CSE 332: Data Structures and Parallelism

Spring 2022

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Lecture 15: Sorting III

Announcements

- Midterm, Friday, November 4
  - In class
  - Coverage: up to, and including QuickSort
- Review session,
  - Tuesday, Nov 1, CSE2 G01, 3 pm – 5 pm
-

Sorting: *The Big Picture*

Simple algorithms: $O(n^2)$

- Insertion sort
- Selection sort
- …

Fancier algorithms: $O(n \log n)$

- Heap sort
- Merge sort
- Quick sort (avg)
- …

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$

- Bucket sort
- Radix sort

“Divide and Conquer”

- **Idea 1**: Divide array in half, *recursively* sort left and right halves, then *merge* two halves
  → known as Mergesort

- **Idea 2 :** Partition array into small items and large items, then recursively sort the two sets
  → known as Quicksort

- Recurrences used to analyze runtime of recursive algorithms

Recurrences

General form:

$T(N) = S(N) + \sum_i a_i T(f_i(N));\quad T(1) = c;$

Important recurrences

$T(N) = T(N-1) + f(N)$

$T(N) = T(aN) + cN,\quad a < 1$

$T(N) = aT(N/b) + N^c$

(for midterm, understand aT(N/a) + N)

Review

- $T(N) = T(N-1) + N^2$; $T(0) = 0$
  - Unroll to get a summation

- $T(N) = T(N/2) + N$; $T(1) = 1$
  - Unroll to get geometric sum
  - $T(N) = N + N/2 + N/4 + N/8 + \ldots + 4 + 2 + 1 = 2N-1$

$$T(N) = 4\,T(N/4) + N; \quad T(1) = 1$$

Quicksort

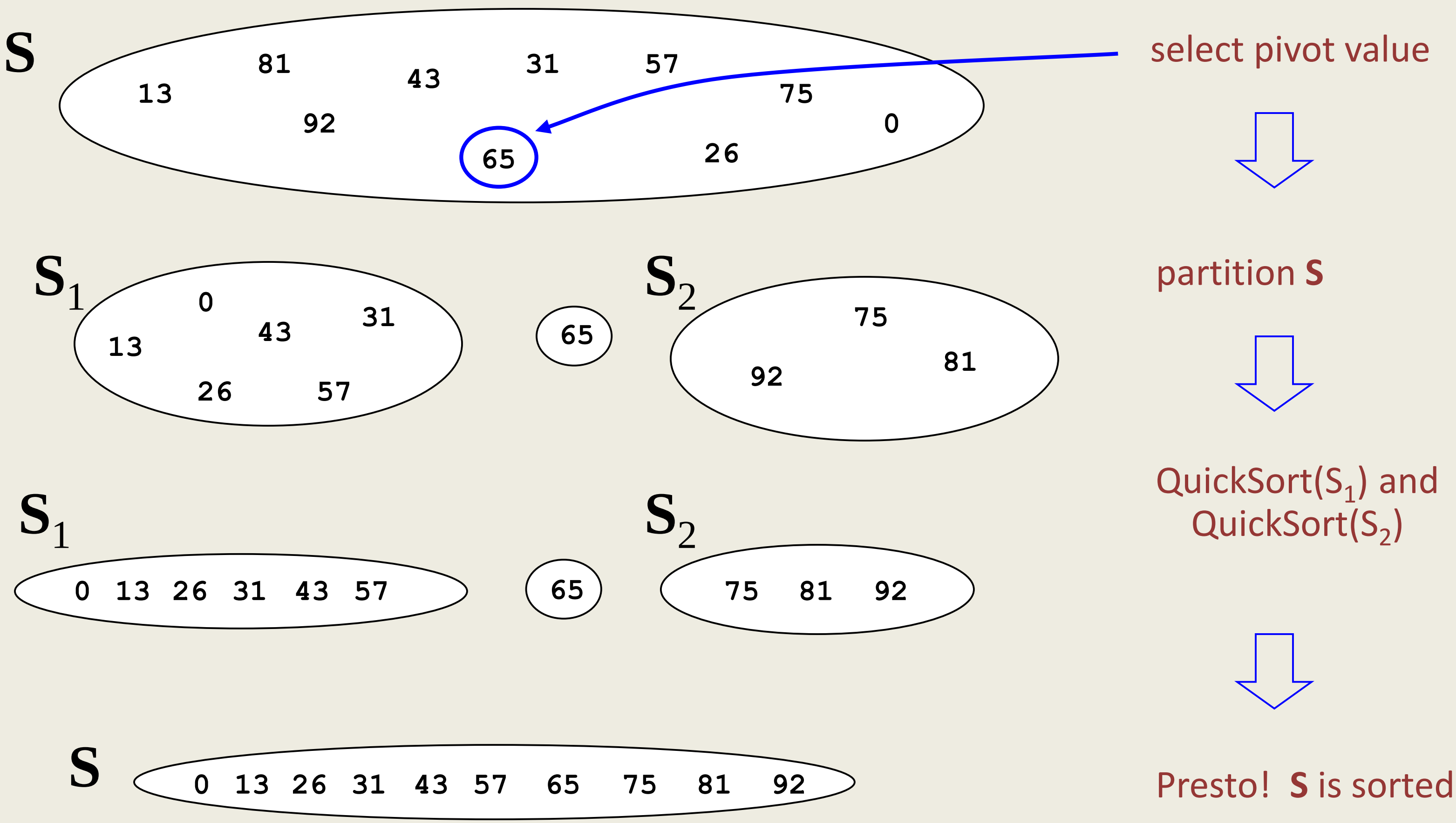
Quicksort uses a divide and conquer strategy, but does not require the O(N) extra space that MergeSort does.

Here's the idea for sorting array **S**:

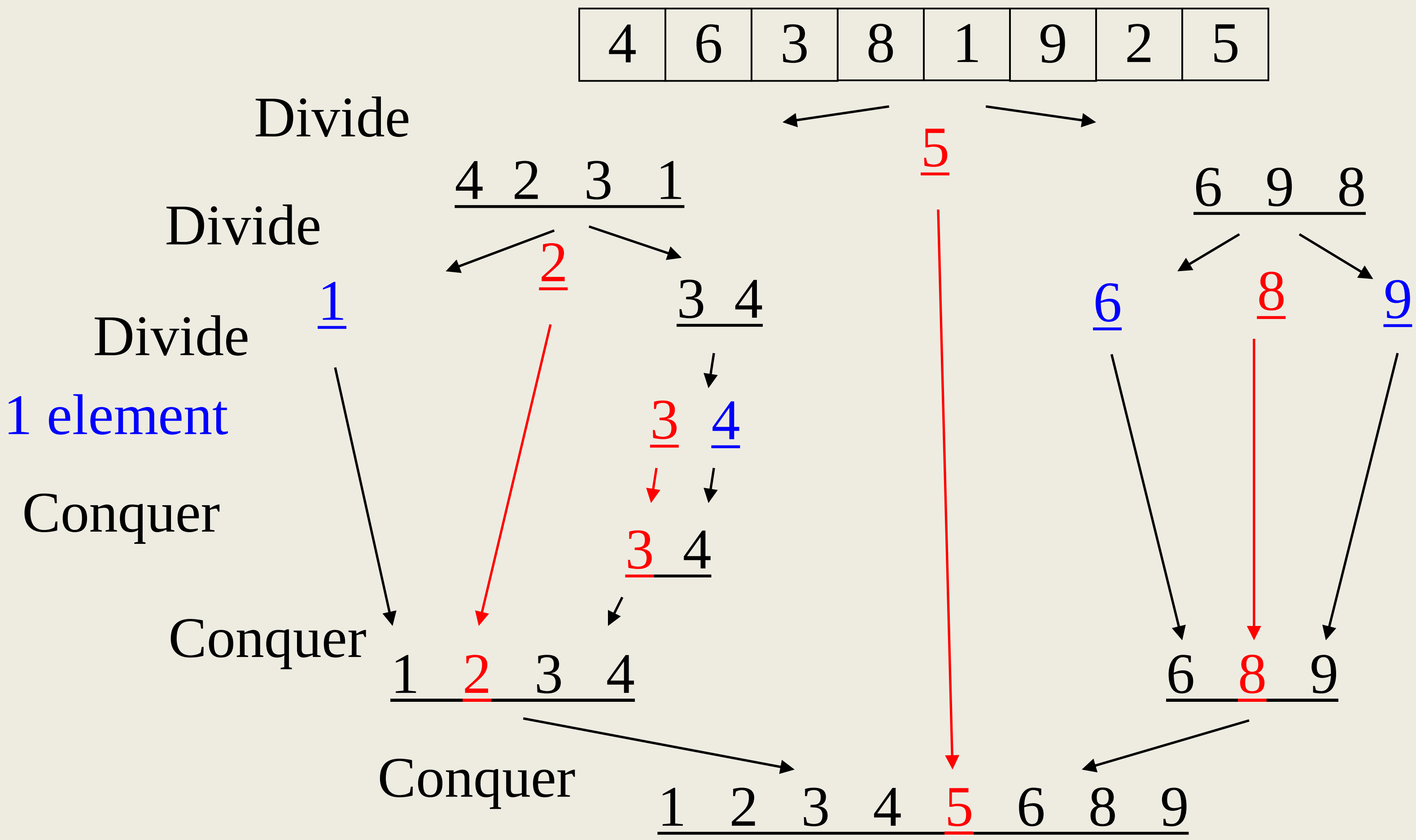
1. Pick an element *v* in **S**. This is the ***pivot*** value.
2. Partition **S**-{*v*} into two disjoint subsets, $\mathbf{S}_1$ and $\mathbf{S}_2$ such that:
   - elements in $\mathbf{S}_1$ are all $\leq v$
   - elements in $\mathbf{S}_2$ are all $\geq v$
3. Return concatenation of QuickSort($\mathbf{S}_1$), *v*, QuickSort($\mathbf{S}_2$)

Recursion ends if Quicksort( ) receives an array of length 0 or 1.

The steps of Quicksort



Quicksort Example



Pivot Picking and Partitioning

The tricky parts are:

- **Picking the pivot**
  - Goal: pick a pivot value so that $|S_1|$ and $|S_2|$ are roughly equal in size.
- **Partitioning**
  - Preferably in-place
  - Dealing with duplicates

Picking the pivot

- Choose the first element in the subarray
- Choose a value that might be close to the middle
  - Median of three
- Choose a random element

Quicksort Partitioning

- Partition the array into left and right sub-arrays such that:
  - elements in left sub-array are $\leq$ pivot
  - elements in right sub-array are $\geq$ pivot
- Can be done in-place with another "two pointer method"
  - Sounds like mergesort, but here we are *partitioning*, not sorting…
  - …and we can do it in-place.
- Lots of work has been invested in engineering quicksort

Quicksort Pseudocode

Putting the pieces together:

```
Quicksort(A[], left, right) {
  if (left < right) {
    medianOf3Pivot(A, left, right);
    pivotIndex = Partition(A, left+1, right-1);

    Quicksort(A, left, pivotIndex – 1);
    Quicksort(A, pivotIndex + 1, right);
  }
}
```

Important Tweak

Insertion sort is actually better than quicksort on small arrays. Thus, a better version of quicksort:

```
Quicksort(A[], left, right) {
  if (right - left ≥ CUTOFF) {
    medianOf3Pivot(A, left, right);
    pivotIndex = Partition(A, left+1, right-1);

    Quicksort(A, left, pivotIndex – 1);
    Quicksort(A, pivotIndex + 1, right);

  } else {
    InsertionSort(A, left, right);
  }
}
```

CUTOFF = 16 is reasonable.

Quicksort run time

- What is the best case behavior?

Worst case run time

- What is the bad case for partitioning?
- Design a bad case input (assume first element is chosen as pivot)

Average case performance

- Assume all permutations of the data are equally likely
  - Or equivalently, a random pivot is chosen

- The math gets messy, but doable

$$T(n) = cn + \frac{1}{n}\sum_{i=0}^{n-1}(T(i) + T(n-1-i))$$

Properties of Quicksort

- O(N^2) worst case performance, but
  O($N \log N$) average case performance.
- Pure quicksort not good for small arrays.
- Iterative version uses a stack
- "In-place," but uses auxiliary storage because of recursive calls.
- Used by Java for sorting arrays of primitive types.

How fast can we sort?

Heapsort and Mergesort have O(N log N) **worst** case running time.

These algorithms, along with Quicksort, also have O(N log N) **average** case running time.

Can we do any better?

Permutations

- Suppose you are given *N* elements
  - Assume no duplicates
- How many possible orderings can you get?
  - Example: a, b, c ($N = 3$)

Permutations

- How many possible orderings can you get?
  - Example: a, b, c (N = 3)
  - (a b c), (a c b), (b a c), (b c a), (c a b), (c b a)
  - 6 orderings = 3•2•1 = 3!   (i.e., "3 factorial")

- For N elements
  - N choices for the first position, (N-1) choices for the second position, …, (2) choices, 1 choice
  - $N(N-1)(N-2)\cdots(2)(1)$= <u>N! possible orderings</u>

Sorting Model

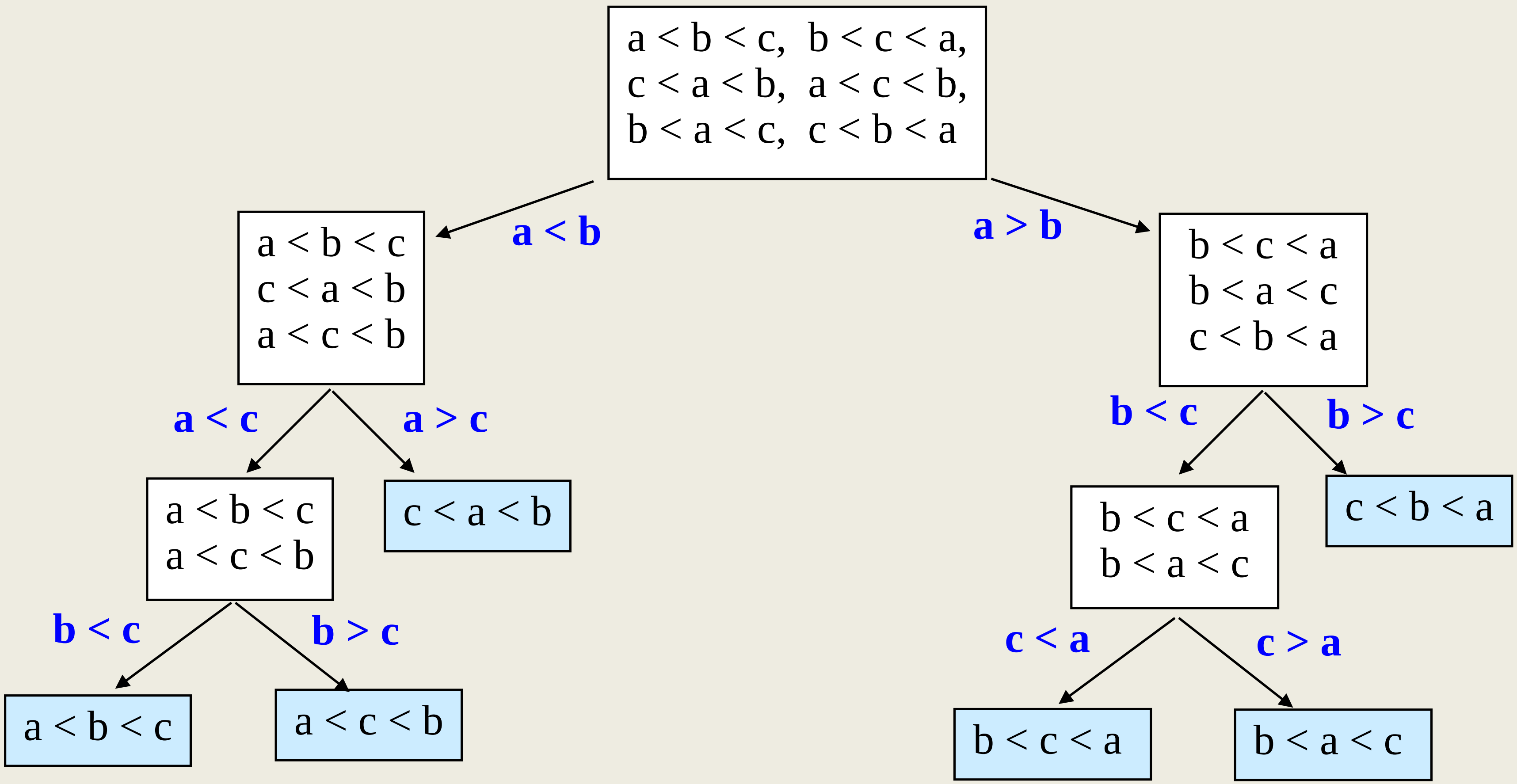
Recall our basic sorting assumption:

We can only compare
two elements at a time.

These comparisons prune the space of possible orderings.

We can represent these concepts in a…

Decision Tree



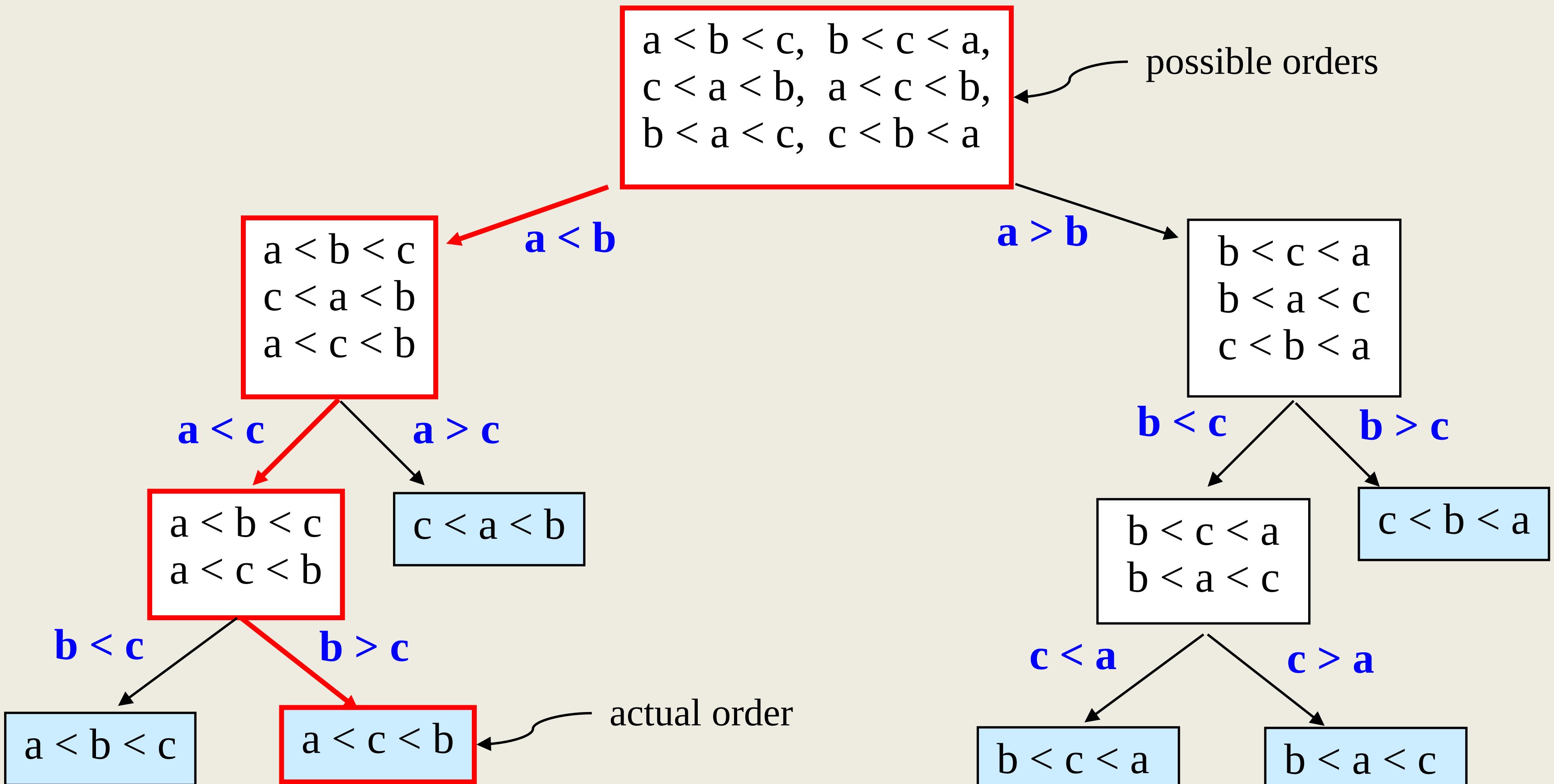
The leaves contain all the possible orderings of a, b, c.

Decision Trees

- A Decision Tree is a Binary Tree such that:
  - Each node = a set of orderings
    - i.e., the remaining solution space
  - Each edge = 1 comparison
  - Each leaf = 1 unique ordering
  - How many leaves for *N* distinct elements?

- Only 1 leaf has the ordering that is the desired correctly sorted arrangement

Decision Tree Example

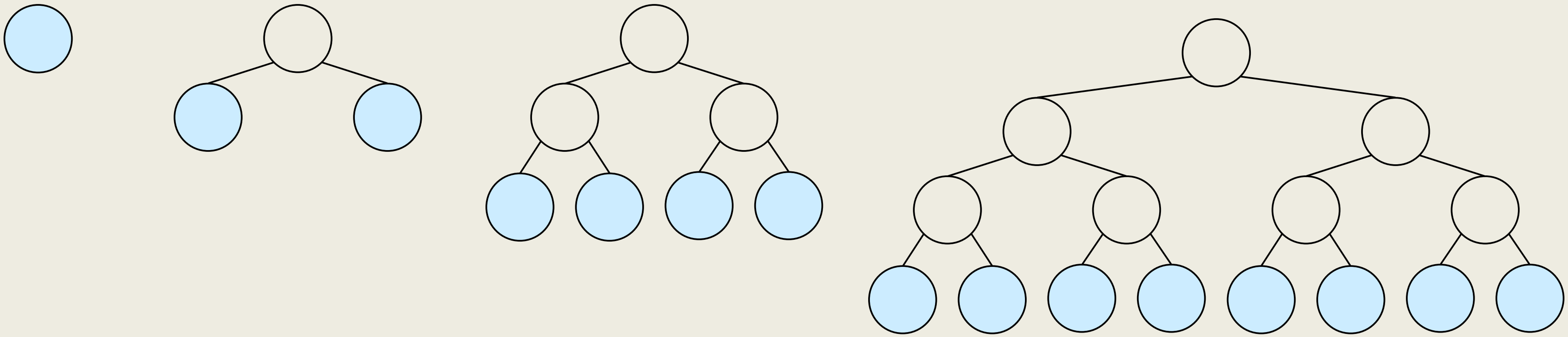


Decision Trees and Sorting

- Every comparison based sorting algorithm corresponds to a decision tree
  - Finds correct leaf by choosing edges to follow
    - i.e., by making comparisons
- We will focus on worst case run time
- Observations:
  - Worst case run time $\geq$ max number of comparisons
  - Max number of comparisons
    = length of the longest path in the decision tree
    = tree height

How many leaves on a tree?

Suppose you have a binary tree of height *h*. How many leaves in a perfect tree?



We can prune a perfect tree to make any binary tree of same height. Can # of leaves increase?

Lower bound on Height

- A binary tree of height h has at most 2^h leaves
  - Can prove by induction
- A decision tree has *N*! leaves. What is its minimum height?

Lower bound on log(n!)

$$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2) \cdots 4 \cdot 3 \cdot 2 \cdot 1 \\ &\geq n \cdot (n-1) \cdot (n-2) \cdots \frac{n}{2} \\ &\geq \frac{n}{2} \cdot \frac{n}{2} \cdot \frac{n}{2} \cdots \frac{n}{2} \\ &\geq \left(\frac{n}{2}\right)^{n/2} \end{aligned}$$

$$\log n! \geq \log \left(\frac{n}{2}\right)^{n/2} = \frac{n}{2} \log \frac{n}{2}$$

$\Omega(N \log N)$

Worst case run time of any comparison-based sorting algorithm is $\boldsymbol{\Omega(N \log N)}$.

Can also show that **average case** run time is also $\boldsymbol{\Omega(N \log N)}$.

Can we do better if we don't use comparisons?

Can we sort in O(n)?

- Suppose keys are integers between 0 and 1000

BucketSort (aka BinSort)

If all values to be sorted are integers between 1 and *B*, create an array **`count`** of size *B*, **increment** counts while traversing the input, and finally output the result.

Example *B*=5. Input = (5,1,3,4,3,2,1,1,5,4,5)

`count` array	
1	
2	
3	
4	
5	

Running time to sort n items?

What about our Ω(n log n) bound?

Dependence on *B*

What if *B* is very large (e.g., 2^{64})?

Fixing impracticality: RadixSort

- RadixSort: generalization of BucketSort for large integer keys
- Origins go back to the 1890 census.
- Radix = "The base of a number system"
  - We'll use 10 for convenience, but could be anything
- Idea:
  - BucketSort on one digit at a time
  - After k^{th} sort, the last k digits are sorted
  - Set number of buckets: *B* = radix.

Radix Sort Example

Input: 478, 537, 9, 721, 3, 38, 123, 67

BucketSort on 1's

0	1	2	3	4	5	6	7	8	9

BucketSort on 10's

0	1	2	3	4	5	6	7	8	9

BucketSort on 100's

0	1	2	3	4	5	6	7	8	9

Output:

Radix Sort Example (1st pass)

Bucket sort
by 1's digit

Input data

478
537
9
721
3
38
123
67

0	1	2	3	4	5	6	7	8	9
	721		3 123				537 67	478 38	9

After 1st pass

721
3
123
537
67
478
38
9

This example uses B=10 and base 10 digits for simplicity of demonstration. Larger bucket counts should be used in an actual implementation.

Radix Sort Example (2^{nd} pass)

After 1^{st} pass

721
3
123
537
67
478
38
9

Bucket sort by 10's digit

0	1	2	3	4	5	6	7	8	9
03 09		721 123	537 38			67	478		

After 2^{nd} pass

3
9
721
123
537
38
67
478

Radix Sort Example (3^{rd} pass)

After 2^{nd} pass

3
9
721
123
537
38
67
478

Bucket sort by 100's digit

0	1	2	3	4	5	6	7	8	9
003 009 038 067	123			478	537		721		

After 3^{rd} pass

3
9
38
67
123
478
537
721

Invariant: after k passes the low order k digits are sorted.

Radixsort: Complexity

In our examples, we had:

- Input size, N
- Number of buckets, B = 10
- Maximum value, $M < 10^3$
- Number of passes, P =

How much work per pass?

Total time?

Choosing the Radix

Run time is roughly proportional to:

$$P(B+N) = \log_B M(B+N)$$

Can show that this is minimized when:

$$B \log_e B \approx N$$

In theory, then, the best base (radix) depends only on N.

For fast computation, prefer $B = 2^b$. Then best b is:

$$b + \log_2 b \approx \log_2 N$$

Example:

- N = 1 million (i.e., ~2^{20}) 64 bit numbers, $M = 2^{64}$
- $\log_2 N \approx 20 \rightarrow b = 16$
- $B = 2^{16} = 65{,}536$ and $P = \log_{(2^{16})} 2^{64} = 4$.

In practice, memory word sizes, space, other architectural considerations, are important in choosing the radix.

Sorting Summary

$O(N^2)$ average, worst case:

- **Selection Sort**, **Bubblesort**, **Insertion Sort**

$O(N \log N)$ average case:

- **Heapsort**: In-place, not stable.
- **BST Sort**: $O(N)$ extra space (including tree pointers, possibly poor memory locality), stable.
- **Mergesort**: $O(N)$ extra space, stable.
- **Quicksort**: claimed fastest in practice, but $O(N^2)$ worst case. Recursion/stack requirement. Not stable.

$\Omega(N \log N)$ worst and average case:

- **Any comparison-based sorting algorithm**

$O(N)$

- **Radix Sort**: fast and stable. Not comparison based. Not in-place. Poor memory locality can undercut performance.