Section 2 Big-O summary

Big-O definition:

- Informal: O(g(n)) is a set of all functions that are upper-bounded by g(n) as n tends to infinity
- To prove that a function, f(n), is in O(g(n)) we need to show that there exist 2 positive constants, c and n_o, such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

Template for a Big-O proof:

1. Identify the claim you are supposed to prove (of the form $f(n) \leq c \cdot g(n)$)
2. Begin scratch work to find c and n_0. Listed below are the suggested steps to follow to arrive at your c and n_0 values:

   i. Identify all the terms with n on the left hand side

   ii. Create inequalities of the form <term> $\leq g(n)$

   iii. Make sure you have the right domain of n when creating the inequalities in (ii), that is, make sure you find the right $n \geq$ <some_natural_num>

   iv. Manipulate and combine those inequalities you found such that you have the left-hand side to be the same as $f(n)$ and the right-hand side to be <some_constant> $\cdot g(n)$

   v. Once you have arrived at an inequality of the form mentioned in (iv), you have successfully found c and n_0 to be <some_constant> and <some_natural_num> respectively.
3. At this point, you should have your c and n_0 values to begin the "actual proof"
4. Listed below are the suggested steps that should help you finish the proof:

   i. Define c and n_0 to be the values you found using your scratch-work

   ii. Find true statements that could help you start your proof. A good starting point is to use some/all of the inequalities you found in (2.ii). They are true statements because they would most likely be some mathematical fact and using such mathematical facts is a great way to begin any proof

   iii. Using the statements you've selected to begin the proof, work towards getting the left-hand side to be equal to $f(n)$ and the right-hand to be equal to $c \cdot g(n)$
5. Et voilà! You've finished your Big-O proof

The template for the Big Omega proof is exactly the same as the one for Big-O proofs. The only difference would be to flip the inequality sign. So the claim you will have to prove would be: $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$. Other inequalities that would change would be: instead of it being <term> $\leq g(n)$, it would be <term> $\geq g(n)$. Other than that, most of your proof should be very similar to how we write Big-O proofs.

Big-Theta Proofs:

Even though we haven't done any Big-Theta proofs in sections, you have all the tools to be able to write Big-Theta proofs.

To show that $f(n) \in \Theta(g(n))$, we need to show that $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$ where there exists some $c_1, c_2 > 0$ and $n_0 > 0$.

In other words, you need to prove that $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$.

Bonus: Disproving a Big Omega relation

To prove: $n \notin \Omega(n^2)$

Recall definition of Big Omega: to show that $f(n) \in \Omega(g(n))$, there exist 2 positive constants, $c \text{ and } n_0$, such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

But to disprove a Big Omega relation, we would need to prove that the opposite of the definition is true.

So negating the definition we have: For every $c > 0 \text{ and } n_0 > 0,$ there exists an $n \geq n_0$ such that $f(n) < c \cdot g(n)$.

Now that we have a claim to prove, we can begin our scratch work to find our n value.

Scratch work:

claim to prove: $n < c \cdot n^2$

$1 < c \cdot n$ [dividing by n on both sides]

$n > \frac{1}{c}$ [dividing by c on both sides]

Now we've found that n must be greater than c for this inequality, $n < c \cdot n^2$, to be true.

Actual Proof:

In our scratch work, we found a range of values n can be such that our claim (negation of the Big Omega definition) holds. But we cannot define n to be a range of values, we need to select a specific value of n. Another problem we face here is that when we try to define n to be a single value, we also need to guarantee that $n \geq n_0$.

So we need to guarantee that:

1) $n > \frac{1}{c}$

2) $n \geq n_0$

One of the ways we can satisfy both these conditions is to use the max function.

So we can say:

$\text{Let } n = \max\{\frac{1}{c}, n_0\} + 1$ (the "+1" is to make sure that n is strictly greater than $\frac{1}{c}$)

Now that we've defined our n, we can follow our usual style of proof to prove our claim.

$n > \frac{1}{c}$ [true statement because of definition of n]

$c \cdot n > 1$ [multiplying c on both sides]

$c \cdot n^2 > n$ [multiplying n on both sides]

Hence, we've shown that $c \cdot g(n) > f(n)$, therefore we've shown that $f(n) \notin \Omega(g(n))$.