



# CSE 332: Data Structures & Parallelism

## Lecture 13: Beyond Comparison Sorting

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# *Today*

- Sorting
  - Comparison sorting
  - Beyond comparison sorting

# The Big Picture

**Simple algorithms:**  
 $O(n^2)$

Insertion sort  
Selection sort  
Shell sort  
...

**Fancier algorithms:**  
 $O(n \log n)$

Heap sort  
Merge sort  
Quick sort (avg)  
...

**Comparison lower bound:**  
 $\Omega(n \log n)$

**Specialized algorithms:**  
 $O(n)$

Bucket sort  
Radix sort

**Handling huge data sets**

External sorting

# *How fast can we sort?*

- Heapsort & mergesort have  $O(n \log n)$  worst-case running time
- Quicksort has  $O(n \log n)$  average-case running times
- These bounds are all tight, actually  $\Theta(n \log n)$
- So maybe we need to dream up another algorithm with a lower asymptotic complexity, such as  $O(n)$  or  $O(n \log \log n)$ 
  - Instead: *prove* that this is *impossible*
    - *Assuming* our comparison *model*: The only operation an algorithm can perform on data items is a 2-element comparison

# *A Different View of Sorting*

- Assume we have  $n$  elements to sort
  - And for simplicity, none are equal (no duplicates)
- How many **permutations** (possible orderings) of the elements?
- Example,  $n=3$ ,

# *A Different View of Sorting*

- Assume we have  $n$  elements to sort
  - And for simplicity, none are equal (no duplicates)
- How many **permutations** (possible orderings) of the elements?
- Example,  $n=3$ , six possibilities
  - $a[0]<a[1]<a[2]$     $a[0]<a[2]<a[1]$     $a[1]<a[0]<a[2]$
  - $a[1]<a[2]<a[0]$     $a[2]<a[0]<a[1]$     $a[2]<a[1]<a[0]$
- In general,  $n$  choices for least element, then  $n-1$  for next, then  $n-2$  for next, ...
  - $n(n-1)(n-2)\dots(2)(1) = n!$  possible orderings

# *Describing every comparison sort*

- A different way of thinking of sorting is that the sorting algorithm has to “find” the right answer among the  $n!$  possible answers
  - Starts “knowing nothing”, “anything is possible”
  - Gains information with each comparison, eliminating some possibilities
    - Intuition: At best, each comparison can eliminate half of the remaining possibilities
  - In the end narrows down to a single possibility

# Counting Comparisons

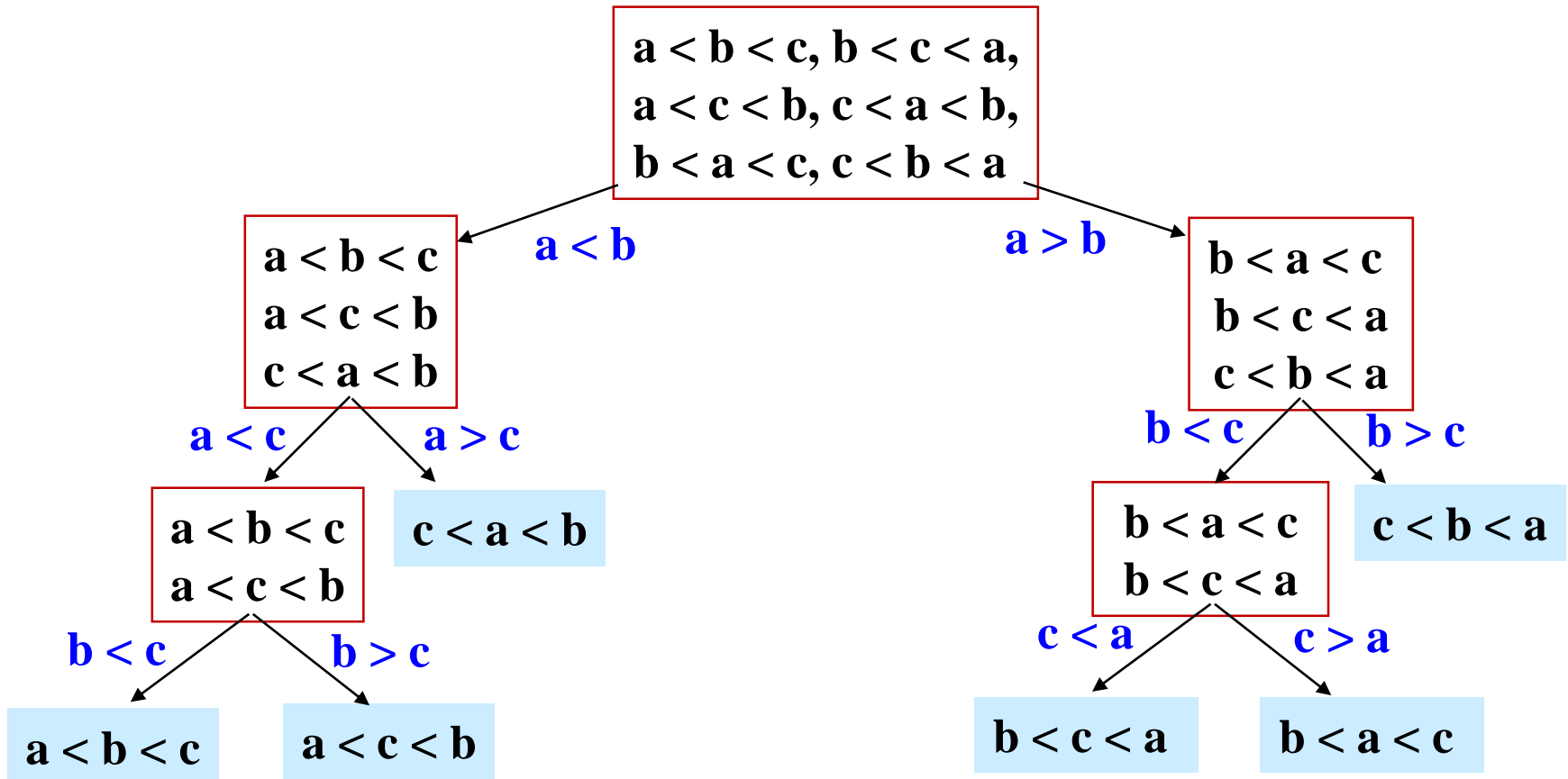
- Don't know what the algorithm is, but it cannot make progress without doing comparisons
  - Eventually does a first comparison “is  $a < b$  ?”
  - Can use the result to decide what second comparison to do
  - Etc.: comparison  $k$  can be chosen based on first  $k-1$  results
- What is the first comparison in:
  - Selection Sort?
  - Insertion Sort?
  - Quicksort?
  - Mergesort?



# Counting Comparisons

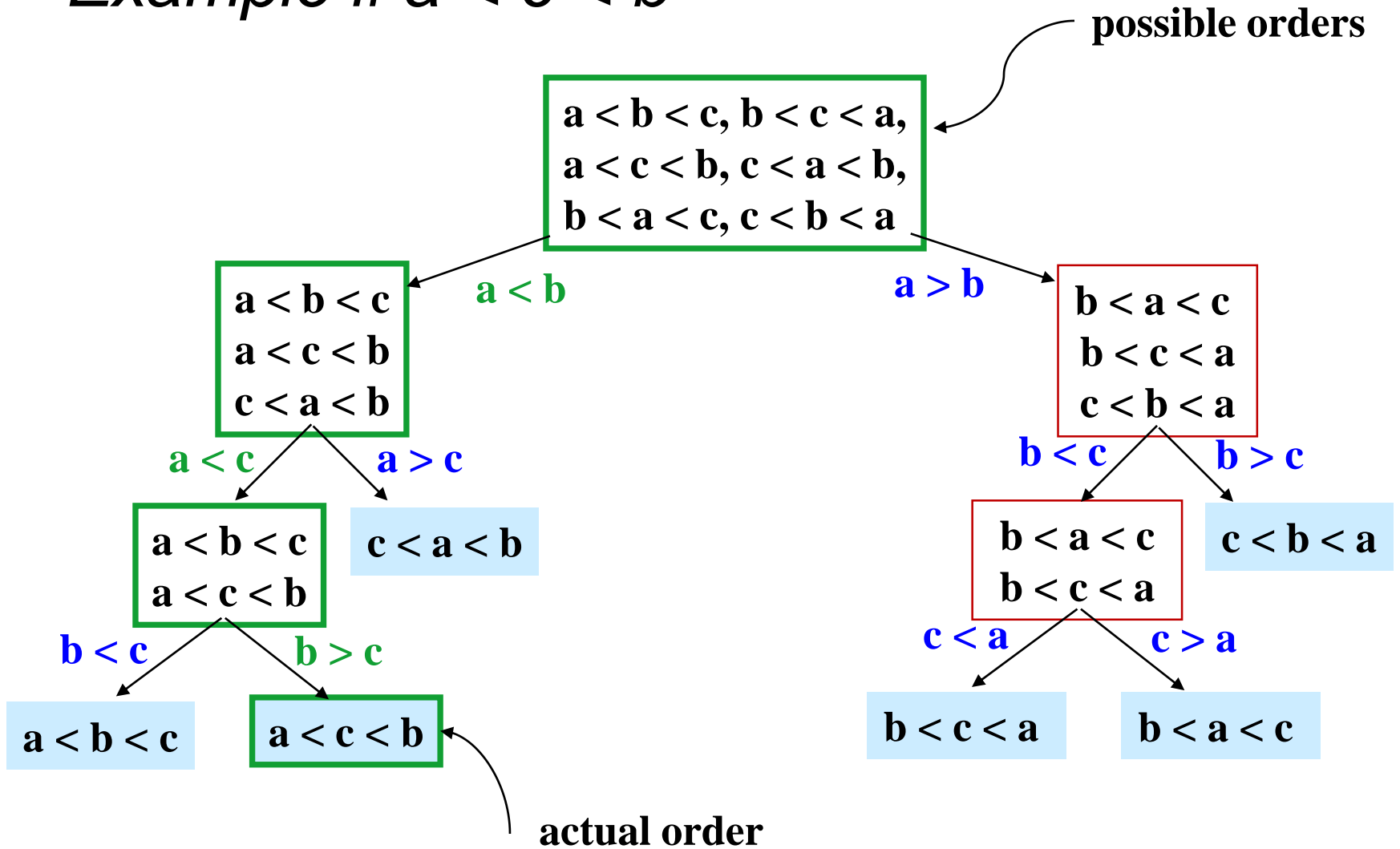
- Don't know what the algorithm is, but it cannot make progress without doing comparisons
  - Eventually does a first comparison “is  $a < b$  ?”
  - Can use the result to decide what second comparison to do
  - Etc.: comparison  $k$  can be chosen based on first  $k-1$  results
- Can represent this process as a *decision tree*
  - Nodes contain “set of remaining possibilities”
  - At root, anything is possible; no option eliminated
  - Edges are “answers from a comparison”
  - The algorithm does not actually build the tree; it's what our *proof* uses to represent “the most the algorithm could know so far” as the algorithm progresses

# One Decision Tree for $n=3$



- The leaves contain all the possible orderings of  $a$ ,  $b$ ,  $c$
- A different algorithm would lead to a different tree

# Example if $a < c < b$



# What the decision tree tells us

- A *binary* tree because each comparison has 2 outcomes
  - Perform only comparisons between 2 elements; binary result
    - Ex: Is  $a < b$ ? Yes or no?
  - We assume no duplicate elements
  - Assume algorithm doesn't ask redundant questions
- Because any data is possible, any algorithm needs to ask enough questions to produce all  $n!$  answers
  - Each answer is a different leaf
  - So the tree must be big enough to have  $n!$  leaves
  - Running *any* algorithm on *any* input will at best correspond to a root-to-leaf path in *some* decision tree with  $n!$  leaves
  - So no algorithm can have worst-case running time better than the height of a tree with  $n!$  leaves
    - Worst-case number-of-comparisons for an algorithm is an input leading to a longest path in algorithm's decision tree

# Where are we

**Proven:** No comparison sort can have worst-case running time better than: **the height of a binary tree with  $n!$  leaves**

- Turns out average-case is same asymptotically
- A comparison sort could be worse than this height, but it cannot be better
- Fine, *how tall is a binary tree with  $n!$  leaves?*

**Now:** Show that a binary tree with  $n!$  leaves has height  $\Omega(n \log n)$

- That is,  $n \log n$  is the lower bound, the height must be at least this, could be more, (in other words your comparison sorting algorithm could take longer than this, but it won't be faster)
- Factorial function grows very quickly

Then we'll conclude that: **(Comparison) Sorting is  $\Omega(n \log n)$**

- This is an amazing computer-science result: proves all the clever programming in the world can't sort in linear time!

## *Lower bound on Height*

- A binary tree of height  $h$  has **at most** *how many* leaves?

$$L \leq \underline{\hspace{4cm}}$$

- A binary tree with  $L$  leaves has height **at least**:

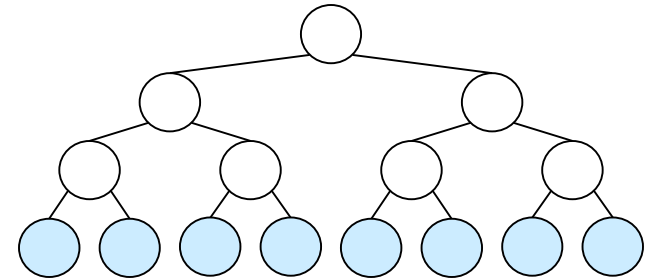
$$h \geq \underline{\hspace{4cm}}$$

- The decision tree has how many leaves:

- So the decision tree has height:

$$h \geq \underline{\hspace{4cm}}$$

# Lower bound on height



- The height of a binary tree with  $L$  leaves is at least  $\log_2 L$
- So the height of our decision tree,  $h$ :

$$h \geq \log_2 (n!)$$

$$= \log_2 (n \cdot (n-1) \cdot (n-2) \dots (2)(1))$$

$$= \log_2 n + \log_2 (n-1) + \dots + \log_2 1$$

$$\geq \log_2 n + \log_2 (n-1) + \dots + \log_2 (n/2)$$

$$\geq (n/2) \log_2 (n/2) \quad \text{each of the } n/2 \text{ terms left is } \geq \log_2 (n/2)$$

$$= (n/2)(\log_2 n - \log_2 2)$$

$$= (1/2)n \log_2 n - (1/2)n$$

$$\text{“=“ } \Omega (n \log n)$$

property of binary trees

definition of factorial

property of logarithms

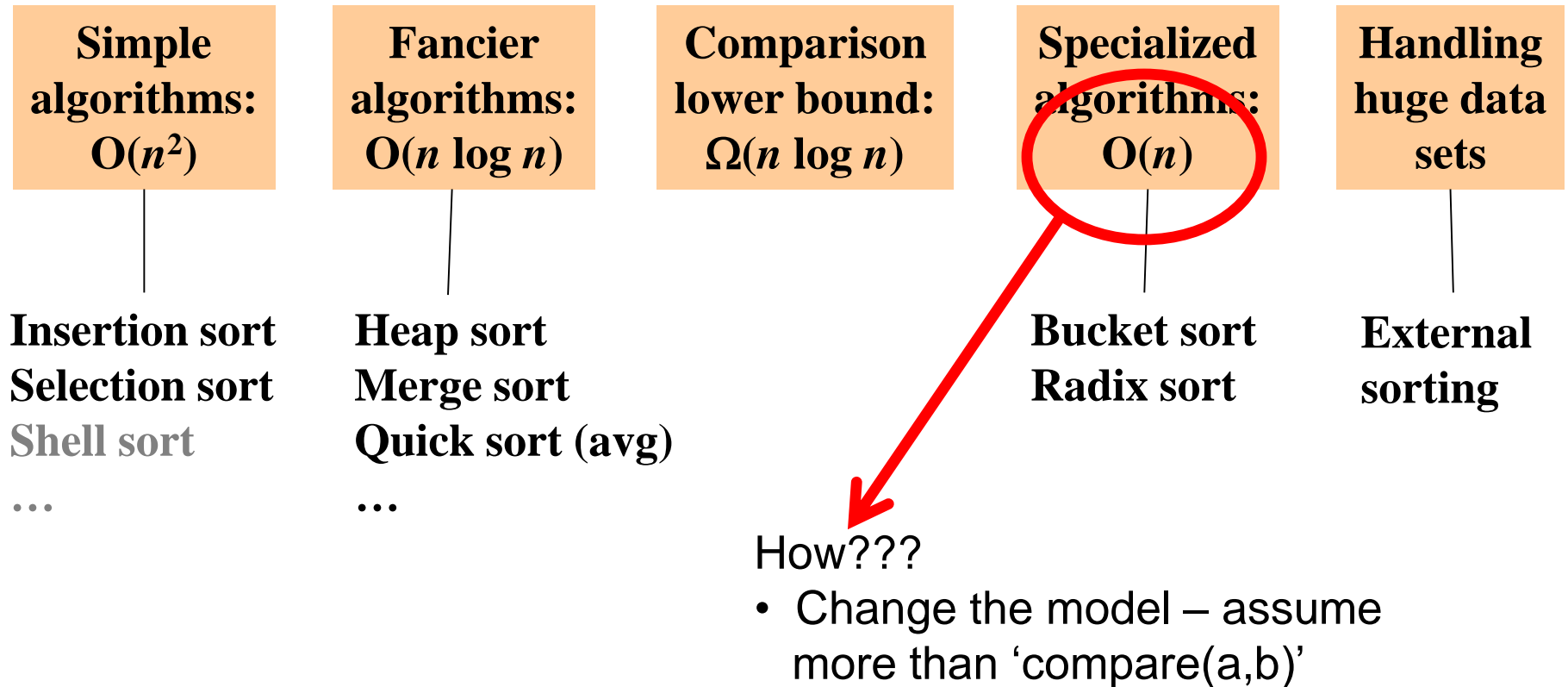
keep first  $n/2$  terms

each of the  $n/2$  terms left is  $\geq \log_2 (n/2)$

property of logarithms

arithmetic

# The Big Picture





# BucketSort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and  $K$  (or any small range),
  - Create an array of size  $K$ , and put each element in its proper bucket (a.k.a. bin)
  - If data is only integers, no need to store more than a *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

count array	
1	
2	
3	
4	
5	

- Example:  
K=5  
Input: (5,1,3,4,3,2,1,1,5,4,5)  
output:

# Analyzing bucket sort

- Overall:  $O(n+K)$ 
  - Linear in  $n$ , but also linear in  $K$
  - $\Omega(n \log n)$  lower bound does not apply because **this is not a comparison sort**
- Good when range,  $K$ , is smaller (or not much larger) than  $n$ 
  - (We don't spend time doing lots of comparisons of duplicates!)
- Bad when  $K$  is much larger than  $n$ 
  - Wasted space; wasted time during final linear  $O(K)$  pass
- For data in addition to integer keys, use list at each bucket

# Bucket Sort with Data

- Most real lists aren't just #'s; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, place at end  $O(1)$  (keep pointer to last element)

Bucket sort illustrates a more general trick: How might you implement a heap for a small range of integer priorities in a similar manner...

count array	
1	→ <b>Rocky V</b>
2	
3	→ <b>Harry Potter</b>
4	
5	→ <b>Casablanca</b> → <b>Star Wars</b>

- Example: Movie ratings: 1=bad,... 5=excellent
- Input=
  - 5: Casablanca
  - 3: Harry Potter movies
  - 1: Rocky V
  - 5: Star Wars

**Result:** 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars  
This result is **stable**; Casablanca still before Star Wars

# Radix sort

- Radix = “the base of a number system”
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128
- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with *least* significant digit, sort with Bucket Sort
    - Keeping sort *stable*
  - Do one pass per digit
- **Invariant:** After  $k$  passes, the last  $k$  digits are sorted
- Aside: Origins go back to the 1890 U.S. census

# Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

**Input:** 478  
537  
9  
721  
3  
38  
143  
67

## First pass:

1. bucket sort by ones digit
2. Iterate thru and collect into a list
  - List is sorted by first digit

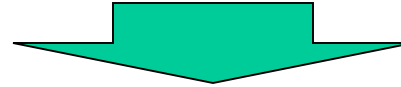
**Order now:** 721

3  
143  
537  
67  
478  
38  
9

# Example

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

Radix = 10



0	1	2	3	4	5	6	7	8	9
3 9		721	537 38	143		67	478		

Order was: 721  
3  
143  
537  
67  
478  
38  
9

Second pass:

**stable** bucket sort by tens digit

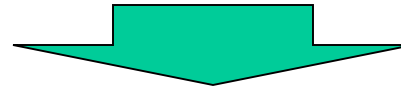
If we chop off the 100's place,  
these #s are sorted

Order now: 3  
9  
721  
537  
38  
143  
67  
478

# Example

0	1	2	3	4	5	6	7	8	9
3		721	537	143		67	478		
9			38						

Radix = 10



0	1	2	3	4	5	6	7	8	9
3	143			478	537		721		
9									
38									
67									

Order was:

3  
9  
721  
537  
38  
143  
67  
478

Order now:

3  
9  
38  
67  
143  
478  
537  
721

Third pass:

**stable** bucket sort by 100s digit

Only 3 digits: We're done!

# RadixSort

- Input: 126, 328, 636, 341, 416, 131, 328

**BucketSort on lsd:**

0	1	2	3	4	5	6	7	8	9

**BucketSort on next-higher digit:**

0	1	2	3	4	5	6	7	8	9

**BucketSort on msd:**

0	1	2	3	4	5	6	7	8	9



# *Analysis of Radix Sort*

Performance depends on:

- Input size:  $n$
- Number of buckets = Radix:  $B$ 
  - e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
- Number of passes = “Digits”:  $P$ 
  - e.g. Ages of people: 3; Phone #: 10; Person’s name: ?
- Work per pass is 1 bucket sort: \_\_\_\_\_
  - Each pass is a Bucket Sort
- Total work is \_\_\_\_\_
  - We do ‘P’ passes, each of which is a Bucket Sort

# *Comparison to Comparison Sorts*

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Approximate run-time:  $15 \cdot (52 + n)$
  - This is less than  $n \log n$  only if  $n > 33,000$
  - Of course, cross-over point depends on constant factors of the implementations plus  $P$  and  $B$ 
    - And radix sort can have poor locality properties
- Not really practical for many classes of keys
  - Strings: Lots of buckets

# *Recap: Features of Sorting Algorithms*

## **In-place**

- Sorted items occupy the same space as the original items. (No copying required, only  $O(1)$  extra space if any.)

## **Stable**

- Items in input with the same value end up in the same order as when they began.

Examples:

- Merge Sort - not in place,      stable
- Quick Sort - in place,      not stable

# *Sorting massive data: External Sorting*

Need sorting algorithms that **minimize disk/tape access** time:

- Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
- Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access

Basic Idea:

- Load chunk of data into Memory, sort, store this “run” on disk/tape
- Use the Merge routine from Mergesort to merge runs
- Repeat until you have only one run (one sorted chunk)
  
- Mergesort can leverage multiple disks
- Weiss gives some examples

# Sorting Summary

- Simple  $O(n^2)$  sorts can be fastest for small  $n$ 
  - selection sort, insertion sort (latter linear for mostly-sorted)
  - good for “below a cut-off” to help divide-and-conquer sorts
- $O(n \log n)$  sorts
  - heap sort, in-place but not stable nor parallelizable
  - merge sort, not in place but stable and works as external sort
  - quick sort, in place but not stable and  $O(n^2)$  in worst-case
    - often fastest, but depends on costs of comparisons/copies
- $\Omega(n \log n)$  is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
  - Bucket sort good for small number of key values
  - Radix sort uses fewer buckets and more phases
- Best way to sort? It depends!