Non-Comparison Sorting Algorithms

CSE 332 Summer 2021

Instructor: Kristofer Wong

Teaching Assistants:

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Announcements

- Midterm
- P1 Grades out. Regrade Requests
- Ex7: videos tonight!
  - Reflection q's cannot be edited after submissions came in, so a new reflection assignment will be posted. ***Disregard the reflections section in the Exercise 7 gradescope assignment. A new one will be posted!***
- Office Hours today pushed to 11:10 so I can release midterm
- Reminder: lecture activities deadline changed to 7am
- Amendment to Insertion/Selection Sort stability

Lecture Outline

- **QuickSort**
  - Comparison with MergeSort
- Comparison Sort Lower Bound
- Non-Comparison Sorts
  - BucketSort
  - RadixSort
- Sorting Wrapup

QuickSort Steps

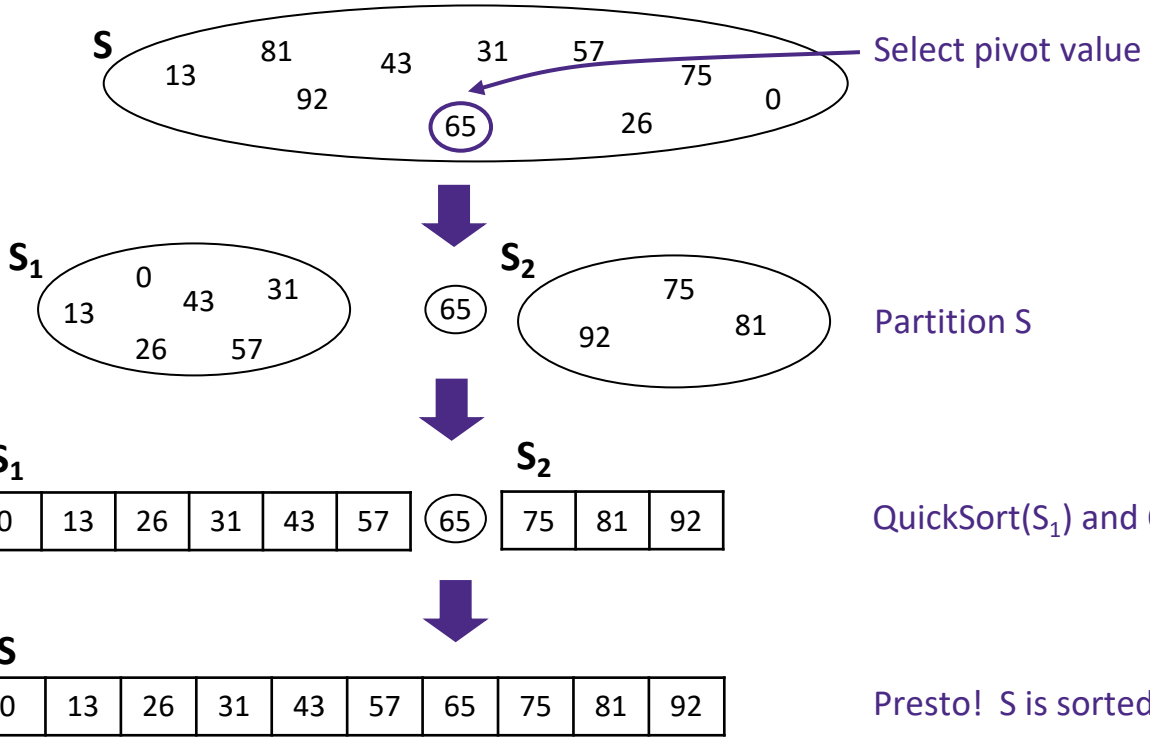
1. Pick the pivot value(s)
   - Any choice is correct; data will end up sorted
   - For efficiency, these value(s) ought to approximate the median

2. Partition all the values into:
   a. The values less than the pivot(s)
   b. The pivot(s)
   c. The values greater than the pivot(s)
   d. .. In linear time? In-place? Stably?

3. Recursively QuickSort(A) and QuickSort(C)

✨*TA-DA!*✨

QuickSort Intuition: Set Partitioning



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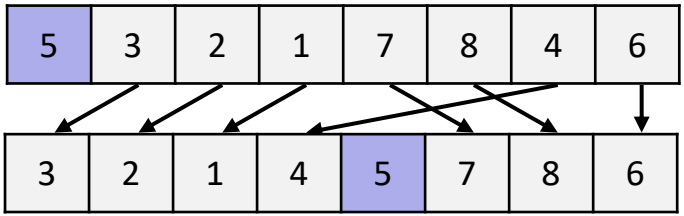
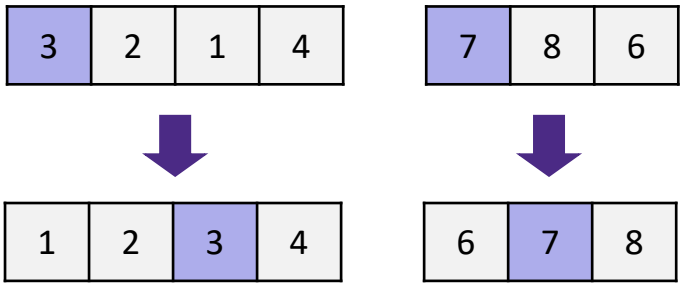
Recursive Call (1 of 3)

Note: for the remainder of this section, our pivot-selection algorithm is "first item in the subarray"

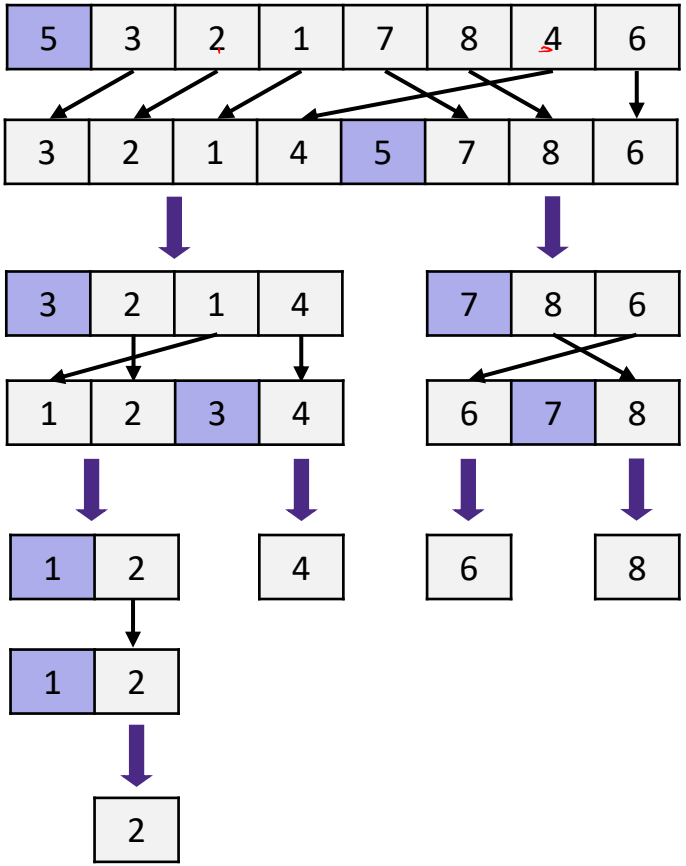
- After partitioning on 5:
  - 5 is in its "correct place" (ie, where it'd be if the array were sorted)



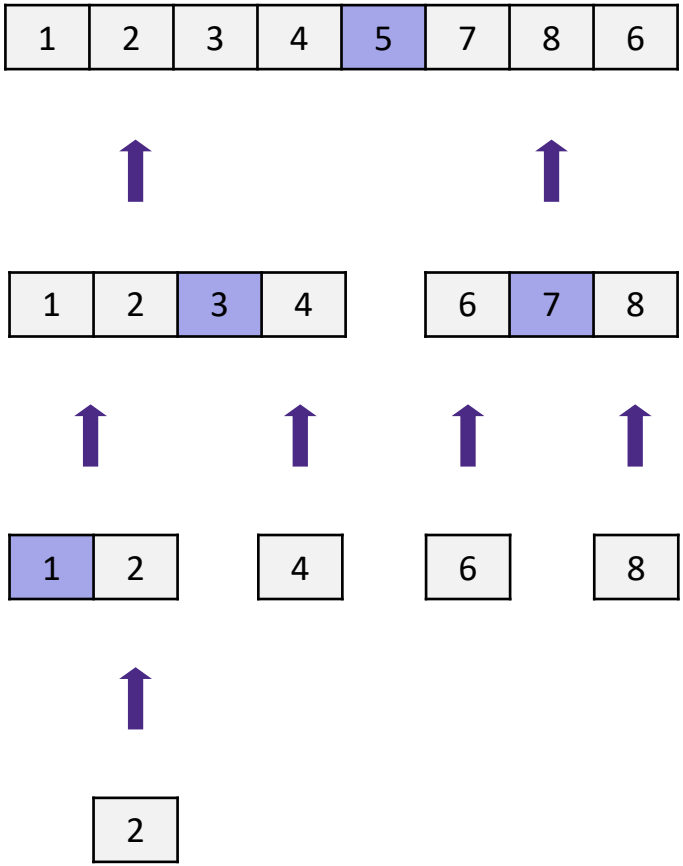
  - Can now sort two halves separately (eg, through recursive use of partitioning)



Recursive Call (2 of 3)



Recursive Call (3 of 3)



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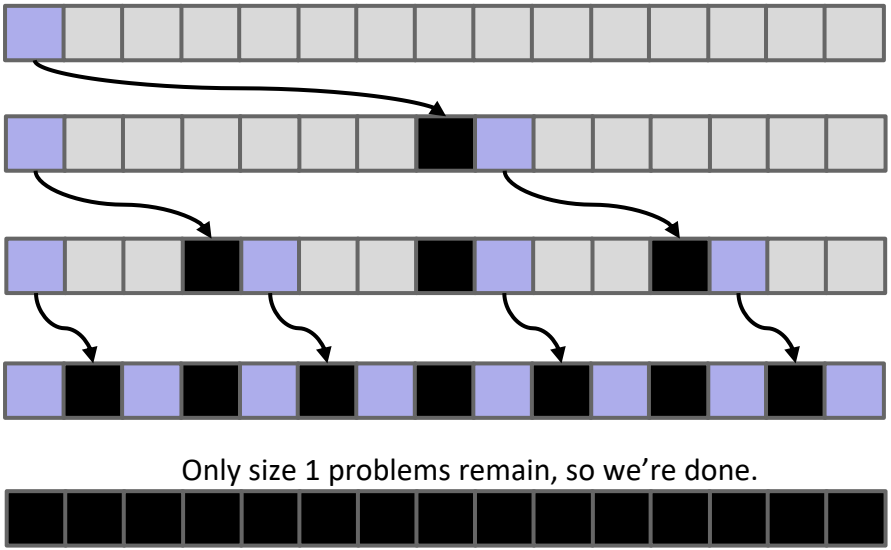
Pivot Selection: Pivot is the Median



$T(0) = T(1) = c_1$

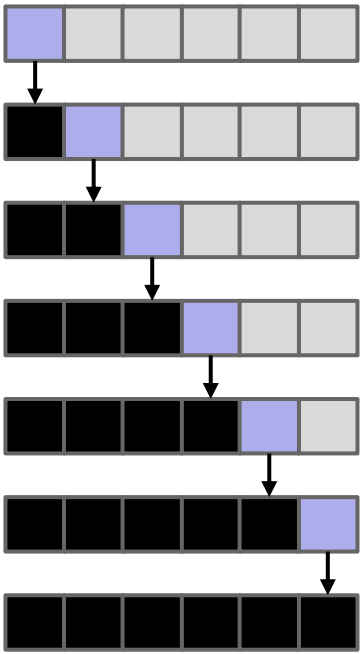
$T(n) = 2T(n/2) + c_2 n$

(partition is linear-time)

Same recurrence as MergeSort:

$O(n \log n)$

Pivot Selection: Pivot is the Min/Max



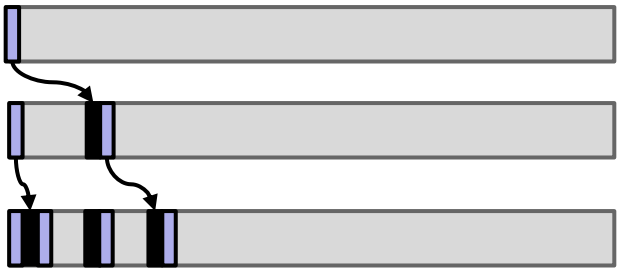
$T(0) = T(1) = c_1$

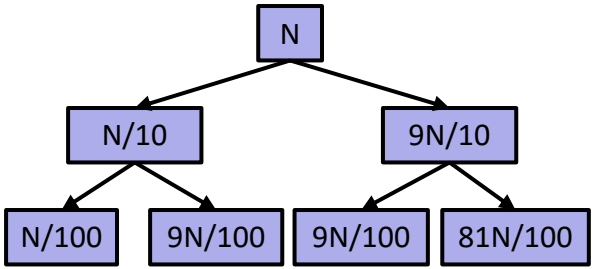
$T(n) = T(n\text{-}1) + c_2 n$

Basically same recurrence as SelectionSort: $O(n^2)$

Pivot Selection: Pivot is Random

- Suppose pivot always ends up *at least 10% from either edge*





- Work at each level: O(N) and Runtime is O(NH)
  - Height is approximately $\log_{10/9} N = O(\log N)$
- Runtime: O(N log N)
  - See proof in text

Pivot Selection Dictates Runtime!

- If pivot lands "somewhere good", Quicksort is Θ(N log N)

- However, the very rare Θ(N^2) cases do happen in practice 👎
  - **Bad ordering**: Array already in (almost-)sorted order and pivot is first or last index
  - **Bad elements**: Array with all duplicates

- Three philosophies for avoiding worst-case behavior:
  1. **Randomness**: pick a random pivot; shuffle before sorting
     - Elegant, but (pseudo)random number generation can be slow
  2. **Smarter Pivot Selection**: calculate or approximate the median
     - Median-of-3: median of `arr[lo]`, `arr[hi-1]`, `arr[(hi+lo)/2]`
  3. **Introspection**: switch to safer sort if recursion goes too deep

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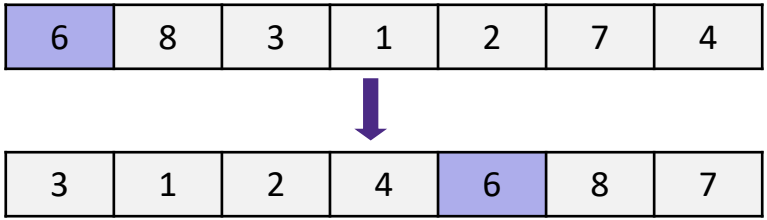
Partitioning: Problem Statement

- Given an array of elements and the 0^{th} value as the pivot, write pseudocode that partitions the array



- Constraints:
  - Must complete in O(N log N) time, but ideally Θ(N)
  - Must use O(N) space, but ideally Θ(1)
  - May use any data structure (eg, BSTs, stacks/queues, etc)
  - Ideally, preserves the elements' relative ordering ("stable")

- Conceptually simple, but hardest part to code up correctly!

Partitioning: Option 1: Three-Pass

- Overview:
  - Copy “less than”s, then copy pivot(s), finally copy “greater-than”s
- Stable! ☺
- Uses extra space ☹
- Constants aren’t great; very slow ☹

Stable Three-Pass Partition

- Allocate an extra array to store output
- Loop: copy less-than-pivot values into output array
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5	550	10	4	10	9	330

Demo Credit: Hannah Tang

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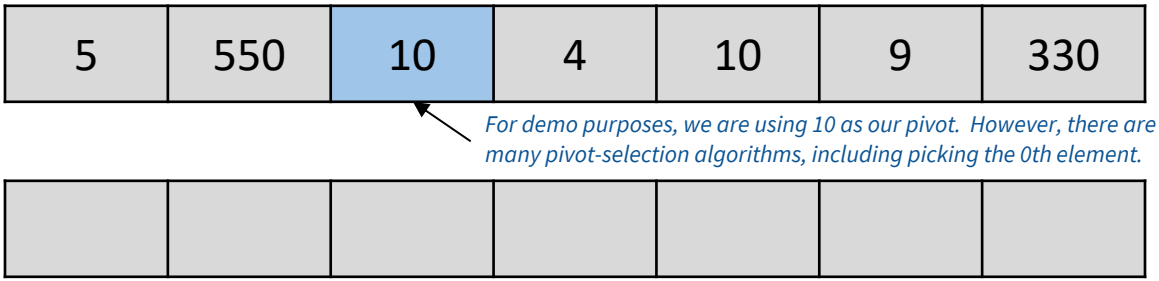

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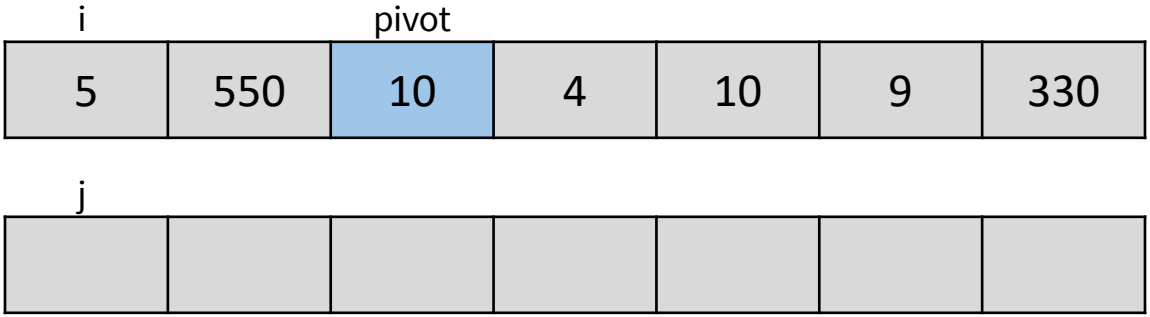

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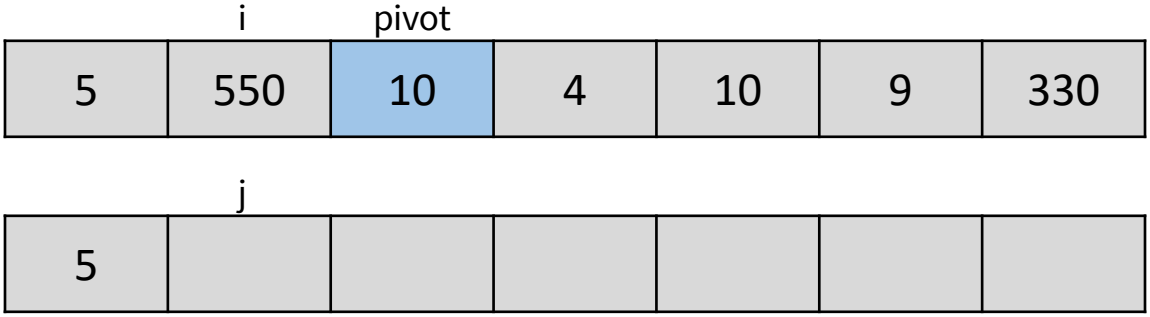

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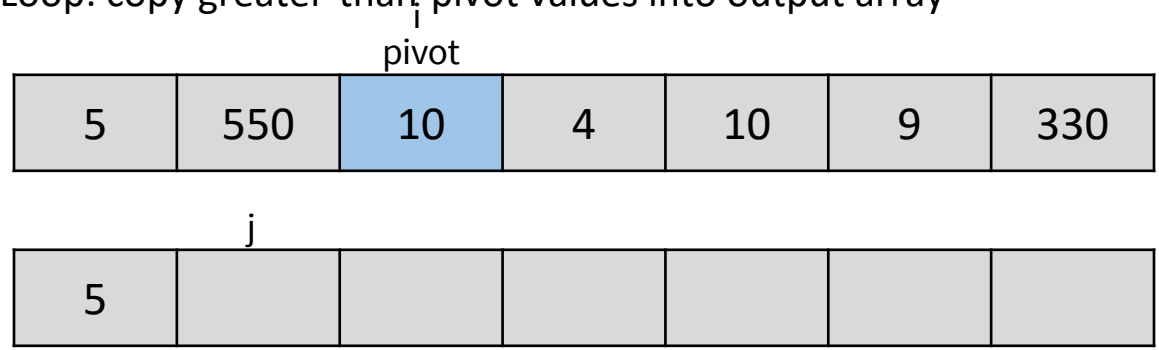

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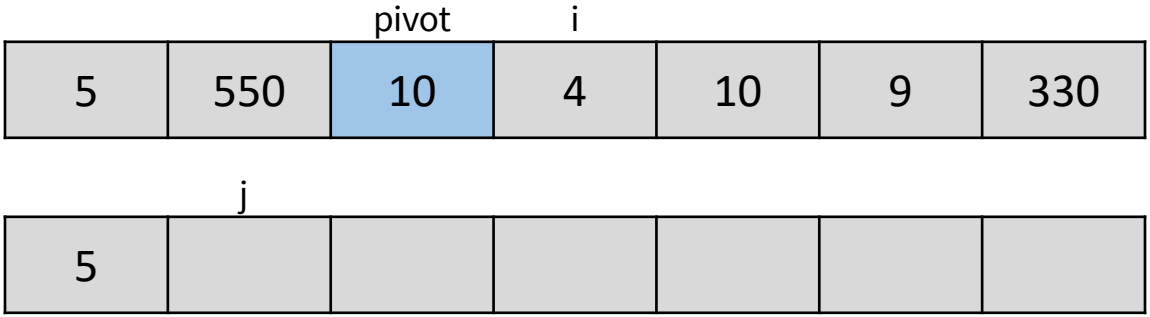

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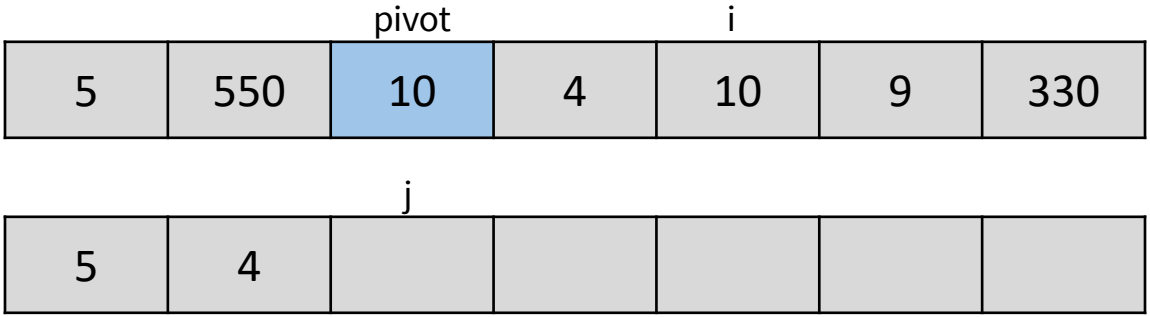

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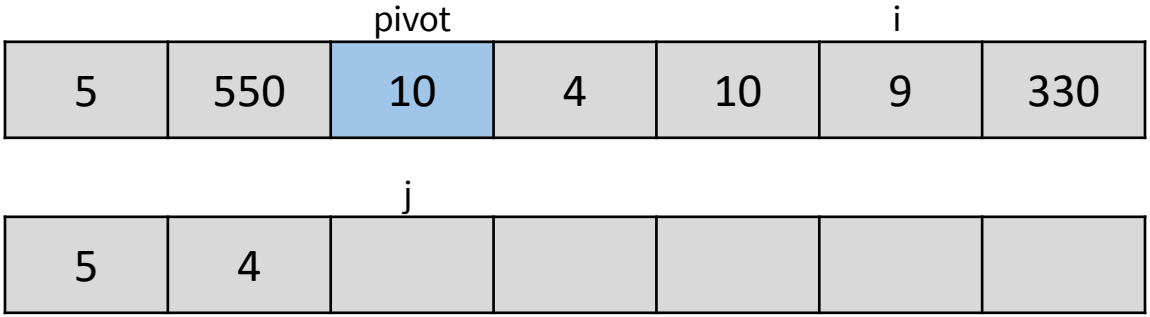

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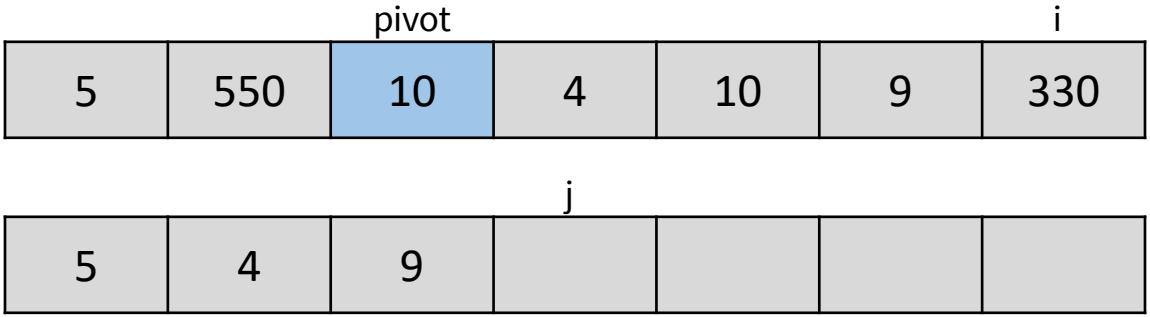

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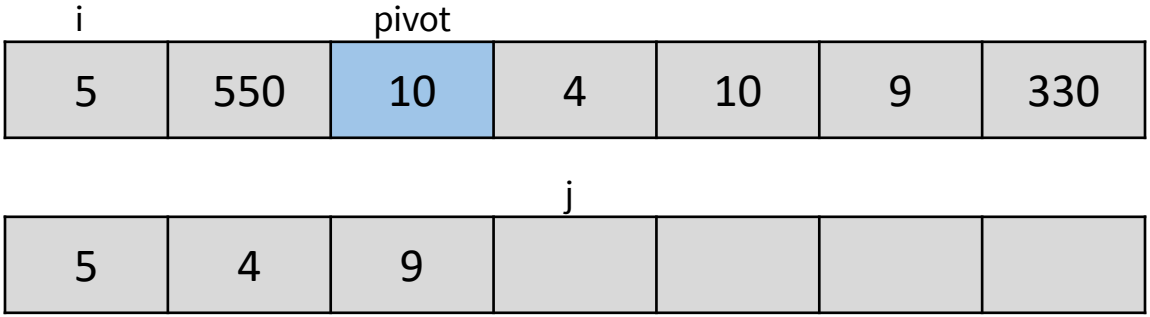

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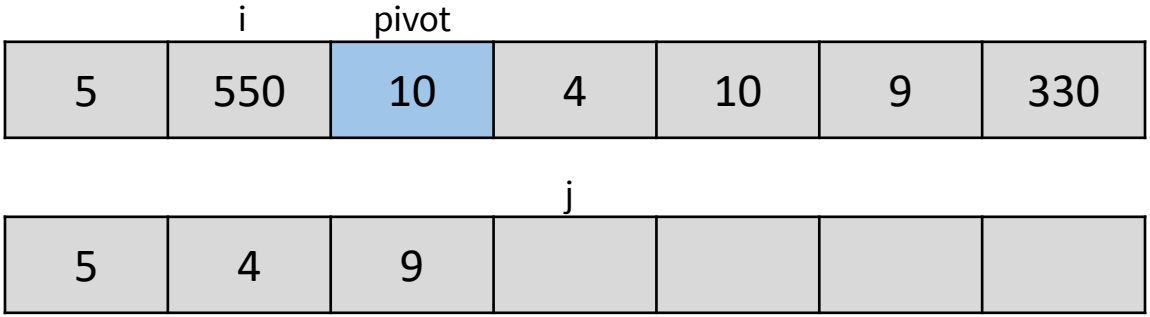

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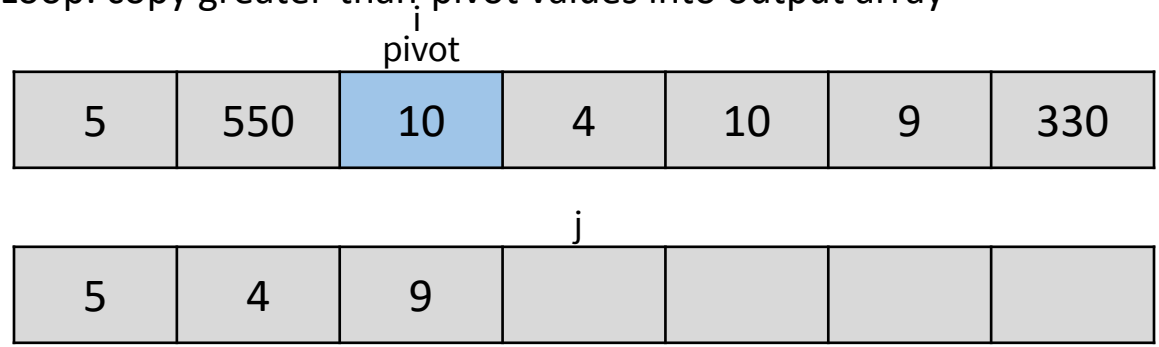

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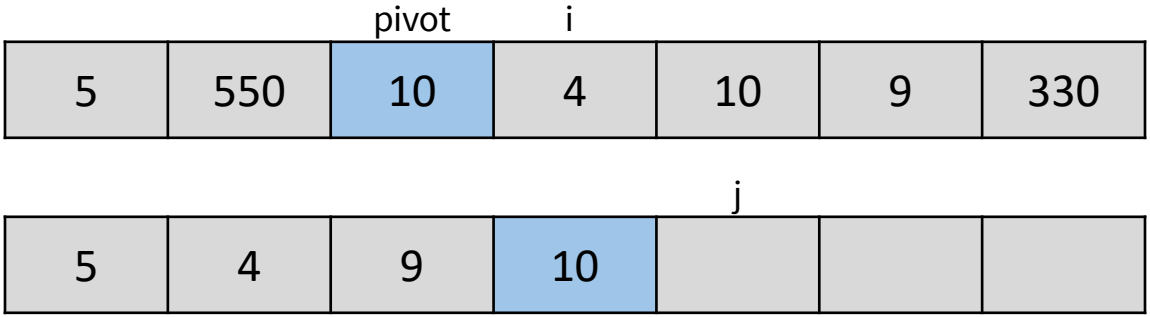

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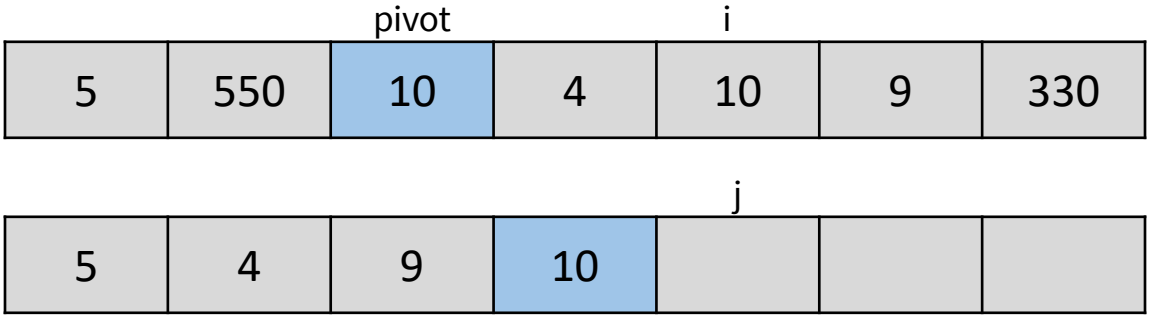

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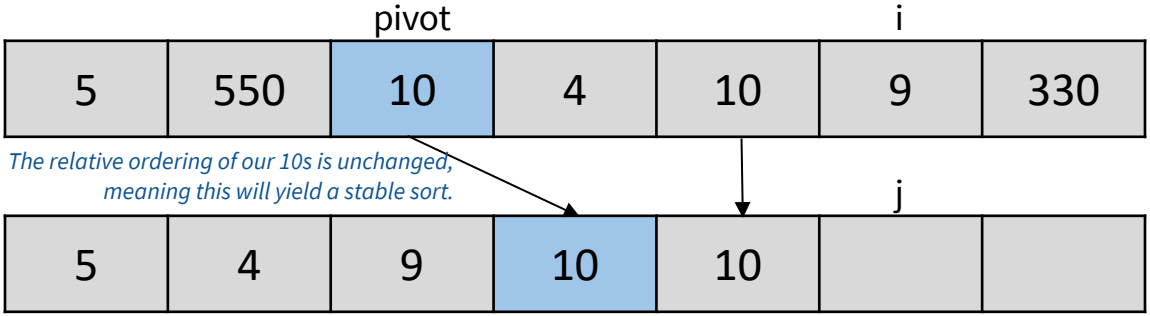

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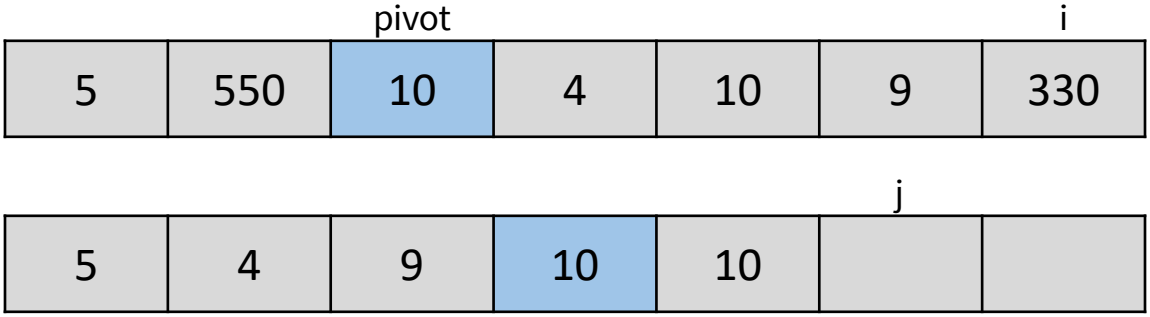

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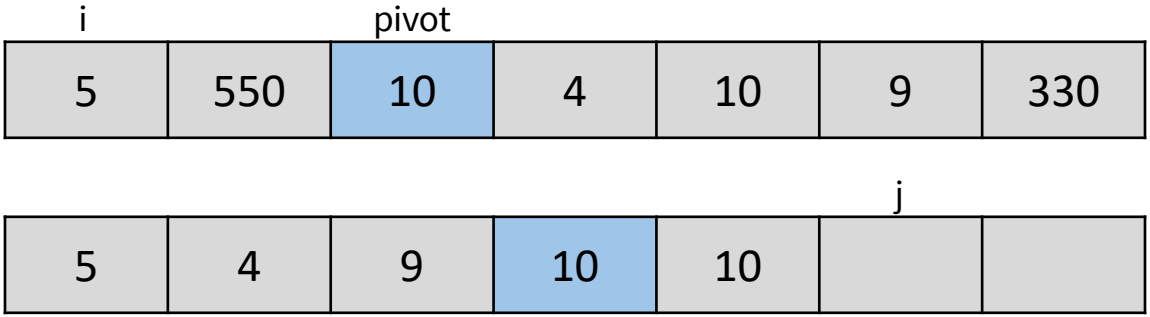

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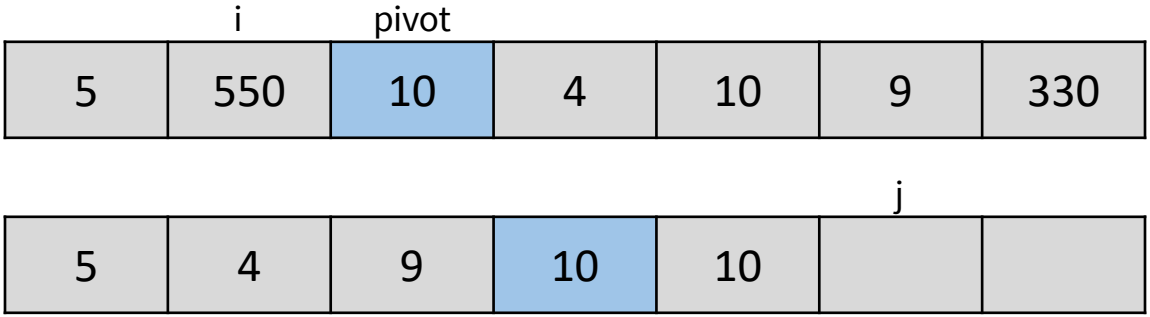

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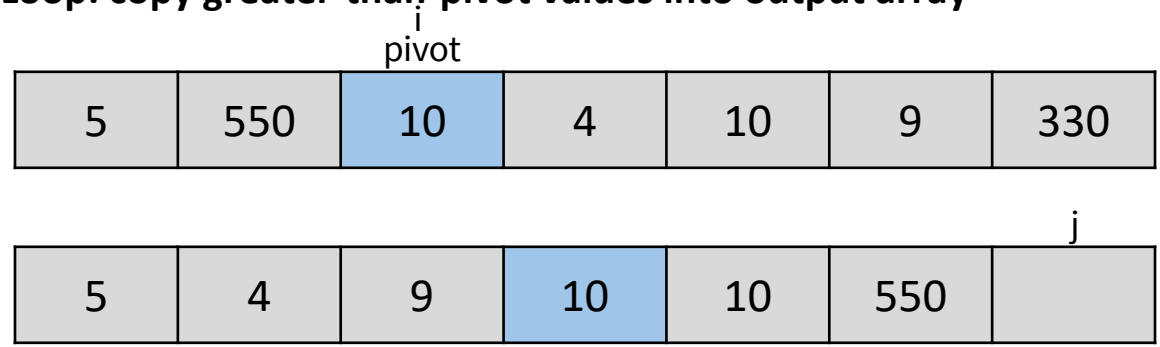

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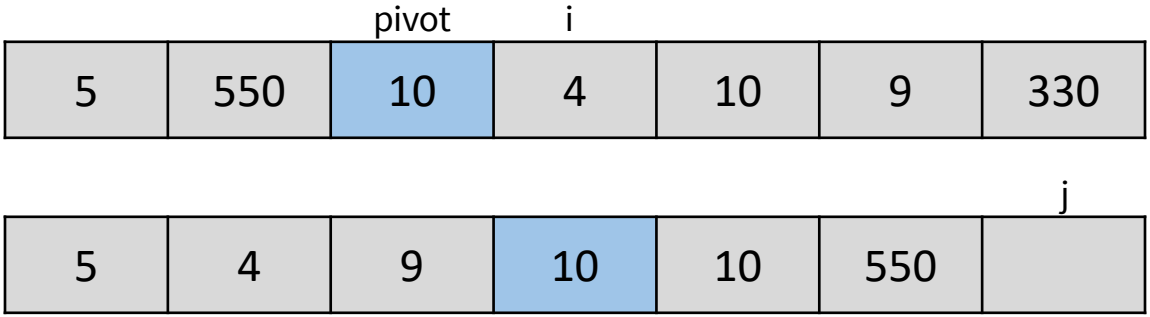

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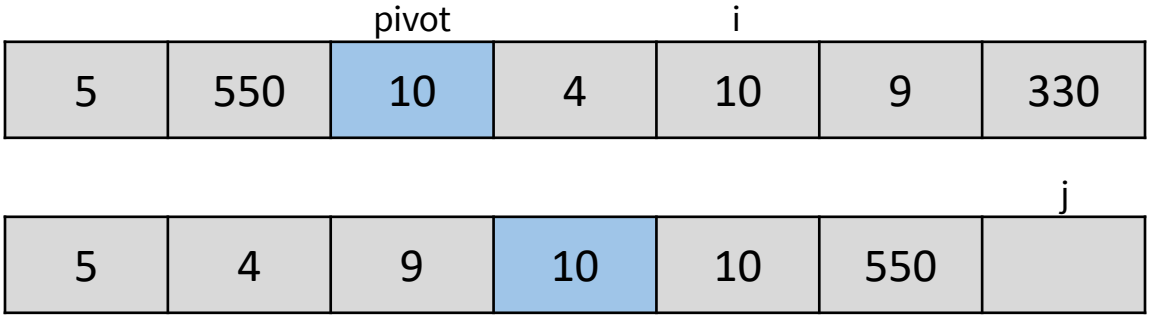

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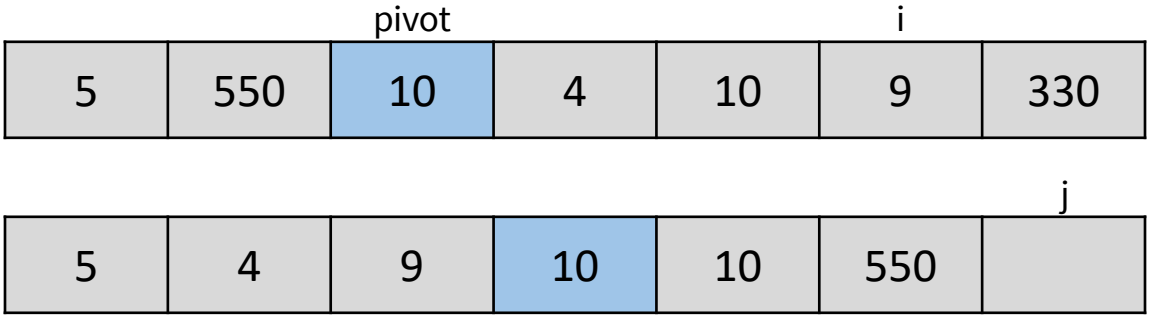

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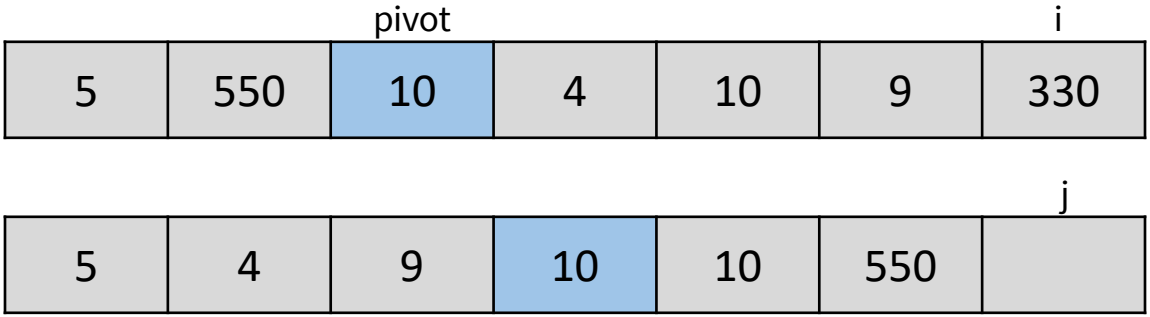

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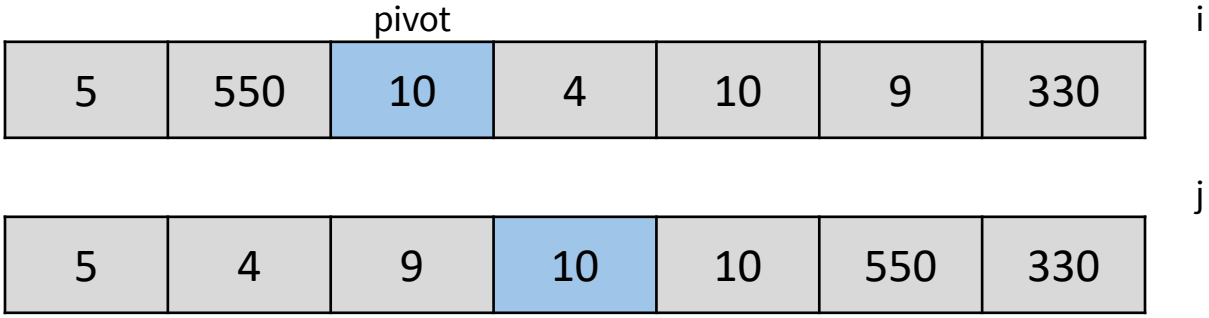


Done partitioning!

Partitioning: Option 2: Hoare Partitioning

- As published in Hoare's original QuickSort paper!

- Intuition:
  - L loves small items (i.e., <pivot) and R loves large items (i.e., >pivot)
  - Walk towards eachother, swapping anything they don't like

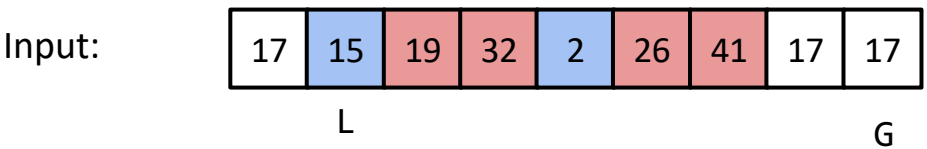
- Algorithm:
  1. Swap pivot with **`arr[lo]`** ("move it out of the way")
  2. Start **`i`** at **`lo+1`**, and **`j`** at **`hi-1`**
  3. Move **`j`** rightward until we hit value **`<pivot`** ("belongs on left")
  4. Move **`i`** leftward until we hit value **`>pivot`** ("belongs on right")
  5. Swap **`arr[i]`** and **`arr[j]`**
  6. When they meet, swap **`arr[0]`** and **`arr[i]`** ("put pivot in correct place")

```
while (i < j)
  if (arr[j] > pivot) j--;
  else if (arr[i] <= pivot) i++;
  else swap(arr[i], arr[j])
```

Hoare Partitioning

Create L and G pointers at left and right ends.

- L pointer is a friend to small items, and hates large or equal items.
- G pointer is a friend to large items, and hates small or equal items.
- Walk pointers towards each other, stopping on a hated item.
  - When both pointers have stopped, swap and move pointers by one.
- When pointers cross, you are done.



Demo Credit: Hannah Tang

Hoare Partitioning

Create L and G pointers at left and right ends.

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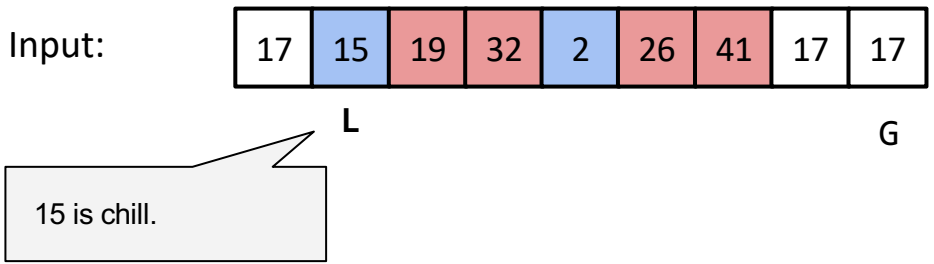

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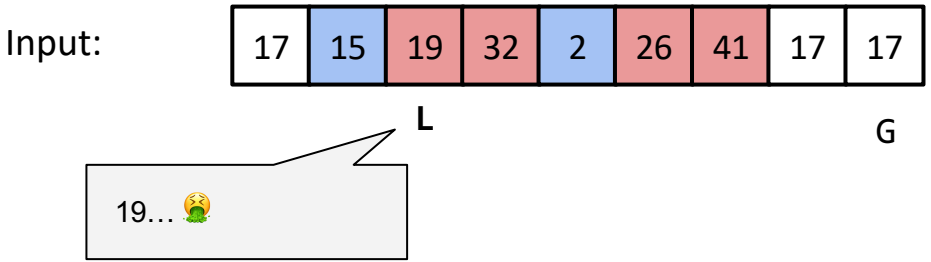

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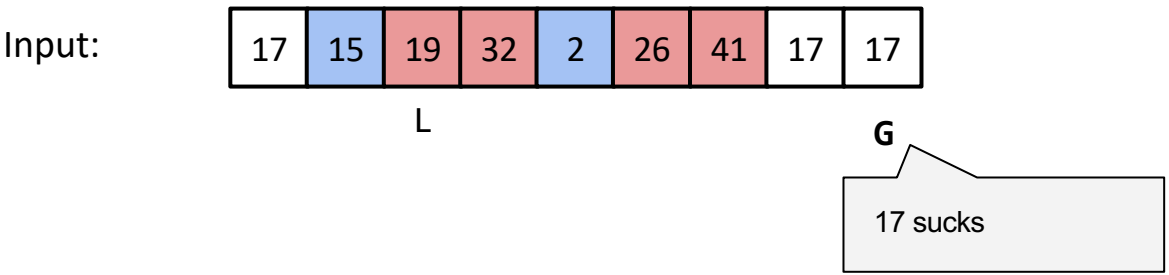

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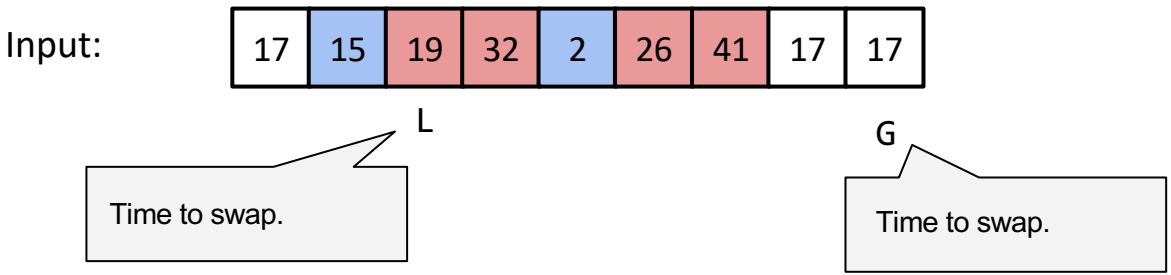

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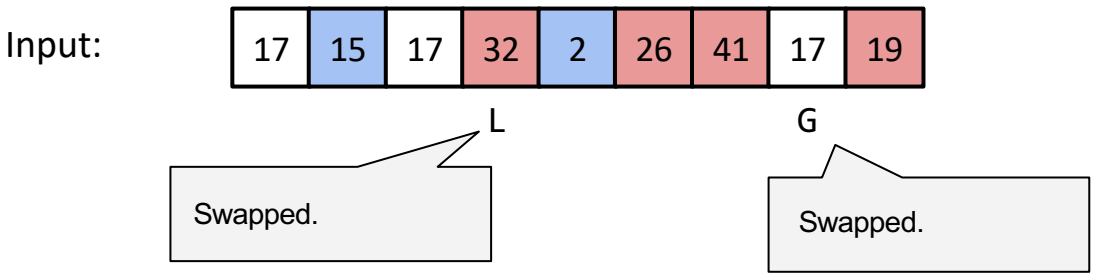

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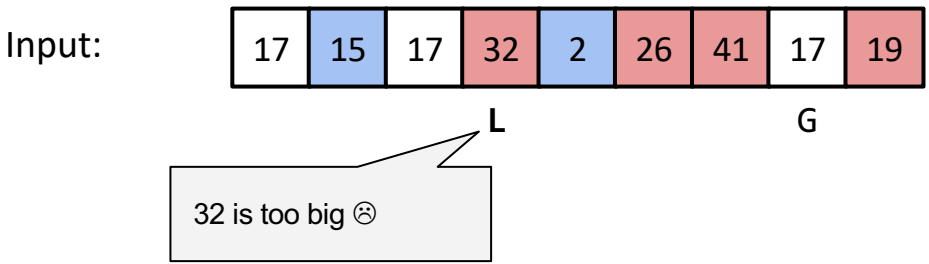

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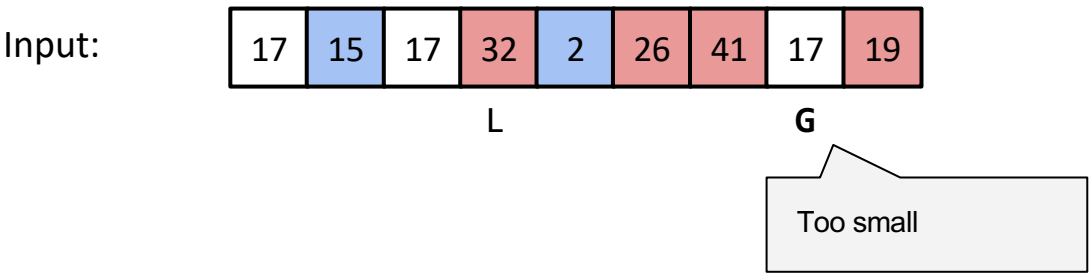

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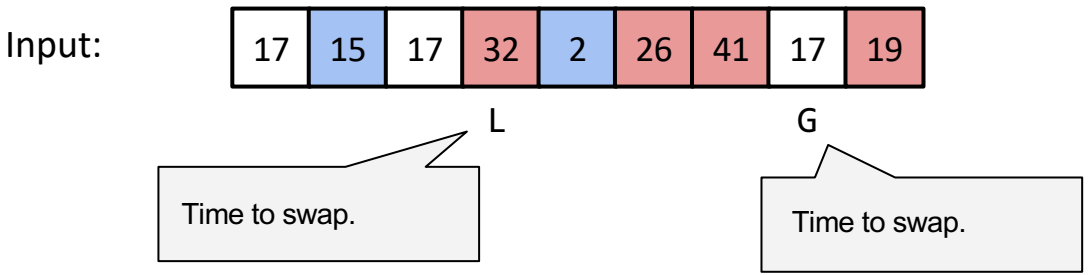

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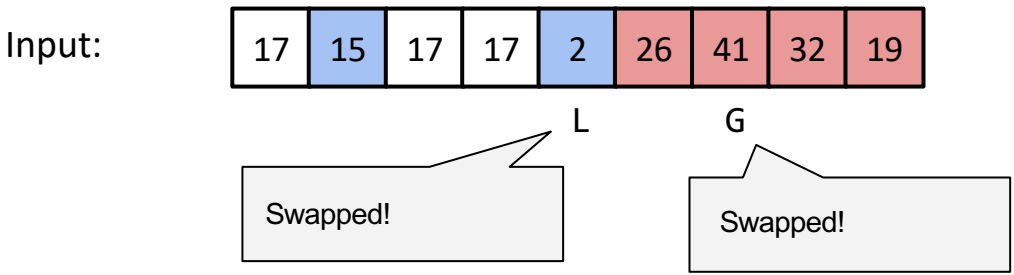

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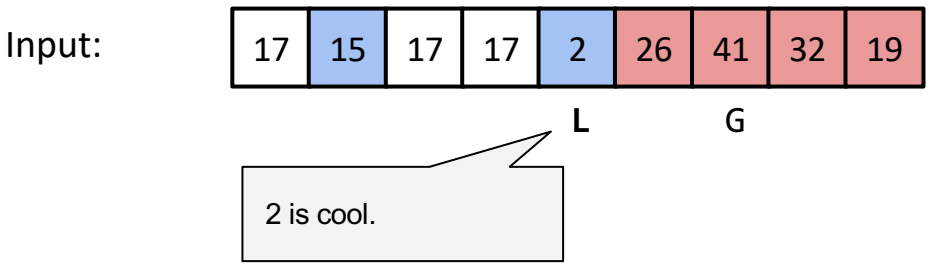

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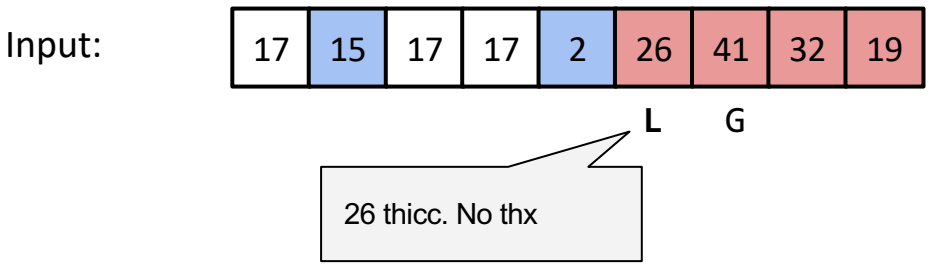

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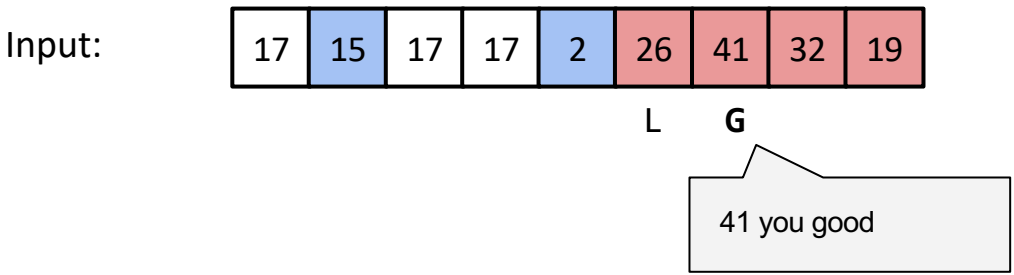

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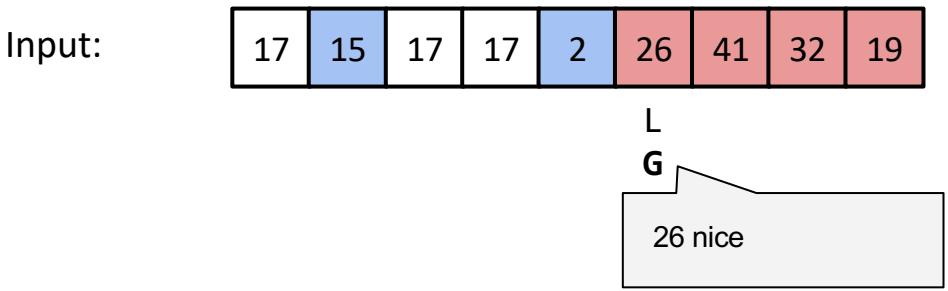

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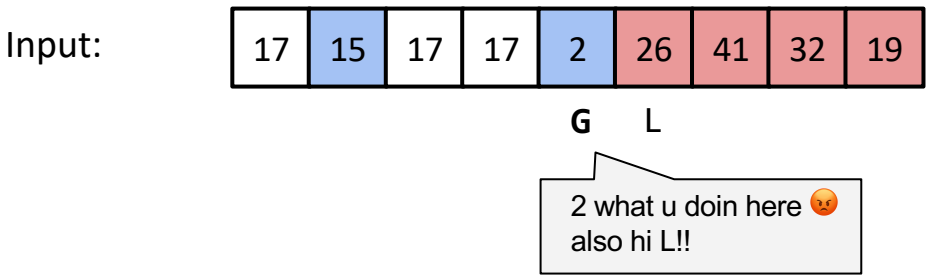

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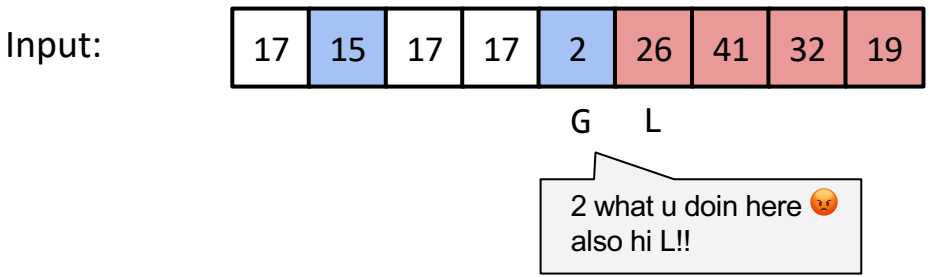

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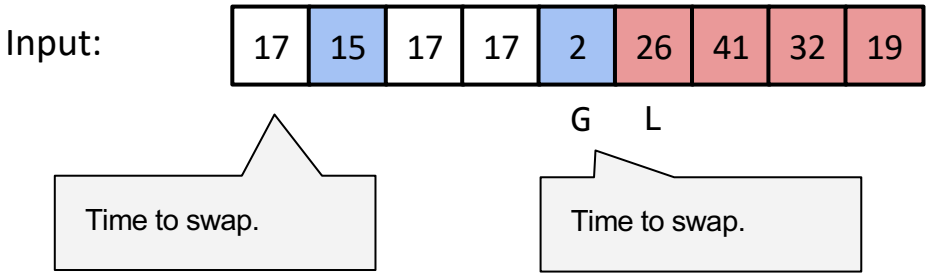

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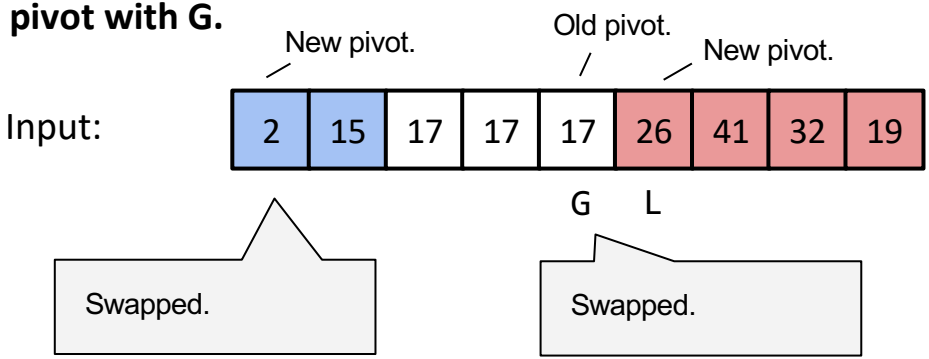


- Partition the following array using Hoare's partitioning algorithm
  - The pivot, 5, has already been moved to the front
  - Sort only by the numbers (eg, 1); the extra letter (eg, 1a) is to help you determine stability

5	1a	6	9	3	7	2	4a	4b	1b

Swap pivot with `arr[lo]` ("move it out of the way")
Start `i` at `lo+1`, and `j` at `hi-1`
  Move `j` rightward until we hit value `<pivot` ("belongs on left")
  Move `i` leftward until we hit value `>pivot` ("belongs on right")
  Swap `arr[i]` and `arr[j]`
When they meet, swap `arr[0]` and `arr[i]` ("put pivot in correct place")

```
while (i < j)
  if (arr[j] > pivot) j--;
  else if (arr[i] <= pivot) i++;
  else swap(arr[i], arr[j])
```

Notes on Hoare Partitioning

- Unstable ☹
- Good constants: single-pass and in-place ☺

Partitioning: Option 3: Three-Way

- Pick *two* pivots
  - Same intuition as median-of-three: it's hard to pick multiple bad pivots simultaneously
- Like Hoare Partitioning, use two pointers walking to the middle
  - But split array into three pieces, not two
  - Good constants: single-pass and in-place; $\log_3 N$ vs $\log_2 N$ ☺
  - Still an unstable sort ☹
- Used in Java's Arrays.sort(), Python's unstable sort, etc
  - Basically the de-facto partition algorithm circa 2020
- Won't be going through this version

QuickSort Steps

1. Pick the pivot value(s)
   - Any choice is correct; data will end up sorted
   - For efficiency, these value(s) ought to approximate the median

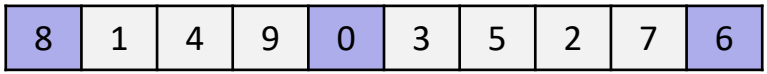
2. Partition all the values into:
   a. The values less than the pivot(s)
   b. The pivot(s)
   c. The values greater than the pivot(s)
   d. … In linear time? In-place? Stably?

3. Recursively QuickSort(A) and QuickSort(C)

✨TA-DA!✨

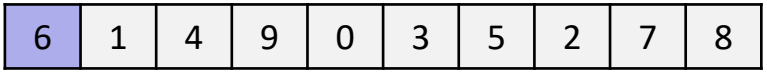
QuickSort: End-to-end Example (1 of 3)

1. Pick pivot (we'll use median-of-3)

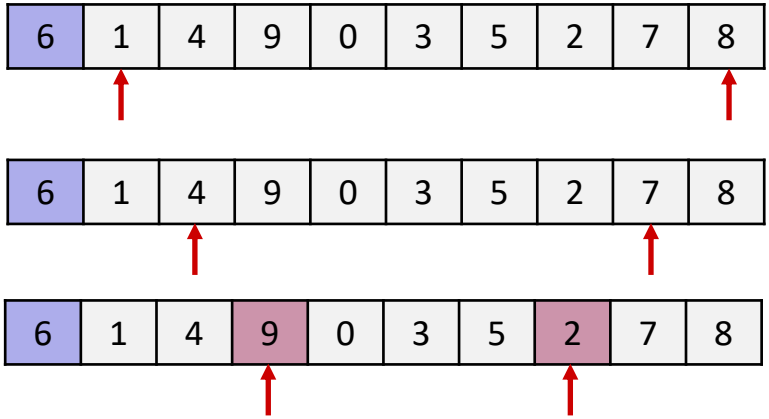


2. Partition (we'll use Hoare Partitioning)

- Move pivot to the beginning position

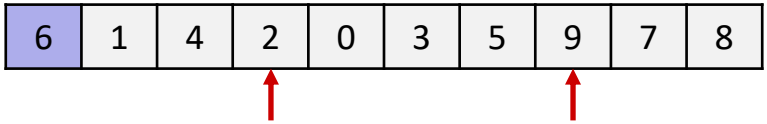


- Let **`lo`** = 1 and **`hi`** = 9; loop until we find "swappable" values

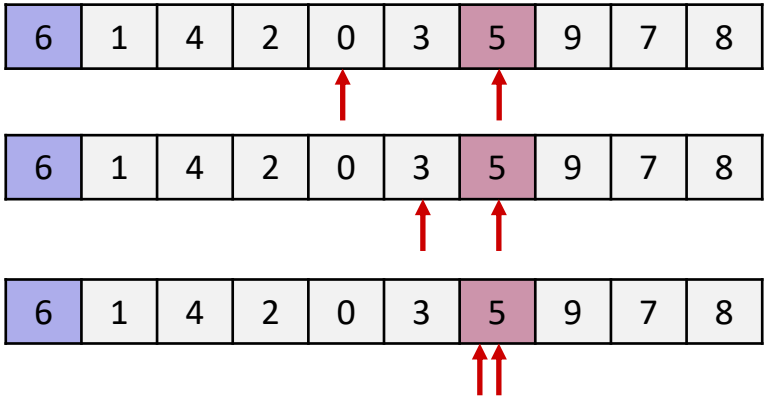


QuickSort: End-to-end Example (2 of 3)

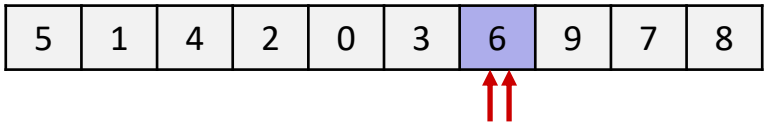
- Swap `lo` = 3 and `hi` = 7



- Keep looping

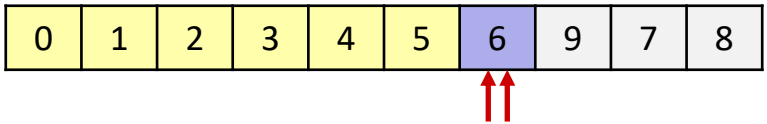


- Done! Swap pivot into position

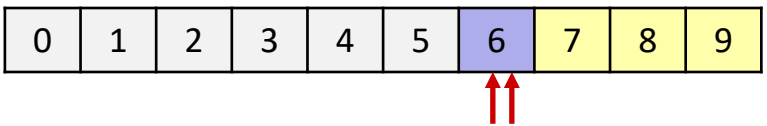


QuickSort: End-to-end Example (3 of 3)

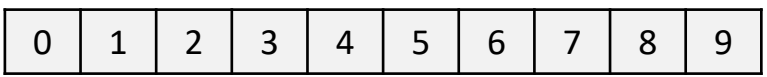
3. Recursively sort left (0 to `lo-1` = 5)



4. Recursively sort right (`hi`+1 = 7 to `arr.length`)



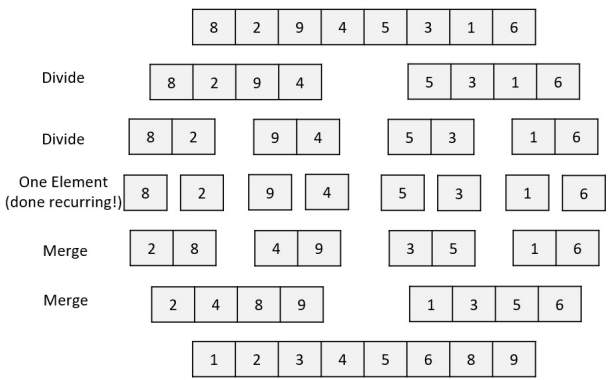
5. Sorted!



QuickSort Optimization: Cutoffs (1 of 2)

- For small *n*, recursion tends to cost more than a quadratic sort
  - Remember: asymptotic complexity applies to large *n*
  - Recursive calls add overhead (which "isn't worth it" for small *n*)

- Recursive calls for small *n* are the most common ("leaf calls")
  - Calls for small *n* are the vast majority of the recursive calls!
  - Image below is for mergesort, but shows this well



QuickSort Optimization: Cutoffs (2 of 2)

- So, switch algorithms for subproblems below a **cutoff** size
  - Eg, Java 12 uses InsertionSort for primitive types when $n < 47$

```
void quickSort(int[] arr, int lo, int hi) {
  if (hi - lo < CUTOFF)
    insertionSort(arr, lo, hi);
  else
    …
}
```

- Switching algorithms after a cutoff is a common technique!
  - E.g. *parallel* algorithms switch to *sequential* after a certain cutoff
  - E.g. MergeSort also uses cutoffs to switch to InsertionSort

- Does not affect asymptotic complexity, just the constants

QuickSort vs MergeSort (1 of 2)

Different algorithms, same problem

- MergeSort:
  - Execution
    - Does its work "on the way up"
      - i.e., in the merge, after the recursive call returns
    - Uses its auxiliary space very effectively:
      - Works well on linked lists
      - Linear merges minimize disk accesses
  - Time: always O(n log n)

- QuickSort:
  - Execution:
    - Does its work "on the way down"
      - i.e., in the partition, before the recursive call
    - Doesn't need auxiliary space
  - Runtime: O(n log n) in best and randomized cases ☺
    - But $O(n^2)$ worst-case ☹

QuickSort vs MergeSort (2 of 2)

- Asymptotic Runtime:
  - QuickSort is O(n log n) in best and randomized cases, but $O(n^2)$ worst-case
  - MergeSort is always O(n log n)

- Constants Matter!
  - QuickSort does fewer copies and more comparisons, so it depends on the relative cost of these two operations
  - Typically, cost of copies is higher so QuickSort really *is* the "quickest"

Lecture Outline

- QuickSort
  - Comparison with MergeSort
- **Comparison Sort Lower Bound**
- Non-Comparison Sorts
  - BucketSort
  - RadixSort
- Sorting Wrapup

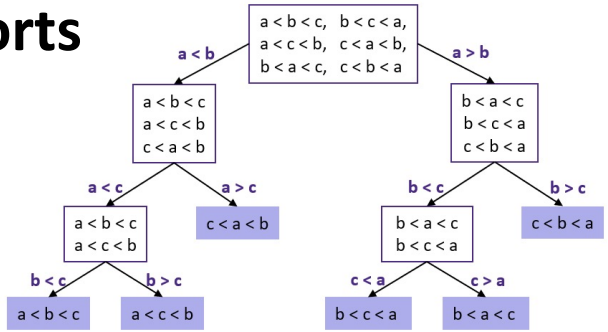
A Different View of Sorting

- Assume we have *n* elements, none are equal (ie, no duplicates)
  - ***n! permutations*** (possible orderings) of the elements. For *n*=3

a < b < c	a < c < b	b < a < c	b < c < a	c < a < b	c < b < a

- Assume an "OptimalSort"
  - Instead of describing how it ***works***, we'll describe what it ***knows and when it knows it***
  - Starts "knowing nothing"; "anything is possible"
  - Each binary: `a < b` or `b < a` comparison gains information, eliminating some possibilities
    - Each comparison eliminates (at most) half of remaining possibilities
  - In the end, narrows down to a single possibility

Representing Comparison Sorts

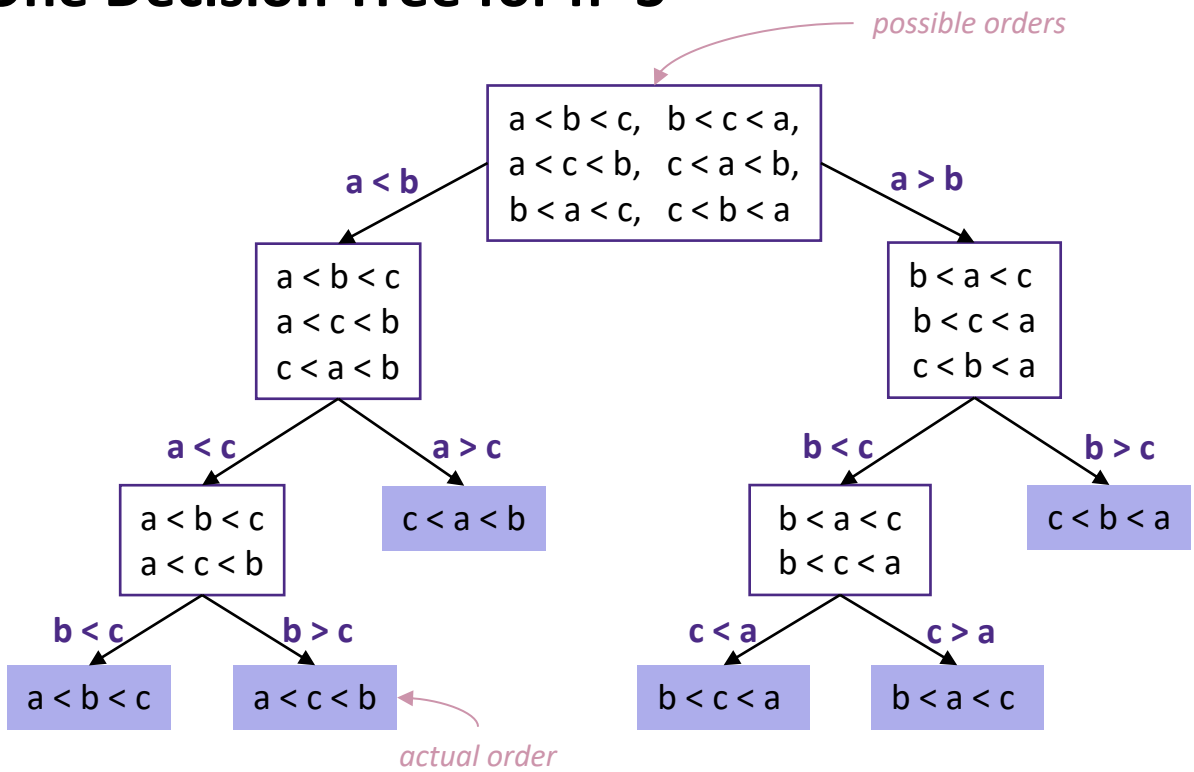


- Let's represent these binary comparisons as a binary tree!

- Called a *Decision Tree*
  - Nodes contain "set of remaining possible orderings"
  - The root contains all possible orderings; anything is possible
  - The leaves contain exactly one specific ordering
  - Edges are "answers from a comparison"

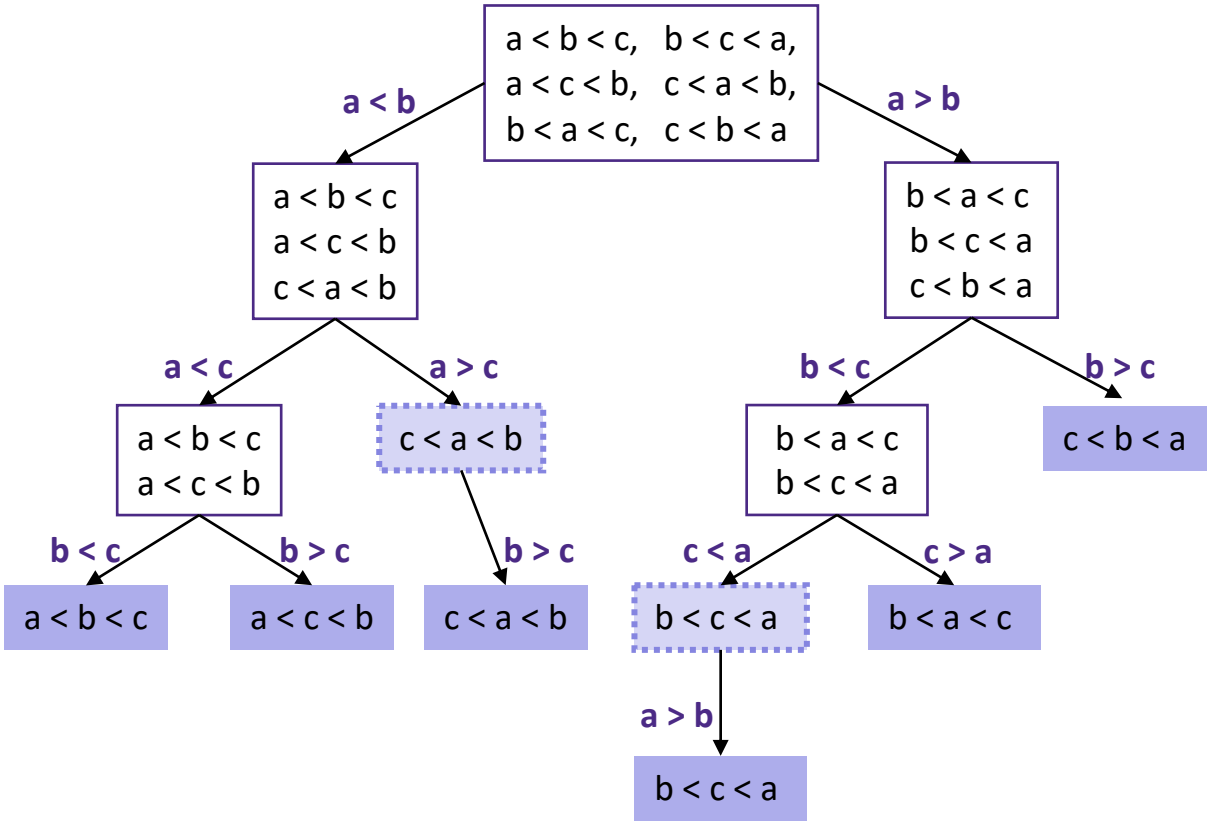
We are not actually building the tree; it's what our proof uses to represent "the most the algorithm could know so far"

One Decision Tree for n=3



- The leaves contain all the possible orderings of a, b, c
- A different algorithm would lead to a different tree

Another Decision Tree for n=3



What the Decision Tree Tells Us

- Because any order is possible, any algorithm needs to ask enough questions to produce all *n*! leaves (ie, orderings)
  - Each answer/ordering may lead to a different leaf
  - ***So the binary tree must be big enough to have n! leaves***

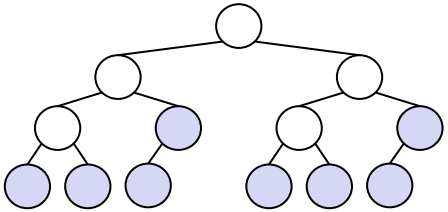
- Running *any* algorithm on *any* input will *at best* correspond to a root-to-leaf path in *some* decision tree with *n*! leaves
  - Path length is the number of comparison operations needed
  - ***So no algorithm can have worst-case running time better than the height of a tree with n! leaves***
    - Because the worst-case number-of-comparisons for an algorithm is an input that yields to a longest path in algorithm's decision tree

How did we get n!?
This is a 312 question. We have n variables and we wish to know how many different ways they can be ordered. In other words, want the number of permutations of size n. A permutation is the choice of r things from a set of n things without replacement and where the order matters:
${}^nP_r = (n!) / (n-r)!$
In our case, r = n because we want the choice of all n things from the set of n things, so: ${}^nP_n = (n!) / (n-n)! = (n!) / 0! = n! / 1 = n!$

Lower Bound on Height (1 of 2)

- A binary tree of height h has at most how many leaves?
  L ≤ _____
- A binary tree with L leaves has height at least:
  h ≥ _____
- The decision tree has how many leaves: _____
- So the decision tree has height:
  h ≥ _____

Lower Bound on Height (2 of 2)



- The height of a binary tree with L leaves is at least $\log_2 L$

- So the height of our decision tree, h:

$h \geq \log_2 (n!)$	property of binary trees
$= \log_2 (n*(n-1)*(n-2)\ldots(2)(1))$	definition of factorial
$= \log_2 n + \log_2 (n-1) + \ldots + \log_2 1$	property of logarithms
$\geq \log_2 n + \log_2 (n-1) + \ldots + \log_2 (n/2)$	keep first n/2 terms
$\geq (n/2) \log_2 (n/2)$	each of the n/2 terms is $\geq$ log2 (n/2)
$= (n/2)(\log_2 n - \log_2 2)$	property of logarithms
$= (1/2)n \log_2 n - (1/2)n$	arithmetic
$\in \Omega (n \log n)$	

So… No comparison based sort can be faster than nlogn.

Lecture Outline

- QuickSort
  - Comparison with MergeSort
- Comparison Sort Lower Bound
- **Non-Comparison Sorts**
  - **BucketSort**
  - RadixSort
- Sorting Wrapup

BucketSort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and *K* (or any small range),
  - Create an array of size *K,* put each element in its **bucket (a.ka. bin)**
  - If data is only integers, can store *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

`count` array	
1	
2	
3	
4	
5	

- Example:

  K=5

  Input: (5,1,3,4,3,2,1,1,5,4,5)

  Output: 1,1,1,2,3,3,4,4,5,5,5

Analyzing BucketSort

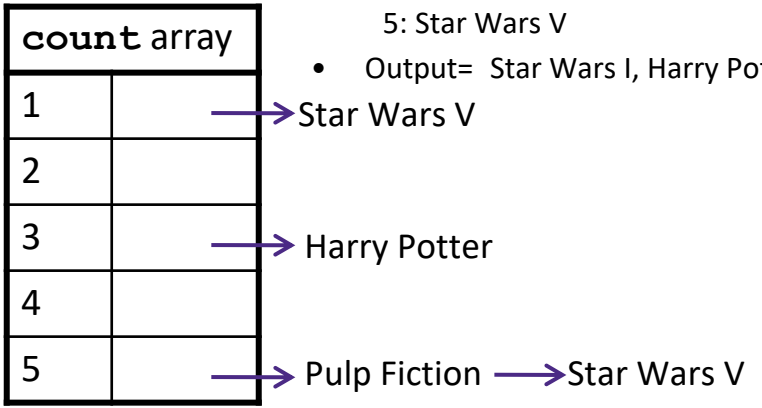
- Overall: $O(n+K)$
  - Linear in n, but also linear in K
  - $\Omega(n \log n)$ doesn't apply because **this is not a comparison sort**

- Good when range, K, is smaller (or not much larger) than n
  - We don't spend time doing lots of comparisons of duplicates!

- Bad when K is much larger than n
  - Wasted space; wasted time during final linear $O(K)$ pass

BucketSort with Data

- Most real lists aren't just #'s; we have data too
  - Make each bucket is a list (say, linked list)
  - To add to a bucket, place at end O(1) (keep pointer to last element)
    - Example: movie ratings (1=bad, … 5=excellent)
    - Input=
      5: Pulp Fiction
      3: Harry Potter movies
      1: Star Wars I
      5: Star Wars V
    - Output= Star Wars I, Harry Potter movies, Pulp Fiction, Star Wars V



Lecture Outline

- QuickSort
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RadixSort

- Radix = "the base of a number system"
  - Examples will use 10 because we are used to that
  - Implementations may use larger numbers
    - For example, for ASCII strings, might use 128
- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with *least* significant digit, sort with Bucket Sort
    - Keeping sort *stable*
  - Do one pass per digit
- **Invariant**: After *k* passes, the last *k* digits are sorted
- Aside: Origins go back to the 1890 U.S. census

RadixSort: Example (1 of 6)

0	1	2	3	4	5	6	7	8	9

Input:

333
143
591
65
332
491

First pass:

1. BucketSort by ones digit
2. Iterate thru and collect into a list

- List is sorted by first digit

RadixSort: Example (2 of 6)

0	1	2	3	4	5	6	7	8	9
	591 491	332	333 143		65				

Input:

333
143
591
65
332
491

First pass:

1. BucketSort by ones digit
2. Iterate thru and collect into a list

- List is sorted by first digit

Order is now:

591
491
332
333
143
65

0	1	2	3	4	5	6	7	8	9
	591 491	332	333 143		65				

Input:

333
143
591
65
332
491

Second pass:

1. BucketSort by **tens** digit, **stably**

Order is now:

0	1	2	3	4	5	6	7	8	9
	591 491	332	333 143		65				

0	1	2	3	4	5	6	7	8	9
			332 333	143		65			591 491

Input:

333
143
591
65
332
491

Second pass:

1. BucketSort by **tens** digit, **stably**

Notice: if we chop off the 100's place, these are now sorted

Order is now:

332
333
143
65
591
491

0	1	2	3	4	5	6	7	8	9
			332 333	143		65			591 491

0	1	2	3	4	5	6	7	8	9

Input:

333
143
591
65
332
491

Third pass:

1. BucketSort by **<u>hundreds</u>** digit, stably

Order is now:

0	1	2	3	4	5	6	7	8	9
			332 333	143		65			591 491

0	1	2	3	4	5	6	7	8	9
65	143		332 333						491 591

Input:

333
143
591
65
332
491

Third pass:

1. BucketSort by **hundreds** digit, stably

Only 3 digits; we're done!

Order is now:

65
143
332
333
491
591

Analysis of Radix Sort

- Performance depends on:
  - Input size: *n*
  - Number of buckets = Radix: *K*
    - e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
  - Number of passes = "Digits": *P*
    - e.g. Ages of people: 3; Phone #: 10; Person's name: ?

- Work per pass is 1 BucketSort: _____
  - *Each pass is a BucketSort!*
- Total work is _____
  - *We do 'P' passes, each of which is a BucketSort!*

Comparison to Comparison Sorts

- Compared to comparison sorts, radix sorts are sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Approximate run-time: 15*(52 + *n*)
  - This is less than *n* log n only if *n* > 33,000
  - Of course, cross-over point depends on constant factors of the implementations plus *P* and *B*
    - And radix sort can have poor locality properties

- Not really practical for many classes of keys
  - Strings: Lots of buckets

Lecture Outline

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Features of Sorting Algorithms

- **In-place**
  - Sorted items occupy the same space as the original items. (No copying required, only O(1) extra space if any.)

- **Stable**
  - Items in input with the same value end up in the same order as when they began.

Sorting: Summary (1 of 3)

- Simple $O(n^2)$ sorts can be fastest for small n
  - SelectionSort, InsertionSort:
    - The latter is linear for mostly-sorted!
    - Good for "below a cut-off" to help divide-and-conquer sorts

- "Fancy" $O(n \log n)$ sorts
  - HeapSort: not parallelizable
  - MergeSort: works as external sort
  - QuickSort: $O(n^2)$ in worst-case; cost of comparisons/copies often makes it fastest

Sorting : Summary (2 of 3)

- $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons

- Non-comparison sorts
  - Bucket sort good for small number of key values
  - Radix sort uses fewer buckets and more phases

- Best way to sort? It depends!

Sorting: Summary (3 of 3)

	Best-Case	Worst-Case	Randomized Case	In-Place?	Stable?	Notes
InsertionSort	$\Theta(N)$	$\Theta(N^2)$	$\Theta(N^2)$	Yes	Yes	Fastest for small or partially-sorted input
SelectionSort	$\Theta(N^2)$	$\Theta(N^2)$	$\Theta(N^2)$	Yes	No	
In-Place HeapSort	$\Theta(N)$	$\Theta(N \log N)$	$\Theta(N \log N)$	Yes	No	Slow in practice
MergeSort	$\Theta(N \log N)$	$\Theta(N \log N)$	$\Theta(N \log N)$	No	Yes	Fastest stable sort
QuickSort *(1st-element pivot + 3-pass partition)*	$\Theta(N \log N)$	$\Theta(N^2)$	$\Theta(N \log N)$	No	Yes	>=2x slower than MergeSort
QuickSort *(Median-of-three pivot + Hoare partition + cutoffs)*	$\Omega(N)$	$O(N^2)$	$\Theta(N \log N)$	Yes	No	Fastest comparison sort
BucketSort	$\Theta(N+K)$	$\Theta(N+K)$	$\Theta(N+K)$	No	Yes	
RadixSort	$\Theta(P(K+N))$	$\Theta(P(K+N))$	$\Theta(P(K+N))$	No	Yes	