Comparison Sorting Algorithms

CSE 332 Summer 2021

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Announcements

- P2 checkpoint 2 tomorrow – google form will release this afternoon
  - QuickSort should be the only thing we haven't covered in lecture yet
- Exercises 7 & 8 out!
  - Ex7 Canvas Groups: Join group and post in *group discussion*
  - Reflection questions subject to change before Wednesday
- Midterm out Wednesday!
- P1 Grades will release

Lecture Outline

- **Intro to Sorting**
- Simple Sorts
  - InsertionSort
  - SelectionSort
- Fancier Sorts
  - HeapSort
  - “Data Structure Sorts”
- Divide & Conquer Sorts
  - MergeSort
  - QuickSort

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- When you play cards, how do you order them in your hand?

- Why do you think we learning about sorting in this class?

Introduction to Sorting (1 of 2)

- Stacks, queues, priority queues, and dictionaries/sets all provide one element at a time

- But often we want "all the items" in some order
  - Alphabetical list of people
  - Population list of countries
  - Search engine results by relevance

- Different sorting algorithms have different asymptotic and constant-factor trade-offs
  - Knowing one way to sort just isn't enough; no single "best sort"
  - **Sorting is an excellent case-study in making trade-offs!**

Introduction to Sorting (2 of 2)

- *Preprocessing* (e.g. sorting) data to make subsequent operations faster is a general technique in computing!
  - Example: Sort the items so that you can:
    - Find the **k**th largest in constant time for any **k**
    - Perform binary search to find an item in logarithmic time
  - Whether preprocessing is beneficial depends on
    - How often the items will change
    - How many items there are

- Preprocessing's benefits depend on how often the items will change and how many items there are
  - **Sorting is an excellent case-study in making trade-offs!**

Comparison Sorting: Definition

- Problem: We have *n* comparable items in an array, and we want to rearrange them to be in increasing order

- Input:
  - An array A of (key, value) pairs
  - A comparison function (consistent and total)
    - Given keys a & b, what is their relative ordering? <, =, >?
    - Ex: keys that implement Comparable or have a Comparator

- Output/Side-Effect:
  - Reorganize the elements of A such that for any index i and j,

    if $i < j$ then $A[i] \leq A[j]$
  - [Usually unspoken] **A** must have all the same items it started with
  - Could also sort in reverse order, of course

Comparison Sort: Variations (1 of 2)

1. Maybe elements are in a linked list
   - Could convert to array and back in linear time, but some algorithms can still "work" on linked lists
2. Maybe if there are ties we should preserve the original ordering
   - Sorts that do this naturally are called **stable sorts**
3. Maybe we must not use more than O(1) "auxiliary space"
   - These are called **in-place sorts**
   - Not allowed to allocate memory proportional to input (i.e., O(n)), but can allocate O(1) # of variables
   - Work is done by swapping around in the array

Comparison Sort: Variations (2 of 2)

4. Maybe we can do more with elements than just compare
   - Comparison sorts assume a binary ‘compare’ operator
   - In special cases we can sometimes get faster algorithms

5. Maybe we have too many items to fit in memory
   - Use an **external sorting** algorithm

Big Picture of Comparison-Based Sorts

- Simple algorithms: O(n^2)
  - InsertionSort, SelectionSort
  - *BubbleSort, ShellSort*

- Fancier algorithms: O(n log n)
  - HeapSort, MergeSort, QuickSort (randomized)

- Comparison-based sorting's lower bound: Ω(n log n)

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  - MergeSort
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InsertionSort

- Idea: At step **k**, insert the **k**th element in the correct position
  - Sort first two elements
  - Now insert 3rd element in order
  - …
- Loop invariant ("when loop index is **i**"):
  - First **i** elements are in sorted order
- Time:

  Best-case: ________ Worst-case: ________
- Characteristics:

  Stable: _______ In-place: _______

SelectionSort

- Idea: At step **k**, select the smallest elt and put it at **k**th position
  - Find smallest element, put it 1st
  - Find next smallest element, put it 2nd
  - …
- Loop invariant ("when loop index is **i**"):
  - First **i** elements are the **i** smallest elements in sorted order
- Time:

  Best-case: ________ Worst-case: ________
- Characteristics:

  Stable: _______ In-place: _______

InsertionSort vs. SelectionSort (1 of 2)

Different algorithms, same problem

- InsertionSort
  - Loop invariant:
    - First `i` elements are in sorted order
  - Characteristics:
    - Stable: yes
  - Time:
    - Worst-case: $O(n^2)$
    - "Average" case: $O(n^2)$

- SelectionSort
  - Loop invariant:
    - First `i` elements are *the `i` smallest elements* in sorted order
  - Characteristics:
    - Stable: no
  - Time:
    - Worst-case: $O(n^2)$
    - "Average" case: $O(n^2)$

InsertionSort vs. SelectionSort (2 of 2)

- InsertionSort has better best-case complexity
  - Best case is when input is "mostly sorted"

- Different constants
  - InsertionSort may do well on small arrays (empirically: N < ~15)
  - Java's built-in sort prefers InsertionSort for arrays <47 items

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- **Fancier Sorts**
  - HeapSort
  - "Data Structure Sorts"

- Divide & Conquer Sorts
  - MergeSort
  - QuickSort

Naïve HeapSort

- Idea: Put everything in a **MIN** heap; successively `deleteMin`
  - `add()` all elements into heap – OR – better yet, use `buildHeap`
  - ```
    for(i=0; i < arr.length; i++)
         arr[i] = deleteMin();
    ```

- Loop invariant ("when loop index is **`i`**"):
  - First **`i`** elements are *the **`i`** smallest elements* in sorted order

- Time: ______

- Characteristics:

  Stable: ______ In-place: ______
    ```

In-place HeapSort

- Idea: Put everything in a **MAX** heap ; successively `deleteMax`
  - insert each `arr[i]` –OR – better yet, use `buildHeap`
  - ```
    for(i=0; i < arr.length; i++)
        arr[arr.length - i] = deleteMax();
    ```
- Loop invariant ("when loop index is **`i`**"): same as naïve version
- Time: ______
- Characteristics:

  Stable: ______ In-place: ______
    ```

Aside: “AVLSort” and “DataStructureSort”

- We can also use a balanced tree to:
  - `add` each element: total time $O(n$ `log` $n)$
  - Do an in-order traversal $O(n)$

- But a balanced tree cannot be made in-place, and constants worse than HeapSort
  - Both are $O(n$ `log` $n)$ in worst, best, and average case
  - Neither sorts parallelizes well

- Don’t even think about trying to sort with a hash table …

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- Why might I care about a sort being stable or in place? Would having these two qualities ever be worth the tradeoff of having a slower algorithm?

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- **Divide & Conquer Sorts**
  - MergeSort
  - QuickSort

Technique: Divide and Conquer

- Very important technique in algorithm design!
  1. Divide problem into smaller parts
  2. Solve the parts independently
     - Recursion
     - Or potentially parallelism!
  3. Combine solution of parts to produce overall solution

- Examples:
  - Sort each half of the array, then combine together
  - Split the array into "small part" and "big part", then sort the parts

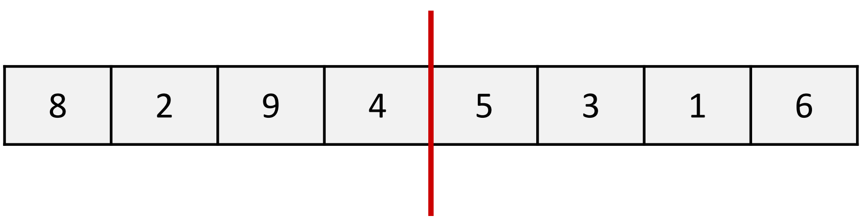
Sorting with Divide and Conquer

- Two great sorting methods are divide-and-conquer!
  - MergeSort:
    - Sort the left half of the elements (recursively)
    - Sort the right half of the elements (recursively)
    - Merge the two sorted halves into a sorted whole
  - QuickSort:
    - Pick a "pivot" element
    - Partition elements into those *less-than* pivot and those *greater-than* pivot
    - Sort the *less-than* elements (recursively)
    - Sort the *greater-than* the elements (recursively)
    - All done! Answer is [*sorted-less-than*] [*pivot*] [*sorted-greater-than*]

MergeSort

- To sort array from position **`lo`** to position **`hi`**:
  - If range is 1 element long, it's sorted! (Base case)
  - Else, split into two halves:
    - "Somehow" sort from **`lo`** to **`(hi+lo)/2`**
    - "Somehow" sort from **`(hi+lo)/2`** to **`hi`**
    - Merge the two halves together

- Merging takes two sorted parts and sorts everything
  - *O*(*n*) time but requires *O*(*n*) auxiliary space…



MergeSort: Merging Example (1 of 10)

- Start with:

arr

8	2	9	4	5	3	1	6

- Return from left and right recursion
  - (pretend it works for now)

arr

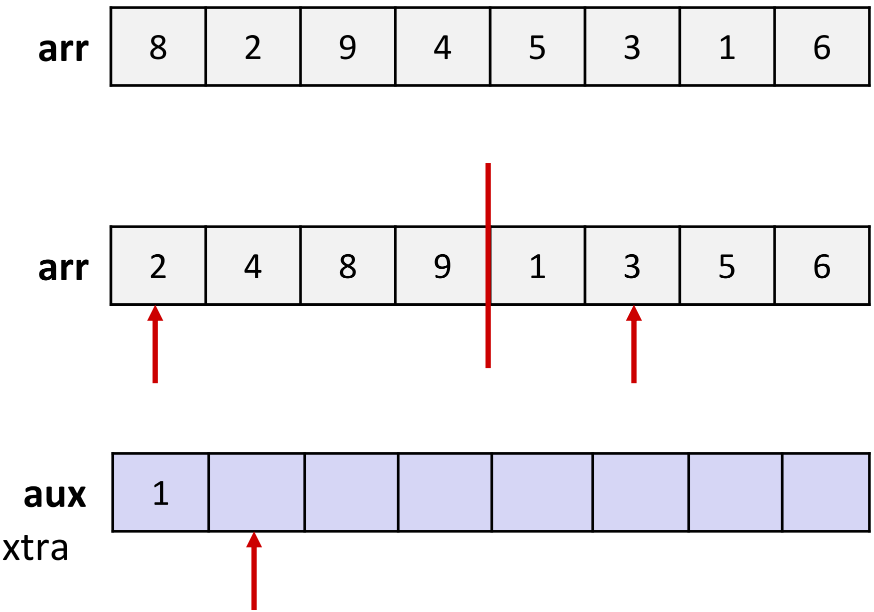
2	4	8	9	1	3	5	6

- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original

aux

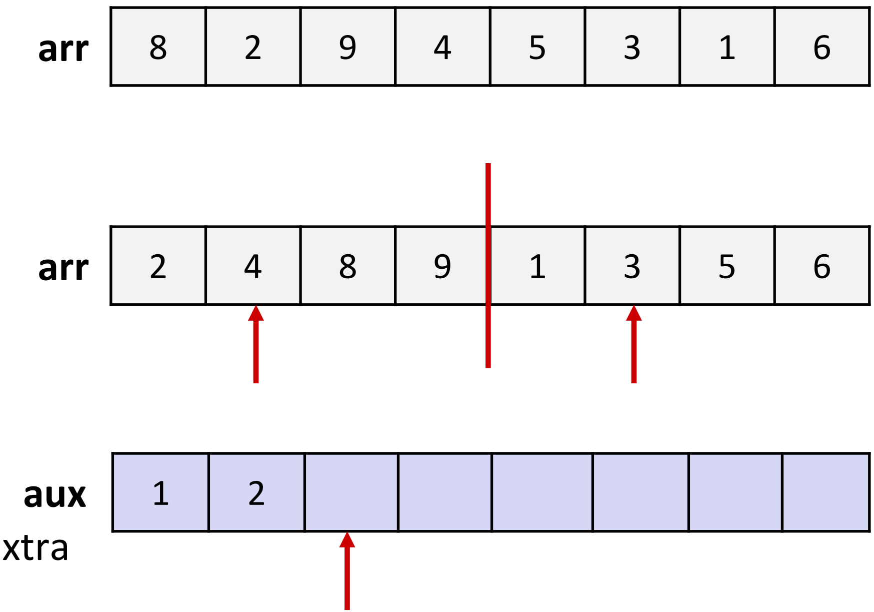
MergeSort: Merging Example (2 of 10)

- Start with:
- Return from left and right recursion
  - (not magic ☺)
- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original

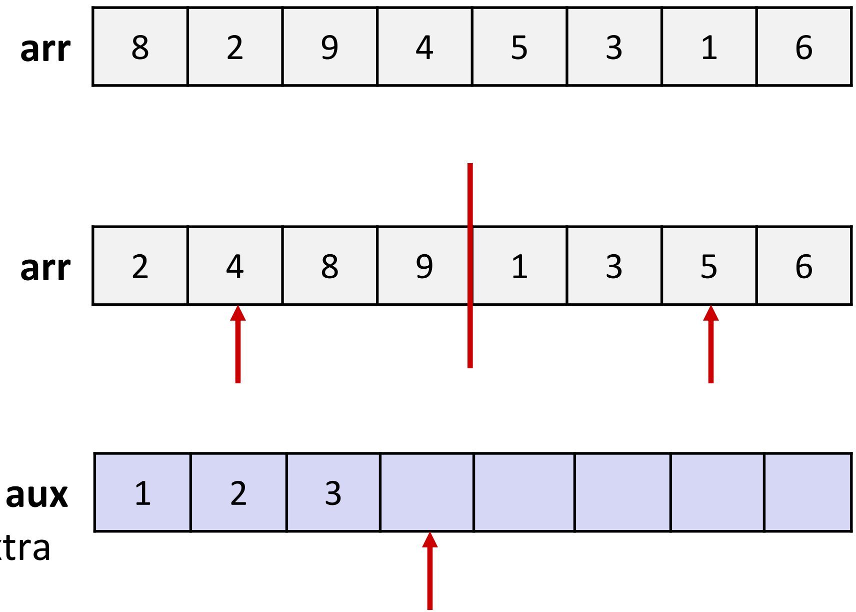


MergeSort: Merging Example (3 of 10)

- Start with:
- Return from left and right recursion
  - (not magic ☺)
- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original



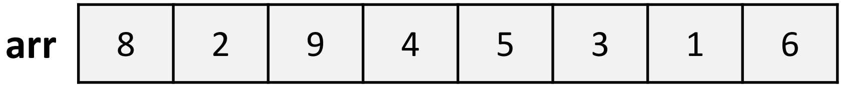
MergeSort: Merging Example (4 of 10)



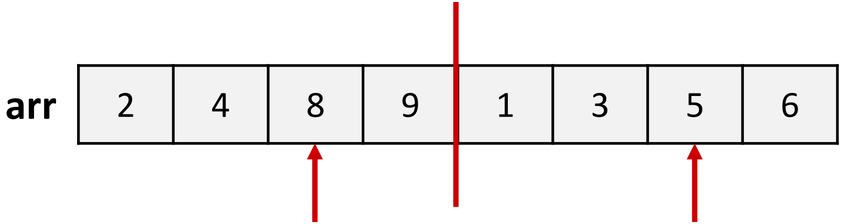
- Start with:
- Return from left and right recursion
  - (not magic ☺)
- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original

MergeSort: Merging Example (5 of 10)

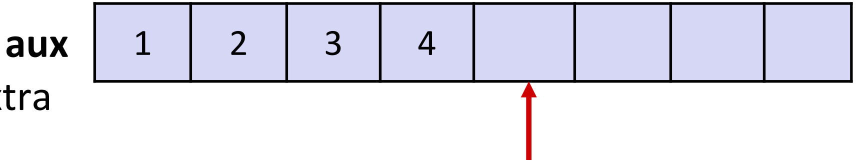
- Start with:



- Return from left and right recursion
  - (not magic ☺)

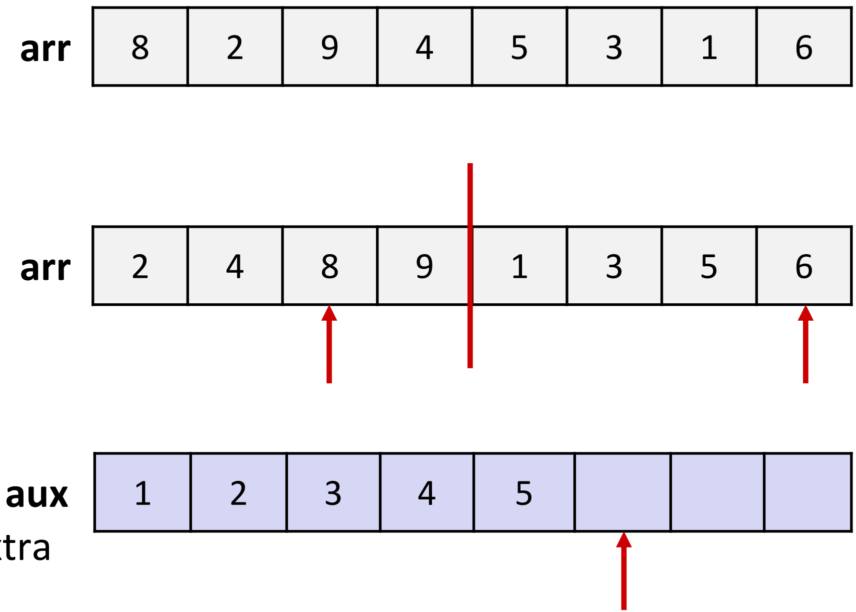


- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original



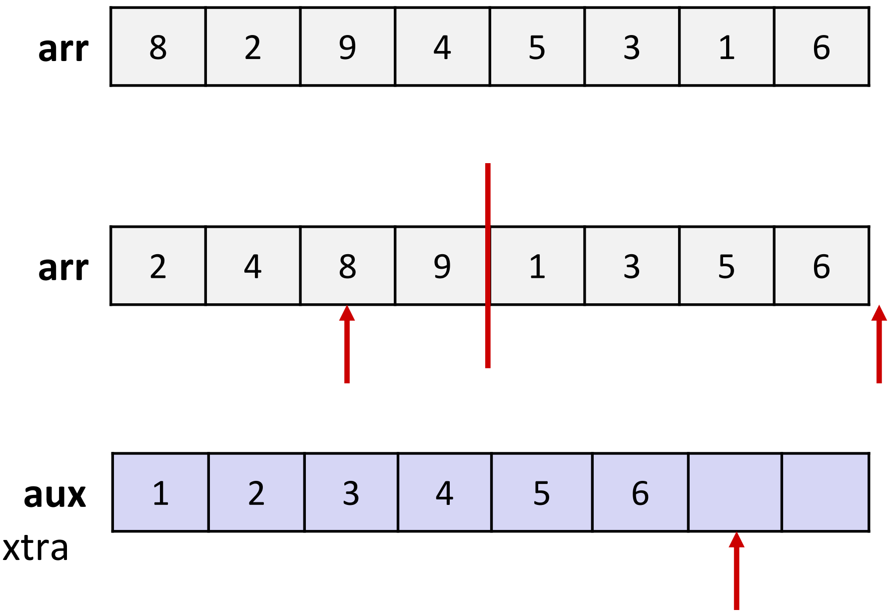
MergeSort: Merging Example (6 of 10)

- Start with:
- Return from left and right recursion
  - (not magic ☺)
- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original



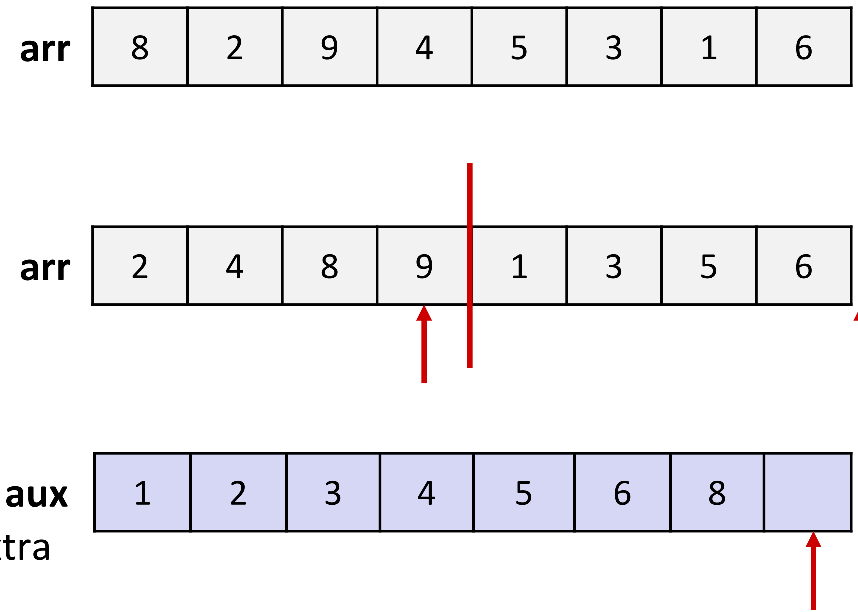
MergeSort: Merging Example (7 of 10)

- Start with:
- Return from left and right recursion
  - (not magic ☺)
- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original



MergeSort: Merging Example (8 of 10)

- Start with:
- Return from left and right recursion
  - (not magic ☺)
- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original



MergeSort: Merging Example (9 of 10)

- Start with:

arr

8	2	9	4	5	3	1	6

- Return from left and right recursion
  - (not magic ☺)

arr

2	4	8	9	1	3	5	6

- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original

aux

1	2	3	4	5	6	8	9

MergeSort: Merging Example (10 of 10)

- Start with:

arr

8	2	9	4	5	3	1	6

- Return from left and right recursion
  - (not magic ☺)

arr

2	4	8	9	1	3	5	6

- Merge
  - Use 3 cursors and an extra auxiliary array
  - When done, copy the extra array back to the original

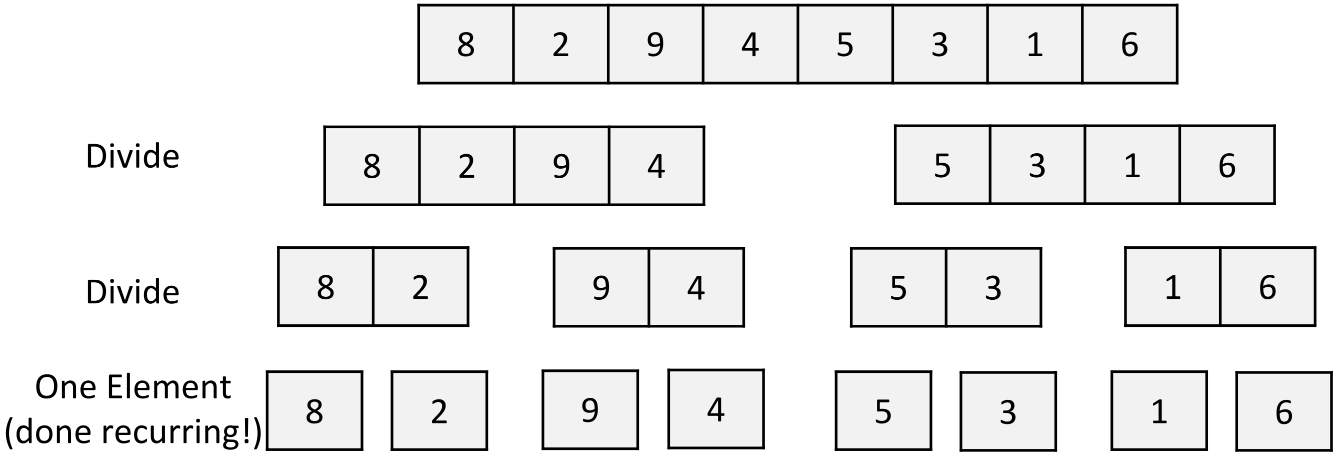
aux

1	2	3	4	5	6	8	9

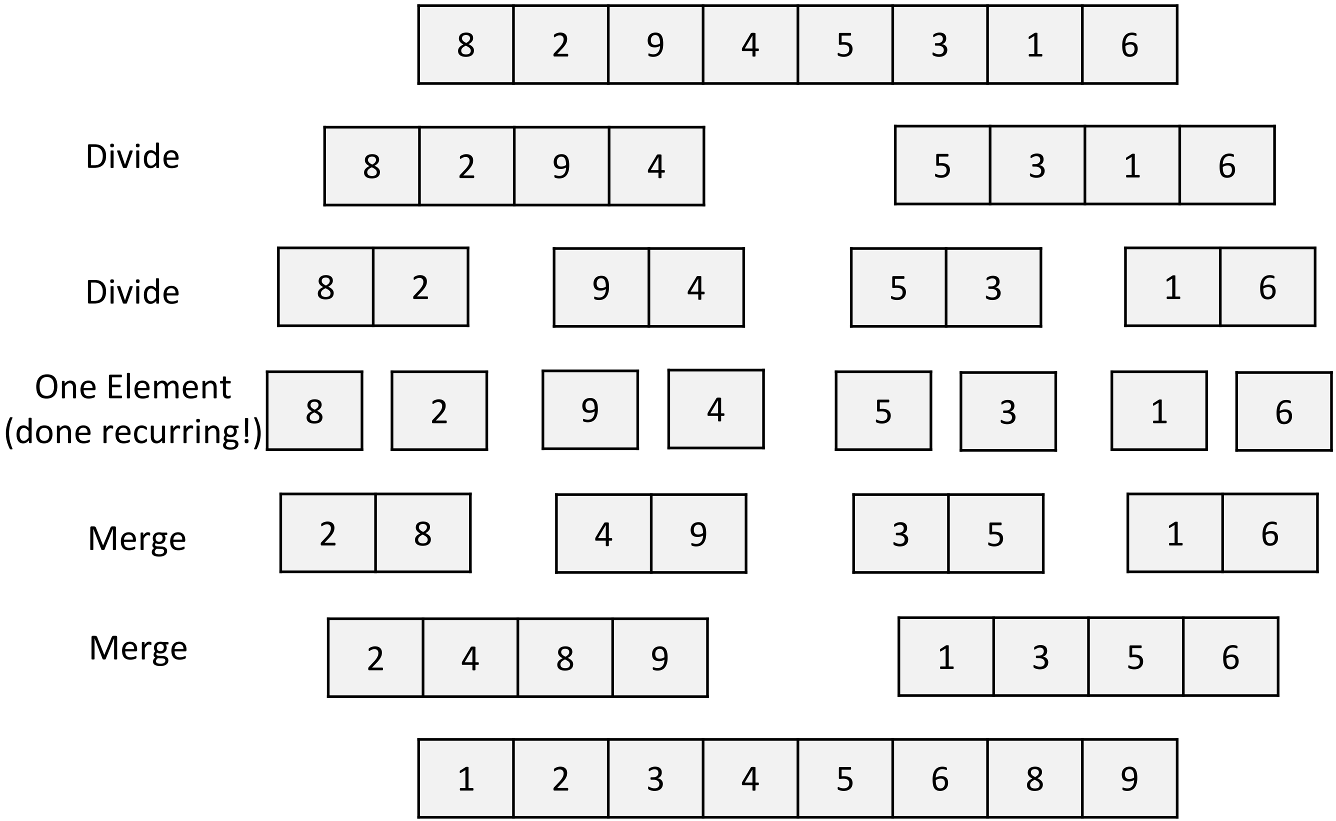
arr

1	2	3	4	5	6	8	9

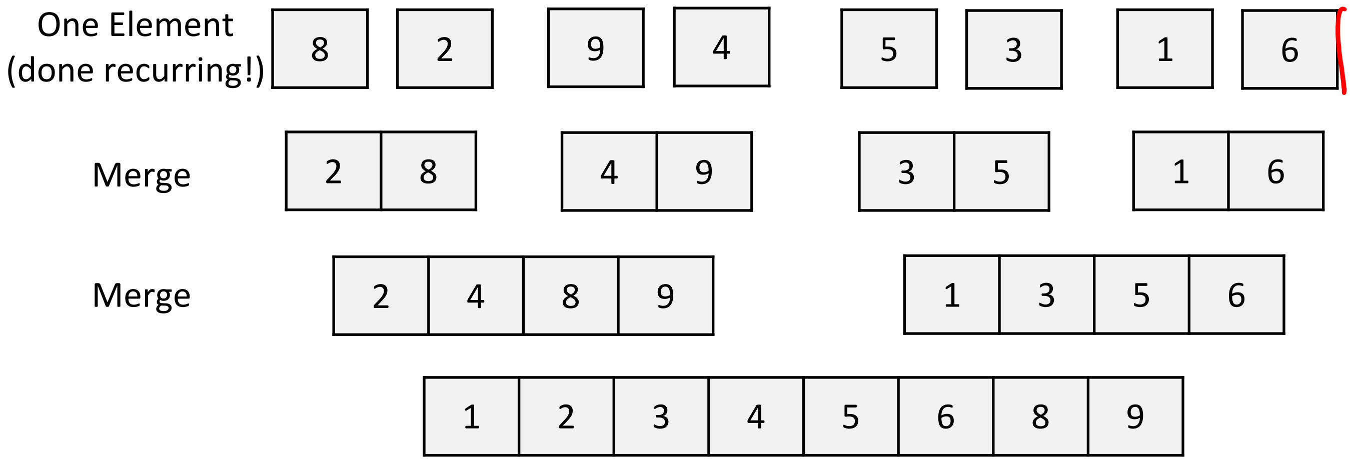
MergeSort: Recursion Example (1 of 3)



MergeSort: Recursion Example (2 of 3)



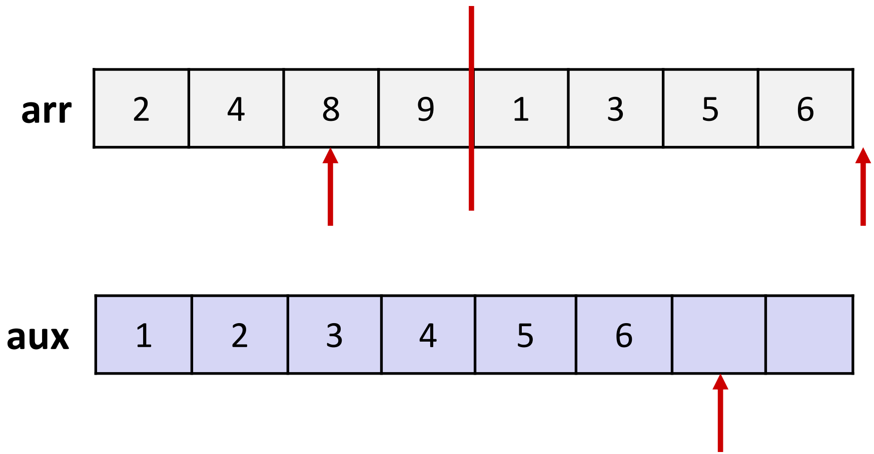
MergeSort: Recursion Example (3 of 3)



When a recursive call ends, its sub-arrays are *each in order*; we just need to merge them *in order together*

Optimizations: Reducing "Dregs Copies" (1 of 2)

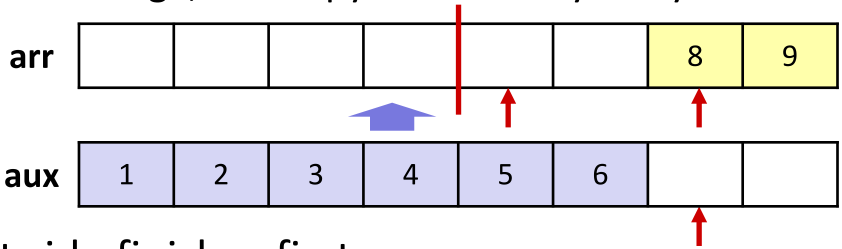
- Remember the final steps of our merge example?



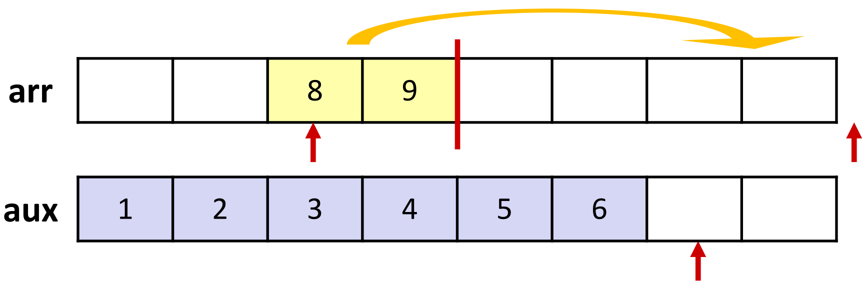
- It's wasteful to copy 8 & 9 to the auxiliary array, and then immediately copy them back into the original array!

Optimizations: Reducing "Dregs Copies" (2 of 2)

- If left side finishes first:
  - Stop the merge, and copy the auxiliary array back to the original



- If right side finishes first:
  - Stop the merge, and copy the dregs directly into right side
  - Then copy auxiliary array back to the original

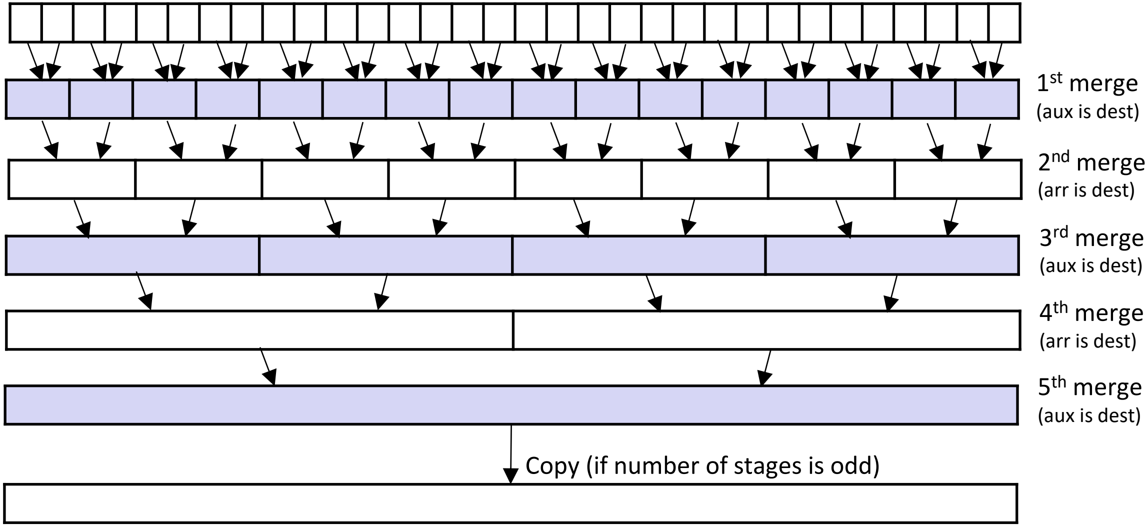


Optimizations: Reducing Temp Arrays (1 of 2)

- Simplest / worst approach:
  - Every divide: allocate two new auxiliary arrays of size **`(hi-lo)/2`**
  - Every merge: allocate another auxiliary array
- Better:
  - Allocate a single auxiliary array of size **`n`** at beginning to use throughout
  - Reuse "slices" of size **`(hi-lo)/2`** within that array at every merge
- Best (but a little tricky):
  - Don't copy back! At 2nd, 4th, 6th, … merges, use the original array as the auxiliary array; at odd-numbered merges, vice-versa
  - If the number of stages is odd, need one final copy at end

Optimizations: Reducing Temp Arrays (2 of 2)

1. Recur down to sub-arrays of size 1 (no copies)
2. As we return from the recursion, switch off arrays



3. Arguably easier to code up without recursion at all

MergeSort: Runtime Analysis (1 of 3)

- MergeSort sorts *n* elements by:
  - Returning immediately if *n*=1
  - Doing 2 subproblems of size *n*/2 + then an *O*(*n*) merge otherwise
- Runtime expression?
  - $T(1) = c_1$
  - $T(n) = 2T(n/2) + c_2 n$

MergeSort: Runtime Analysis (2 of 3)

$T(1) = c_1$

First expansion

$T(n) = 2T(n/2) + c_2 n$

Second expansion

$= 2(2T(n/4) + c_2 n/2) + c_2 n$

$= 4T(n/4) + 2c_2 n$

Third expansion

$= 4(2T(n/8) + c_2 n/4) + 2c_2 n$

$= 8T(n/8) + 3c_2 n$

k^{th} expansion

$= 2^k T(n/2^k) + kc_2 n$

If I want $n/2^k = 1$, let $k = \log n$

Then $T(n) = 2^k T(n/2^k) + kc_2 n$

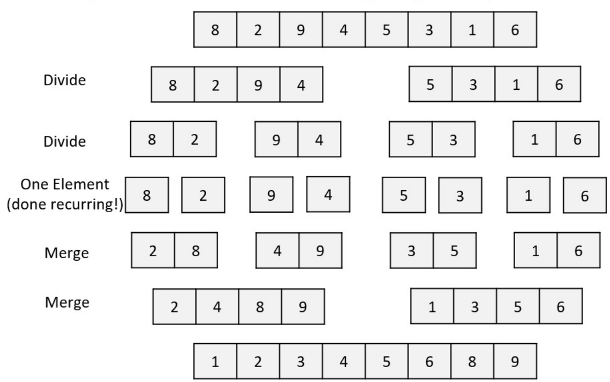
$= 2^{\log n} T(1) + \log n \, c_2 n$

$= c_1 n + c_2 n \log n$

$= O(n \log n)$

MergeSort: Runtime Analysis (3 of 3)

- More intuitively, this recurrence comes up often enough you should "just know" it's $O(n \log n)$
- MergeSort's runtime is relatively easy to intuit
  - Best, worst, and "average" all have the same runtime
  - The recursion "tree" will have $\log n$ height and at each level we do a *total* amount of merging equal to n



MergeSort: Characteristics

- Execution:
  - Merge sorted subarrays as it "recurs upward" (ie, returns from recursive calls)
- Characteristics:
  - Stable: yes
  - In-place: no
- Time: always O(n log n)

```
mergeSort(arr, startIdx, endIdx) {
  if (startIdx == endIdx
      || startIdx + 1 == endIdx) {
    return;
  }

  midIdx = (endIdx - startIdx)/2
    + startIdx;
  mergeSort(arr, startIdx, midIdx);
  mergeSort(arr, midIdx, endIdx);
  merge(arr, startIdx, midIdx,
        endIdx);
}
```

MergeSort: Final Thoughts

- We've discussed arrays, but you may need to sort linked lists
  - One approach:
    - Convert to array: O(n)
    - Sort: O(n log n)
    - Convert back to list: O(n)
  - Alternatively: MergeSort works well on linked lists
    - HeapSort and QuickSort do not ☹
    - InsertionSort and SelectionSort can work, but they're slower

- *MergeSort is the best choice for external sorting*
  - *Linear merges minimize new disk accesses*

QuickSort Steps

1. Pick the pivot value(s)
   - Any choice is correct; data will end up sorted
   - For efficiency, these value(s) ought to approximate the median

2. Partition all the values into:
   a. The values less than the pivot(s)
   b. The pivot(s)
   c. The values greater than the pivot(s)
   d. .. In linear time? In-place? Stably?

3. Recursively QuickSort(A) and QuickSort(C)

✨*TA-DA!*✨

QuickSort Steps

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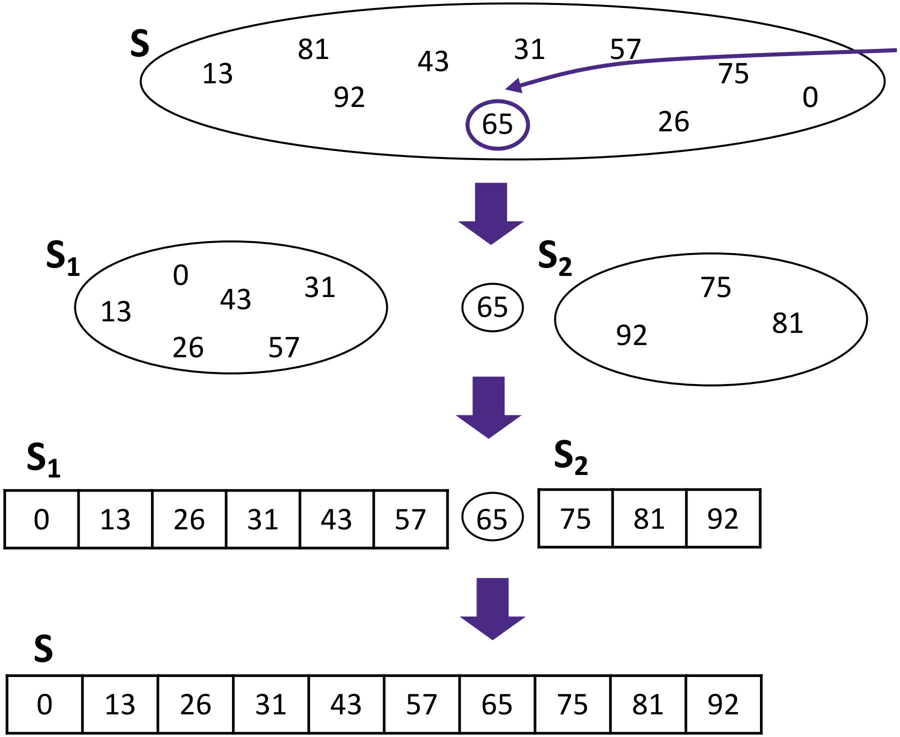
3. **Recursively QuickSort(A) and QuickSort(C)**

✨*TA-DA!*✨

QuickSort Intuition: Set Partitioning



Select pivot value

Partition S

QuickSort(S_1) and QuickSort(S_2)

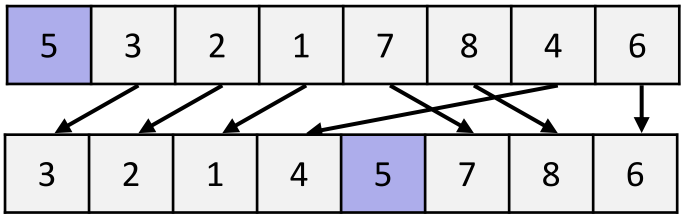
Presto! S is sorted

[Weiss]

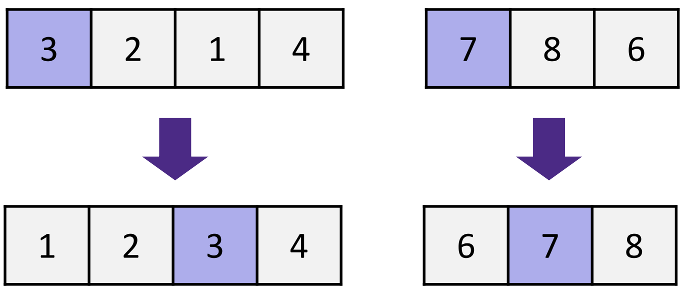
Note: for the remainder of this section, our pivot-selection algorithm is “first item in the subarray”

Recursive Call (1 of 3)

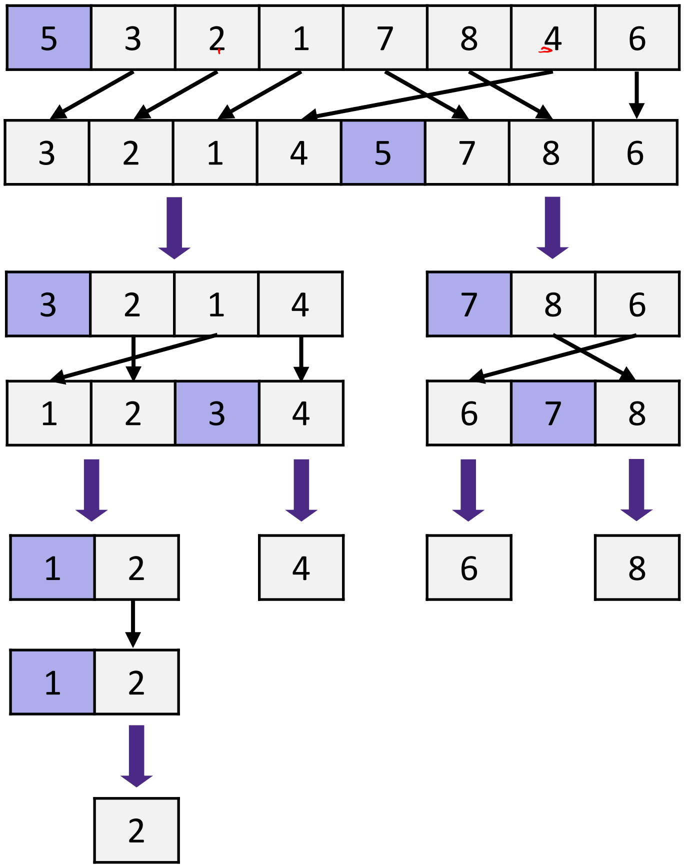
- After partitioning on 5:
  - 5 is in its “correct place” (ie, where it’d be if the array were sorted)



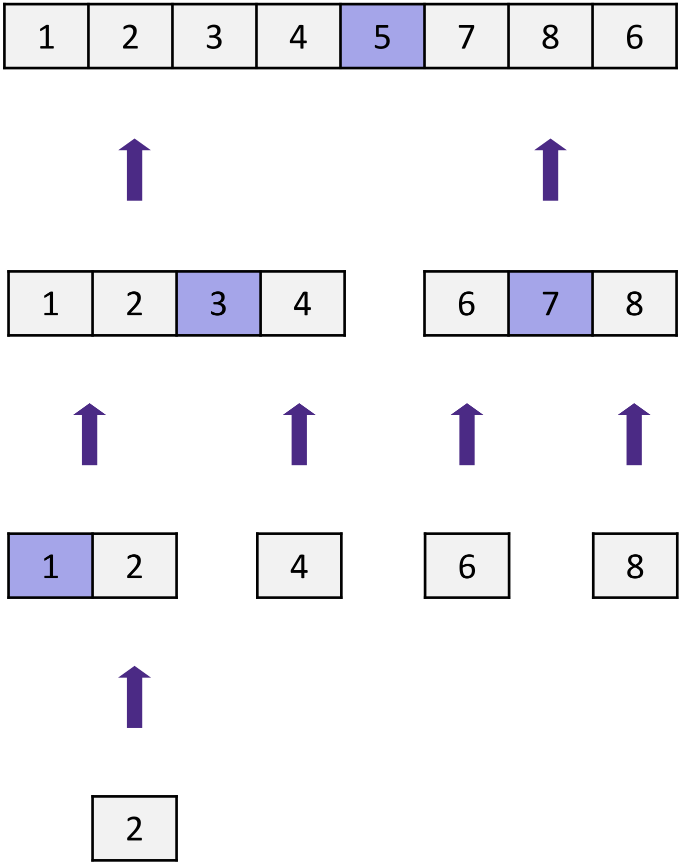
  - Can now sort two halves separately (eg, through recursive use of partitioning)



Recursive Call (2 of 3)



Recursive Call (3 of 3)



QuickSort Steps

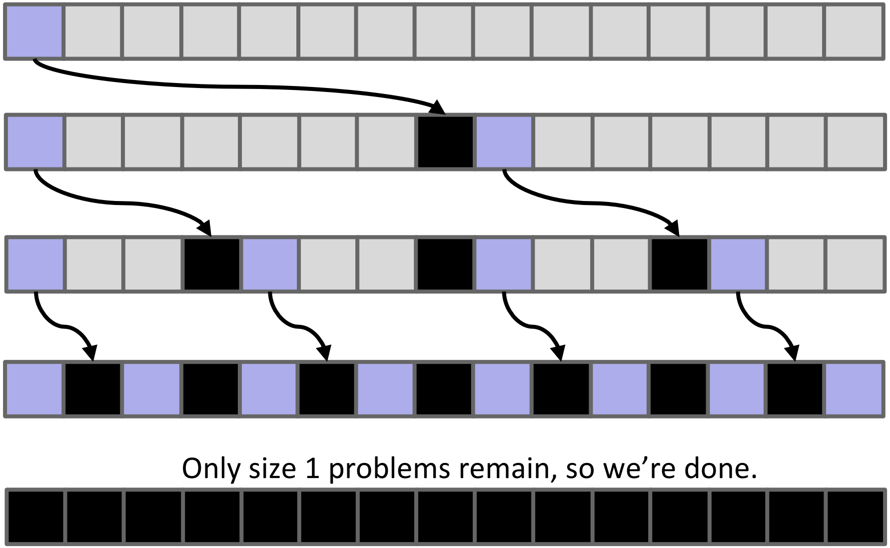
1. **Pick the pivot value(s)**
   - **Any choice is correct; data will end up sorted**
   - **For efficiency, these value(s) ought to approximate the median**

2. Partition all the values into:
   a. The values less than the pivot(s)
   b. The pivot(s)
   c. The values greater than the pivot(s)
   d. … In linear time? In-place? Stably?

3. Recursively QuickSort(A) and QuickSort(C)

✨*TA-DA!*✨

Pivot Selection: Pivot is the Median



$T(0) = T(1) = c_1$

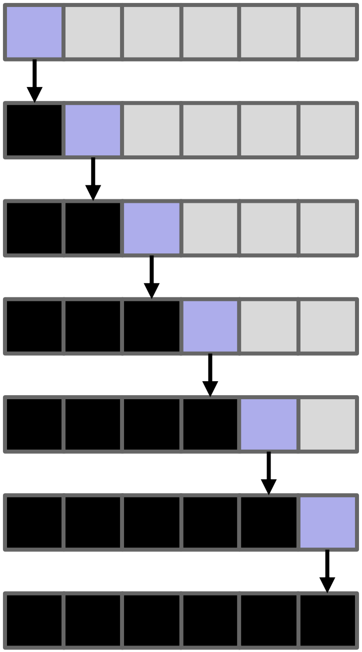
$T(n) = 2T(n/2) + c_2\, n$

(partition is linear-time)

Same recurrence as MergeSort:

$O(n \textbf{ log } n)$

Pivot Selection: Pivot is the Min/Max



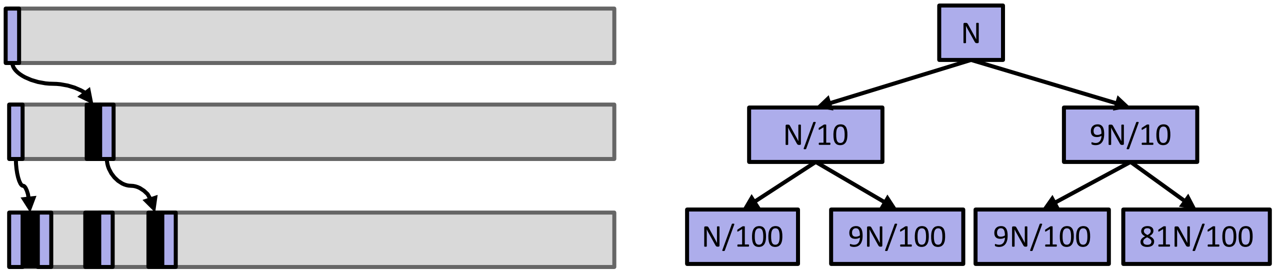
$T(0) = T(1) = c_1$

$T(n) = T(n\text{-}1) + c_2 n$

Basically same recurrence as SelectionSort: $O(n^2)$

Pivot Selection: Pivot is Random

- Suppose pivot always ends up *at least 10% from either edge*



- Work at each level: O(N) and Runtime is O(NH)
  - Height is approximately $\log_{10/9} N = O(\log N)$
- Runtime: O(N log N)
  - See proof in text

Pivot Selection Dictates Runtime!

- If pivot lands "somewhere good", Quicksort is Θ(N log N) 🥂
- However, the very rare Θ(N^2) cases do happen in practice 👎
  - **Bad ordering**: Array already in (almost-)sorted order and pivot is first or last index
  - **Bad elements**: Array with all duplicates
- Three philosophies for avoiding worst-case behavior:
  1. **Randomness**: pick a random pivot; shuffle before sorting
     - Elegant, but (pseudo)random number generation can be slow
  2. **Smarter Pivot Selection**: calculate or approximate the median
     - Median-of-3: median of `arr[lo]`, `arr[hi-1]`, `arr[(hi+lo)/2]`
  3. **Introspection**: switch to safer sort if recursion goes too deep

Avoiding Worst-Case Pivots

- Example worst-cases:
  - **Bad ordering**: Array already in (almost-)sorted order and pivot is first or last index
  - **Bad elements**: Array with all duplicates

- Three philosophies for avoiding worst-case behavior:
  1. **Randomness**: pick a random pivot; shuffle before sorting
     - Elegant, but (pseudo)random number generation can be slow
  2. **Smarter Pivot Selection**: calculate or approximate the median
     - Median-of-3: median of `arr[lo]`, `arr[hi-1]`, `arr[(hi+lo)/2]`
  3. **Introspection**: switch to safer sort if recursion goes too deep
     - … what algorithm might be safer in the presence of badly-ordered elements?