

CSE 332: Data Structures & Parallelism Lecture 5: Algorithm Analysis II

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Today

- Analyzing Recursive Code
- Solving Recurrences

Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic
- Assignment
- Access one Java field or array index
- Etc.

(This is an approximation of reality: a very useful "lie".)

Consecutive statements Sum of time of each statement

Loops Num iterations * time for loop body

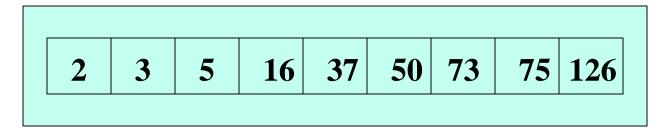
Conditionals Time of condition plus time of

slower branch

Function Calls Time of function's body

Recursion Solve recurrence equation

Linear search



Find an integer in a sorted array

Analyzing Recursive Code

- Computing run-times gets interesting with recursion
- Say we want to perform some computation recursively on a list of size n
 - Conceptually, in each recursive call we:
 - Perform some amount of work, call it w(n)
 - Call the function recursively with a smaller portion of the list
- So, if we do w(n) work per step, and reduce the problem size in the next recursive call by 1, we do total work:

$$T(n)=w(n)+T(n-1)$$

With some base case, like T(1)=5=O(1)

Example Recursive code: sum array

Recursive:

Recurrence is some constant amount of work
 O(1) done n times

```
int sum(int[] arr) {
   return help(arr,0);
}
int help(int[]arr,int i) {
   if(i==arr.length)
      return 0;
   return arr[i] + help(arr,i+1);
}
```

Each time **help** is called, it does that O(1) amount of work, and then calls **help** again on a problem one less than previous problem size.

Recurrence Relation: T(n) = O(1) + T(n-1)

Solving Recurrence Relations

Say we have the following recurrence relation:

$$T(n)=6$$
 "ish" + $T(n-1)$
 $T(1)=9$ "ish" \leftarrow base case

- Now we just need to solve it; that is, reduce it to a closed form.
- Start by writing it out:

```
T(n)=6+T(n-1)
=6+6+T(n-2)
=6+6+6+T(n-3)
=6+6+6+...+6+T(1) = 6+6+6+...+6+9
=6k+T(n-k)
=6k+9, where k is the # of times we expanded T()
```

We expanded it out n-1 times, so

$$T(n)=6k+T(n-k)$$

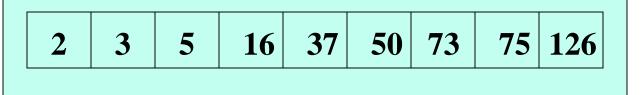
=6(n-1)+T(1) = 6(n-1)+9
=6n+3 = O(n)

Or When does n-k=1?
Answer: when k=n-1

Best case:

Binary search

Worst case:



Find an integer in a sorted array

Can also be done non-recursively but "doesn't matter" here

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Binary search

```
Best case: 9 "ish" steps = O(1)
Worst case: T(n) = 10 "ish" + T(n/2) where n is hi-lo
```

- $O(\log n)$ where n is array.length
- Solve recurrence equation to know that...

Solving Recurrence Relations

Determine the recurrence relation. What is the base case?

$$T(n) = 10 + T(n/2)$$
 $T(1) = 15$

2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.

3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

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Solving Recurrence Relations

Determine the recurrence relation. What is the base case?

$$T(n) = 10 + T(n/2)$$
 $T(1) = 15$

2. "Expand" the original relation to find an equivalent general expression in terms of the number of expansions.

```
 T(n) = 10 + 10 + T(n/4) 
 = 10 + 10 + 10 + T(n/8) 
 = ... 
 = 10k + T(n/(2^k))  (where k is the number of expansions)
```

3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

```
- n/(2^k) = 1 means n = 2^k means k = \log_2 n

- So T(n) = 10 \log_2 n + 15 (get to base case and do it)

- So T(n) is O(\log n)
```

sum array again

Two "obviously" linear algorithms: T(n) = O(1) + T(n-1)

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}</pre>
```

Recursive:

Recurrence isc + c + ... + cfor n times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

What about a binary version of sum?

```
int sum(int[] arr) {
    return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
    if(lo==hi) return 0;
    if(lo==hi-1) return arr[lo];
    int mid = (hi+lo)/2;
    return help(arr,lo,mid) + help(arr,mid,hi);
}
```

What about a binary version of sum?

```
int sum(int[] arr) {
    return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
    if(lo==hi)         return 0;
    if(lo==hi-1)         return arr[lo];
    int mid = (hi+lo)/2;
    return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is T(n) = O(1) + 2T(n/2)

- -1+2+4+8+... for log n times
- $-2^{(\log n)}-1$ which is proportional to n (by definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than O(n) – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls at the same time
 - Like having a friend do half the work for you!

```
int sum(int[]arr) {
    return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
    if(lo==hi)         return 0;
    if(lo==hi-1)         return arr[lo];
    int mid = (hi+lo)/2;
    return(help(arr,lo,mid))+(help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed, the recurrence is now T(n) = O(1) + 1T(n/2)
 - O(log n): same recurrence as for find

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$$T(n) = O(1) + T(n/2)$$
 logarithmic $O(\log n)$
 $T(n) = O(1) + 2T(n/2)$ linear $O(n)$
 $T(n) = O(1) + T(n-1)$ linear $O(n)$
 $T(n) = O(n) + T(n-1)$ quadratic $O(n^2)$
 $T(n) = O(1) + 2T(n-1)$ exponential $O(2^n)$
 $T(n) = O(n) + T(n/2)$ linear $O(n)$
 $T(n) = O(n) + 2T(n/2)$ loglinear $O(n \log n)$

Note big-Oh can also use more than one variable

Example: can sum all elements of an n-by-m matrix in O(nm)