

Spring 2022

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Lecture 16: Sorting IV

Announcements

- Midterm, Friday, November 4
- Next week
  - Graph Theory
  - Graph Theory
  - Veterans Day

Sorting: *The Big Picture*

Simple algorithms: O(n^2)

Insertion sort
Selection sort
…

Fancier algorithms: O(n log n)

Heap sort
Merge sort
Quick sort (avg)
…

Comparison lower bound: Ω(n log n)

Specialized algorithms: O(n)

Bucket sort
Radix sort

How fast can we sort?

Heapsort and Mergesort have O(N log N) **worst** case running time.

These algorithms, along with Quicksort, also have O(N log N) **average** case running time.

Can we do any better?

Permutations

- How many possible orderings can you get?
  - Example: a, b, c (N = 3)
  - (a b c), (a c b), (b a c), (b c a), (c a b), (c b a)
  - 6 orderings = 3•2•1 = 3! (i.e., “3 factorial”)

- For N elements
  - N choices for the first position, (N-1) choices for the second position, …, (2) choices, 1 choice
  - $N(N-1)(N-2)\cdots(2)(1)$= <u>N! possible orderings</u>

Sorting Model

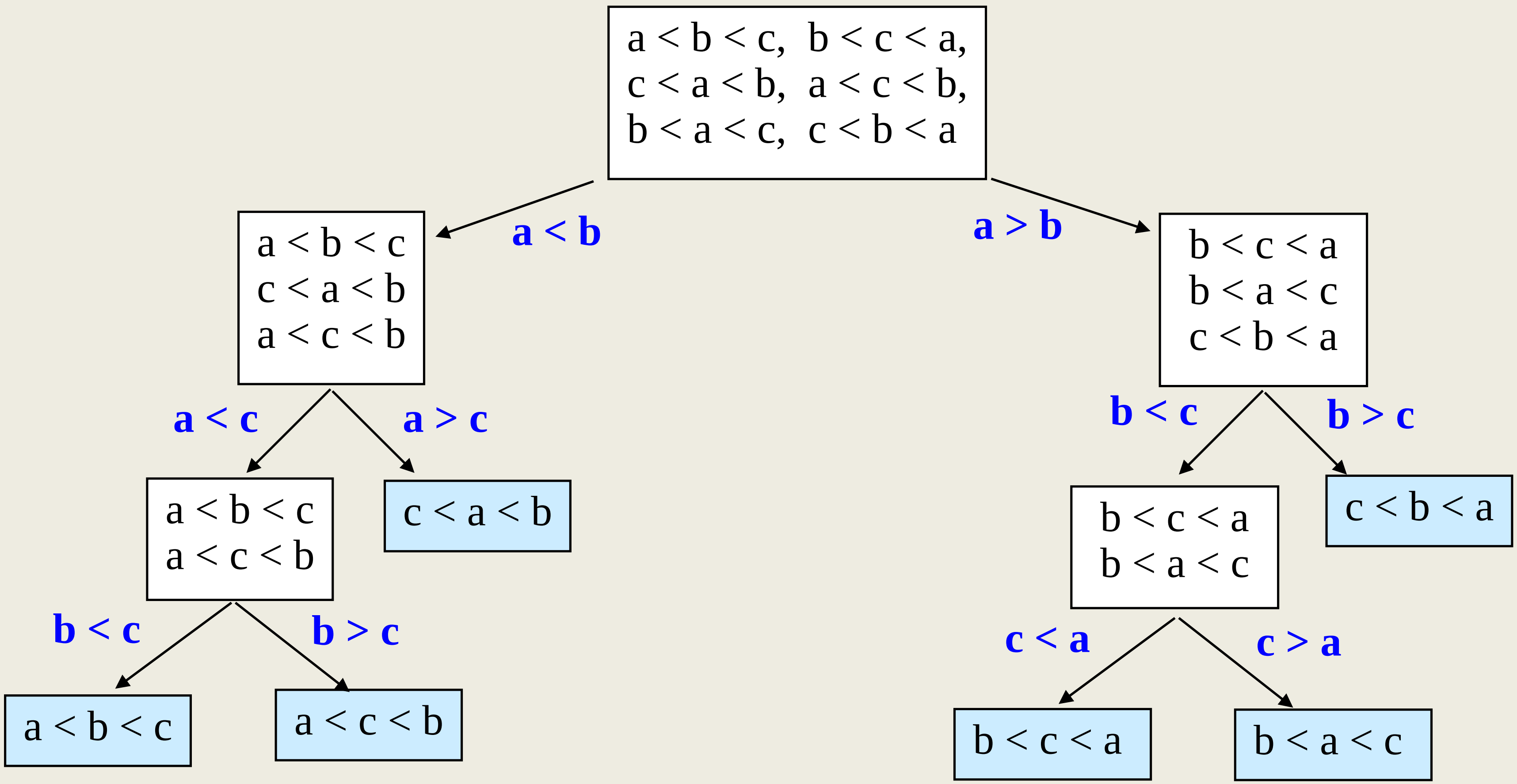
Recall our basic sorting assumption:

We can only compare
two elements at a time.

These comparisons prune the space of possible orderings.

We can represent these concepts in a…

Decision Tree



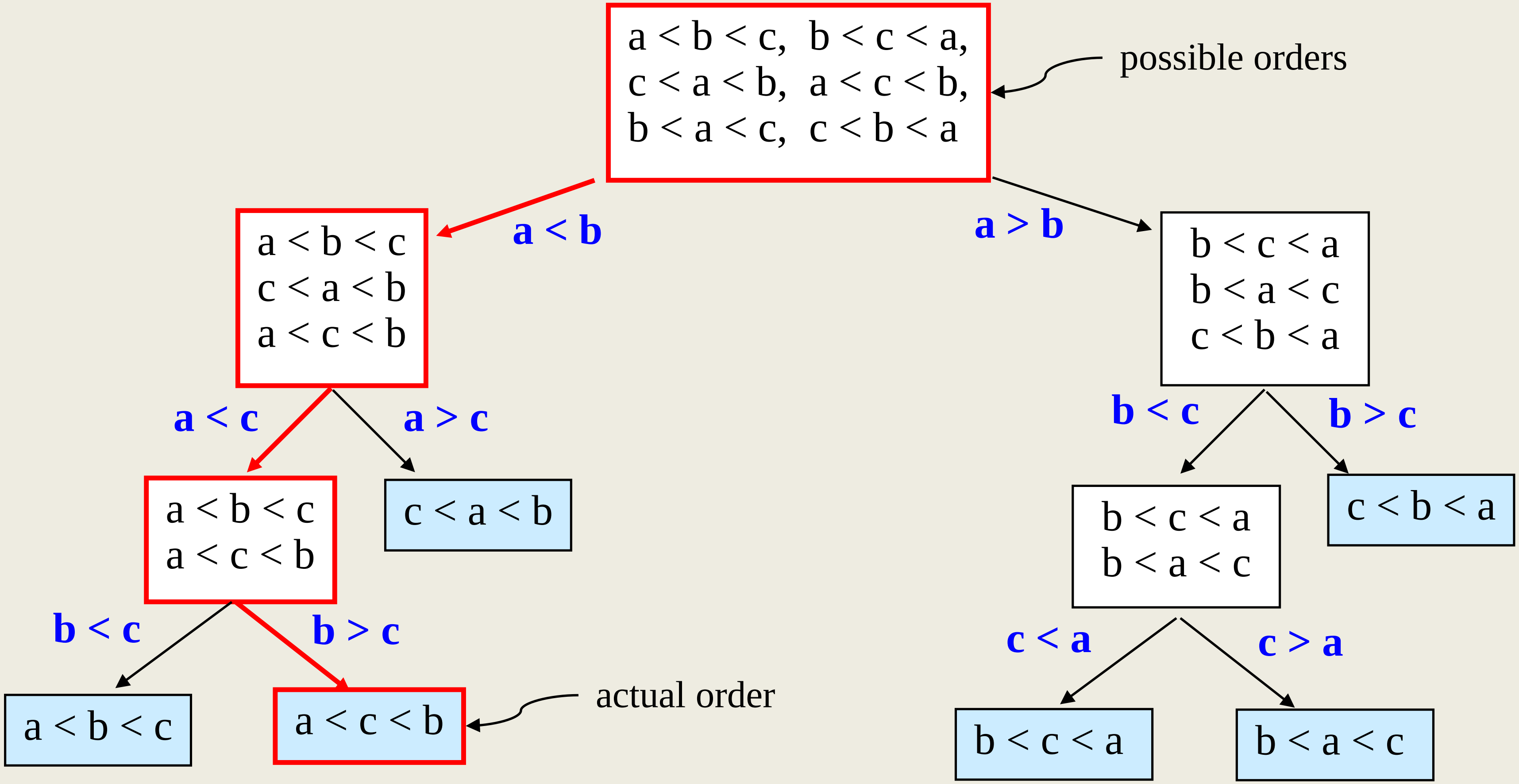
The leaves contain all the possible orderings of a, b, c.

Decision Trees

- A Decision Tree is a Binary Tree such that:
  - Each node = a set of orderings
    - i.e., the remaining solution space
  - Each edge = 1 comparison
  - Each leaf = 1 unique ordering
  - How many leaves for *N* distinct elements?
- Conceptual Tool
- Only 1 leaf has the ordering that is the desired correctly sorted arrangement

Decision Tree Example

Suppose the order is: a < c < b

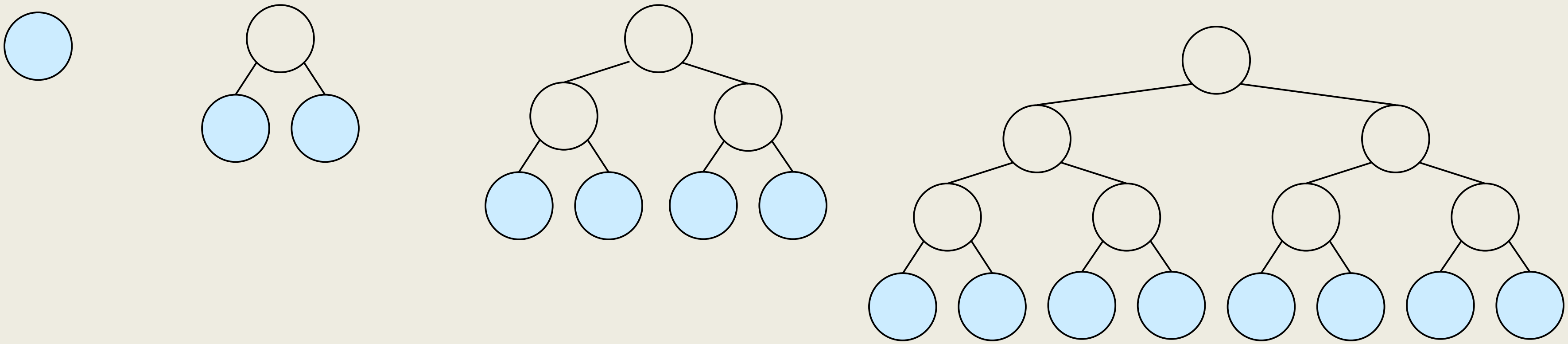


Decision Trees and Sorting

- Every comparison based sorting algorithm corresponds to a decision tree
  - Finds correct leaf by choosing edges to follow
    - i.e., by making comparisons
- We will focus on worst case run time
- Observations:
  - Worst case run time ≥ max number of comparisons
  - Max number of comparisons
    = length of the longest path in the decision tree
    = tree height

How many leaves on a tree?

Suppose you have a binary tree of height h. How many leaves in a perfect tree?



We can prune a perfect tree to make any binary tree of same height. Can # of leaves increase?

Lower bound on Height

- A binary tree of height h has at most 2^h leaves
- A decision tree has N! leaves. What is its minimum height?

Lower bound on log(n!)

$$
\begin{aligned}
n! &= n \cdot (n-1) \cdot (n-2) \cdots 4 \cdot 3 \cdot 2 \cdot 1 \\
&\geq n \cdot (n-1) \cdot (n-2) \cdots \frac{n}{2} \\
&\geq \frac{n}{2} \cdot \frac{n}{2} \cdot \frac{n}{2} \cdots \frac{n}{2} \\
&\geq \left(\frac{n}{2}\right)^{n/2}
\end{aligned}
$$

$$\log n! \geq \log\left(\frac{n}{2}\right)^{n/2} = \frac{n}{2} \log \frac{n}{2}$$

$\Omega(N \log N)$

Worst case run time of any comparison-based sorting algorithm is $\boldsymbol{\Omega(N \log N)}$.

Can also show that **average case** run time is also $\boldsymbol{\Omega(N \log N)}$.

Can we do better if we don’t use comparisons?

Can we sort in O(n)?

- Suppose keys are integers between 0 and 1000

BucketSort (aka BinSort)

If all values to be sorted are integers between 1 and *B*, create an array **`count`** of size *B*, **increment** counts while traversing the input, and finally output the result.

Example *B*=5. Input = (5,1,3,4,3,2,1,1,5,4,5)

count array	
1	
2	
3	
4	
5	

Running time to sort n items?

What about our Ω(n log n) bound?

Dependence on *B*

What if *B* is very large (e.g., 2^{64})?

Definition

- A sort is said to be stable if the order of elements with equal key is preserved.

Fixing impracticality: RadixSort

- RadixSort: generalization of BucketSort for large integer keys

- Origins go back to the 1890 census.

- Radix = "The base of a number system"
  - We'll use 10 for convenience, but could be anything

- Idea:
  - BucketSort on one digit at a time
  - After k^{th} sort, the last k digits are sorted
  - Set number of buckets: *B* = radix.

Radix Sort Example

Input: 478, 537, 9, 721, 3, 38, 123, 67

BucketSort on 1's

0	1	2	3	4	5	6	7	8	9

BucketSort on 10's

0	1	2	3	4	5	6	7	8	9

BucketSort on 100's

0	1	2	3	4	5	6	7	8	9

Output:

Radix Sort Example (1^{st} pass)

Bucket sort
by 1's digit

Input data

478
537
9
721
3
38
123
67

0	1	2	3	4	5	6	7	8	9
	72<u>1</u>		<u>3</u> 12<u>3</u>				53<u>7</u> 6<u>7</u>	47<u>8</u> 3<u>8</u>	<u>9</u>

After 1^{st} pass

721
3
123
537
67
478
38
9

This example uses B=10 and base 10 digits for simplicity of demonstration. Larger bucket counts should be used in an actual implementation.

Radix Sort Example (2^{nd} pass)

After 1^{st} pass

721
3
123
537
67
478
38
9

Bucket sort by 10's digit

0	1	2	3	4	5	6	7	8	9
03 09		721 123	537 38			67	478		

After 2^{nd} pass

3
9
721
123
537
38
67
478

Radix Sort Example (3rd pass)

After 2nd pass

3
9
721
123
537
38
67
478

Bucket sort by 100's digit

0	1	2	3	4	5	6	7	8	9
003 009 038 067	123			478	537		721		

After 3rd pass

3
9
38
67
123
478
537
721

Invariant: after k passes the low order k digits are sorted.

Radix Sort Complexity

In our examples, we had:

- Input size, N
- Number of buckets, B = 10
- Maximum value, M < 10^3
- Number of passes, P =

How much work per pass?

Total time?

Sorting Summary

$O(N^2)$ average, worst case:

- **Selection Sort**, **Bubblesort**, **Insertion Sort**

$O(N \log N)$ average case:

- **Heapsort**: In-place, not stable.
- **BST Sort**: $O(N)$ extra space (including tree pointers, possibly poor memory locality), stable.
- **Mergesort**: $O(N)$ extra space, stable.
- **Quicksort**: claimed fastest in practice, but $O(N^2)$ worst case. Recursion/stack requirement. Not stable.

$\Omega(N \log N)$ worst and average case:

- **Any comparison-based sorting algorithm**

$O(N)$

- **Radix Sort**: fast and stable. Not comparison based. Not in-place.