AVL Trees

CSE 332 Summer 2021

Instructor: Kristofer Wong

Teaching Assistants:

Alena Dickmann	Arya GJ	Finn Johnson
Joon Chong	Kimi Locke	Peyton Rapo
Rahul Misal	Winston Jodjana	

Announcements

- Clarifying urgent P1 announcement from Ed
- Gradescope in lecture activities
- Fill out the P2 partner survey!!!
  - There will be 1 group of 3
- Friday's lecture

No formal activity today

- PLEASE do the activities today anyway. They are very helpful for gaining intuition.

- What is the impact that the order of elements have on the resultant BST's **structure** and **ordering**?

Lecture Outline

- AVL Tree
  - **Bounding a BST's height**
  - Find
  - Add
  - Remove
  - Wrapup

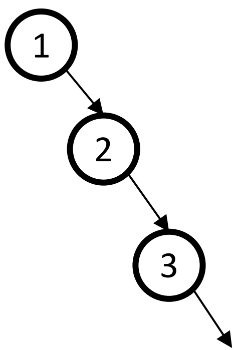
Why does BST height matter? (1 of 2)

	BST, Randomized	BST, Worst
Find	Θ(h) aka Θ(log N)	Θ(h) aka Θ(N)
Add	Θ(h) aka Θ(log N)	Θ(h) aka Θ(N)
Remove	Θ(h) aka Θ(log N)	Θ(h) aka Θ(N)

- For a BST with *n* items:
  - Randomized height is Θ(log n) – see text for proof (pgs 120-122)
  - Worst case height is Θ(n)
- Simple cases, such as inserting in order, lead to worst case structure!

Why does BST height matter? (2 of 2)

- Insert keys 1, 2, 3, 4, 5, 6, 7, 8, 9 into an empty BST
  - The resultant tree is a "linked list"
  - What is the big-Oh *aggregate* runtime for n add()s of sorted input?



Aggregate Runtime for n adds: O(n^2)

(not a happy place)

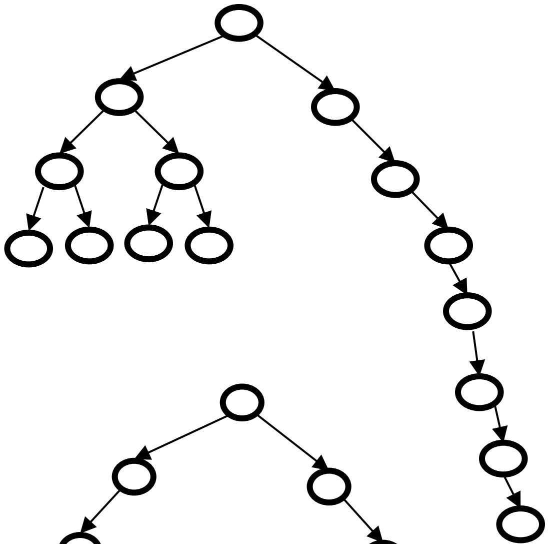
Goal: Balance the BST

- Require a **Balance Condition** that:
  1. Ensures height is always O(log n)
  2. Is easy to maintain

Potential BST Balance Conditions

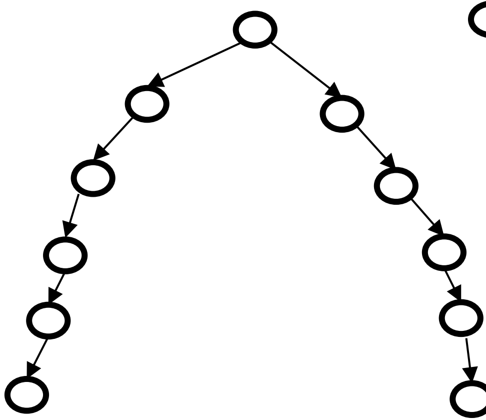
- Left and right subtrees of the *root* have equal number of nodes

Too weak!
Height mismatch example:

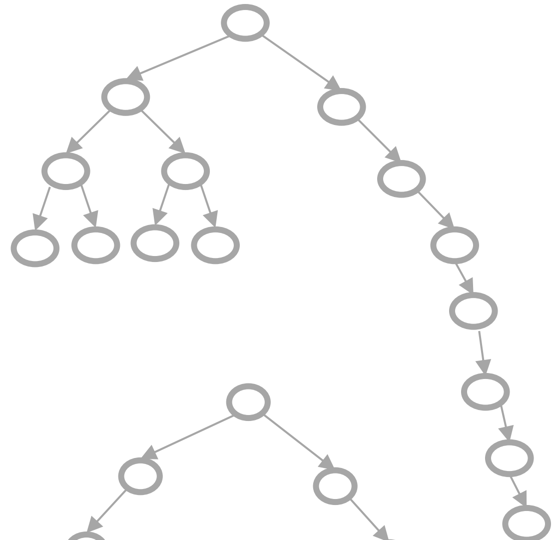


- Left and right subtrees of the *root* have equal *height*

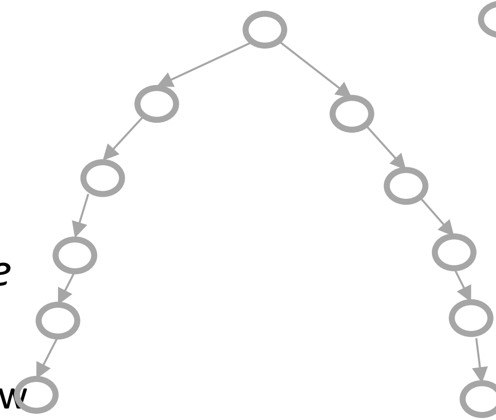
Too weak!
Double chain example:



The AVL Balance Condition (1 of 2)



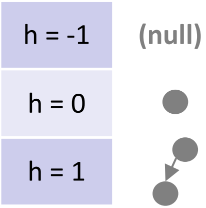
- Left and right subtrees of the *root* have equal number of nodes



- Left and right subtrees of the *root* have equal *height*

- Left and right subtrees of *every node* have *heights* **differing by at most 1**
  - ***NOTE:*** *height* here is different from how we defined it in the past…

The AVL Balance Condition (2 of 2)



> Left and right subtrees of *every node* have *heights* **differing by at most 1**

Definition: **balance**(*node*) = height(*node*.left) – height(*node*.right)

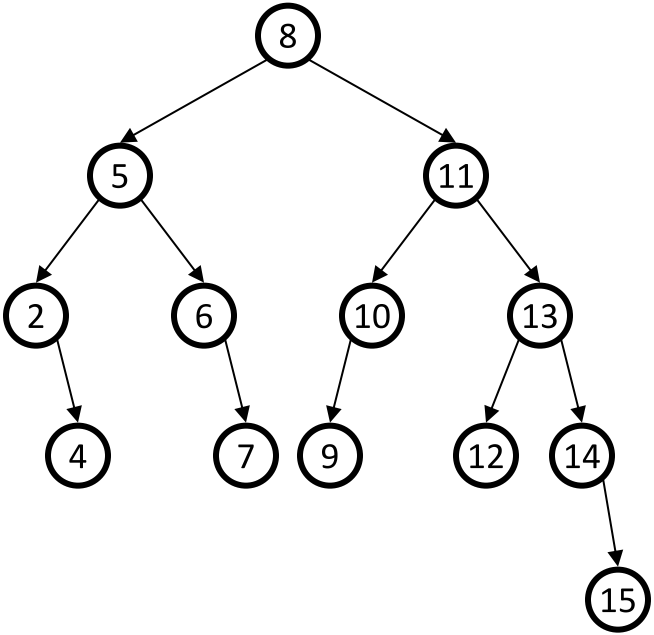
AVL property: **for every node *x*, –1 ≤ balance(*x*) ≤ 1**

Results:

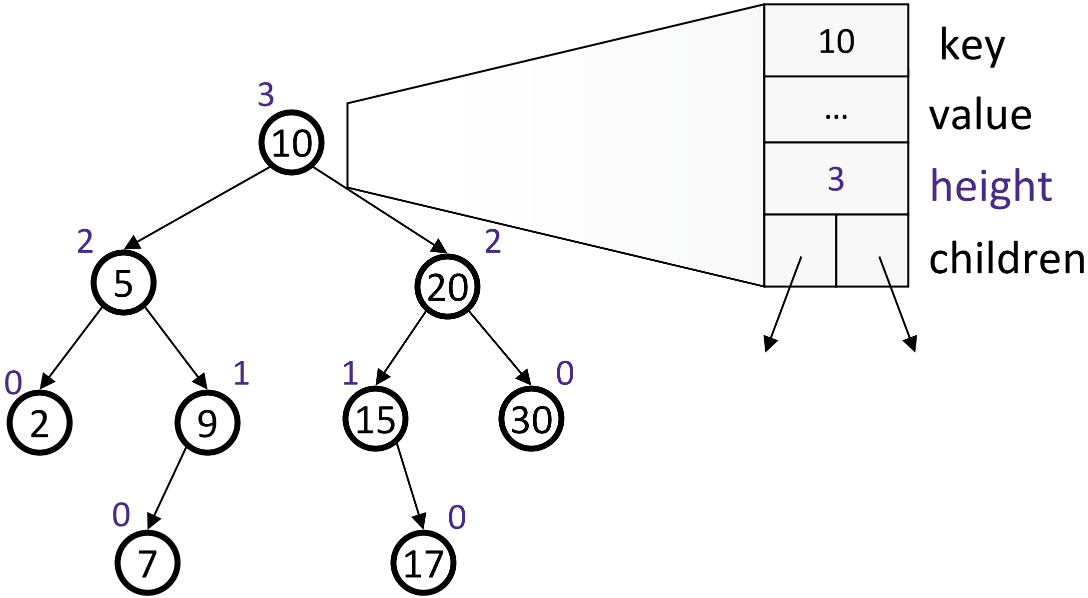
- Ensures shallow depth: h ∈ Θ(log n)
  - Will prove this by showing that an AVL tree of height *h* must have a number of nodes *exponential* in *h*
- Efficient to maintain using rotations

The AVL Tree Data Structure

- Structural properties
  - Binary tree property (0, 1, or 2 children)
  - Heights of left and right subtrees for every node differ by at most 1
- Ordering property
  - Same as for BST

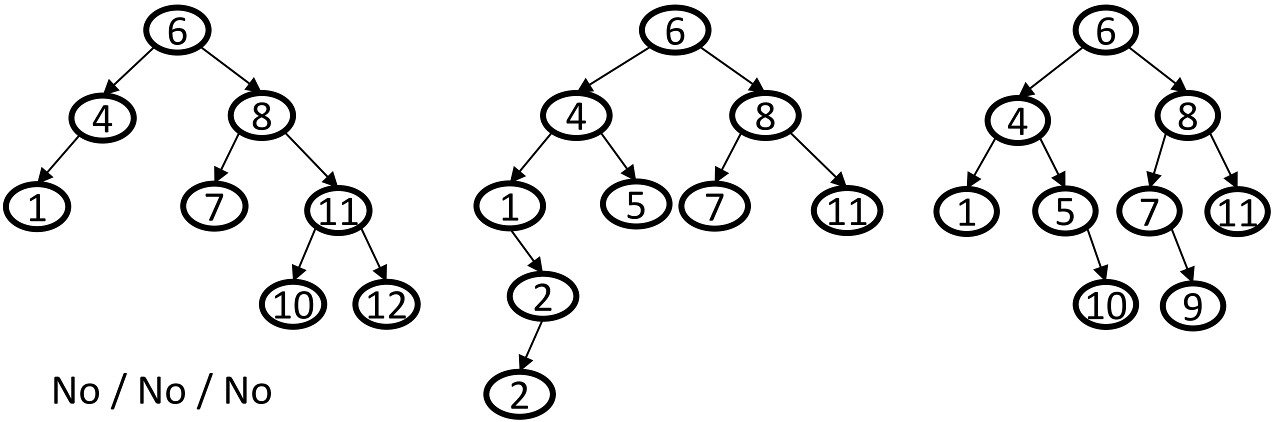


Implementation detail…



<not gradescope> activity

- Are the following trees AVL trees?



A. No / No / No
B. Yes / No / No
C. Yes / Yes / No
D. Yes / Yes / Yes
E. Yes / No / Yes

Lecture Outline

- AVL Tree
  - Bounding a BST's height
  - **Find**
  - Add
  - Remove
  - Wrapup

AVL Find

- Surprise! You already know this one

- 🎉🎉🎉 find() is O(log n)! 🎉🎉🎉
  - Proof to come Friday..

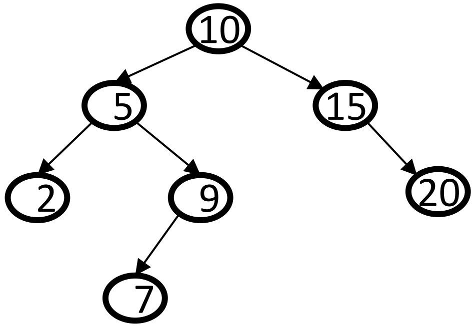
Lecture Outline

- AVL Tree
  - Bounding a BST's height
  - Find
  - **Add**
  - Remove
  - Wrapup

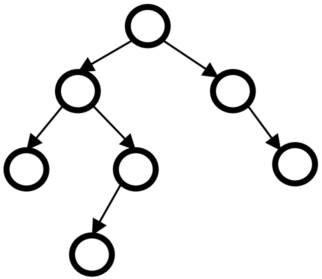
Problems with adding elements

- But as we add() and remove elements(), we need to:
  - 👎 Track heights
  - 👎 Detect imbalance
  - 👎 Restore balance

What needs to happen when we insert(8)?



AVL add(): Overall Approach



- Our overall algorithm looks like:
  1. Insert the new node as in a BST (a new leaf)
  2. For each node on the path from the root to the new leaf:
     - The insertion may (or may not) have changed the node's height
     - Detect height imbalance and perform a *rotation* to restore balance

- Fact that makes it a bit easier:
  - Imbalances only occur along the path from the new leaf to the root
  - There must be a deepest element that is unbalanced
  - After rebalancing this deepest node, every node above it is also rebalanced
  - Therefore, *at most one node needs to be rebalanced*

AVL add(): Overall Approach

- Fact that makes it a bit easier:
  - Imbalances only occur along the path from the new leaf to the root
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  - After rebalancing this deepest node, every node above it is also rebalanced
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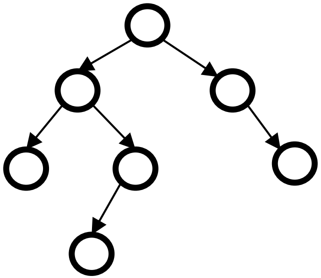
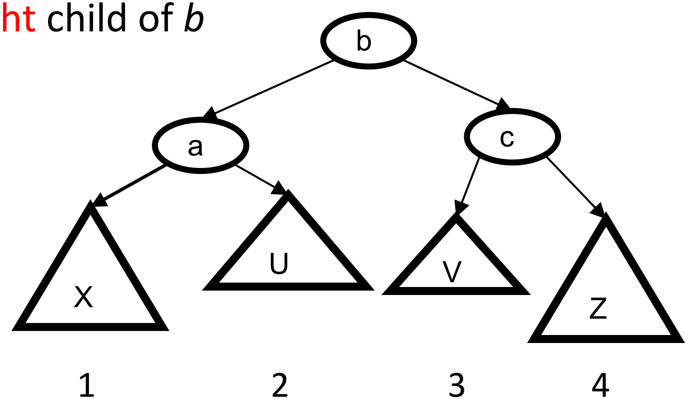


AVL add(): Cases

- Let *b* be the deepest node where an imbalance occurs

- There are four cases to consider. The insertion is in the:
  1. left subtree of the left child of *b*
  2. right subtree of the left child of *b*
  3. left subtree of the right child of *b*
  4. right subtree of the right child of *b*



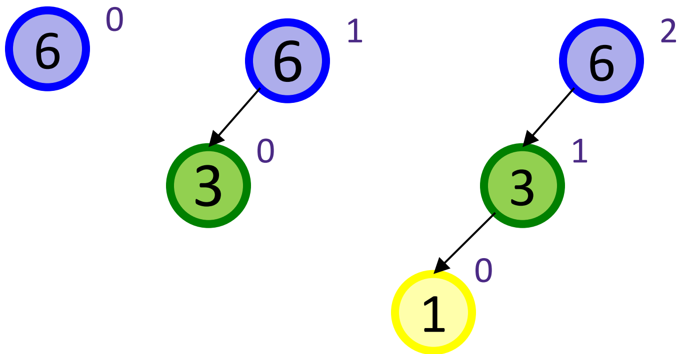
Case #1: Example

The insertion is in the:
1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*

add(6)

add(3)

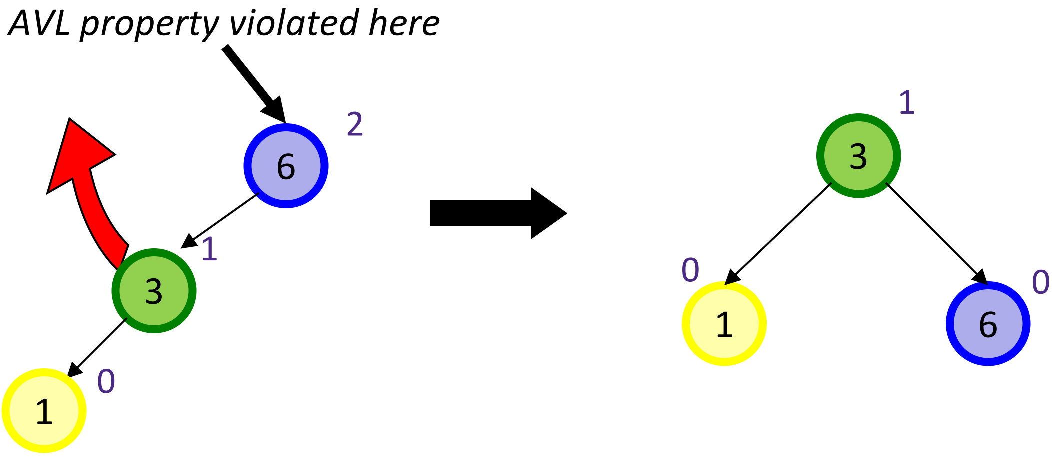
add(1)



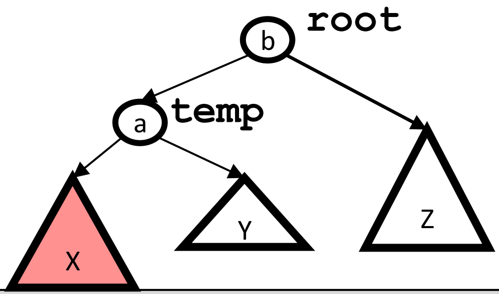
- Last add() violates balance property
- What is the only way to fix this?

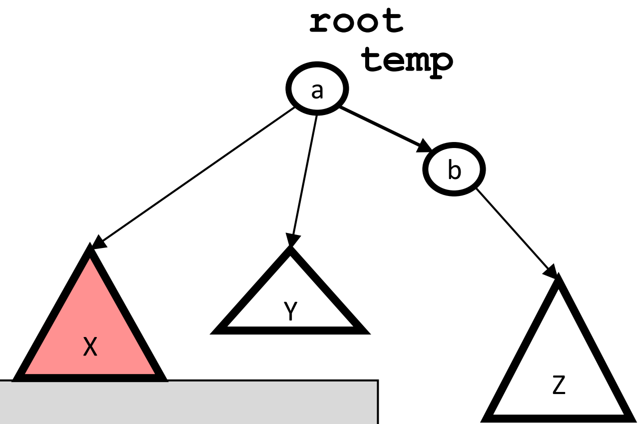
Case #1 Fix: Apply "Single Rotation"

- ***Single rotation:***
  - Move child of unbalanced node into parent position
  - Parent becomes the "other" child



Case #1: Pseudocodea






```
void rotateRight(Node root) {
  Node temp = root.left
  root.left = temp.right
  temp.right = root
  root.height = max(root.right.height(),
                    root.left.height()) + 1
  temp.height = max(temp.right.height(),
                    temp.left.height()) + 1
  root = temp
}
```

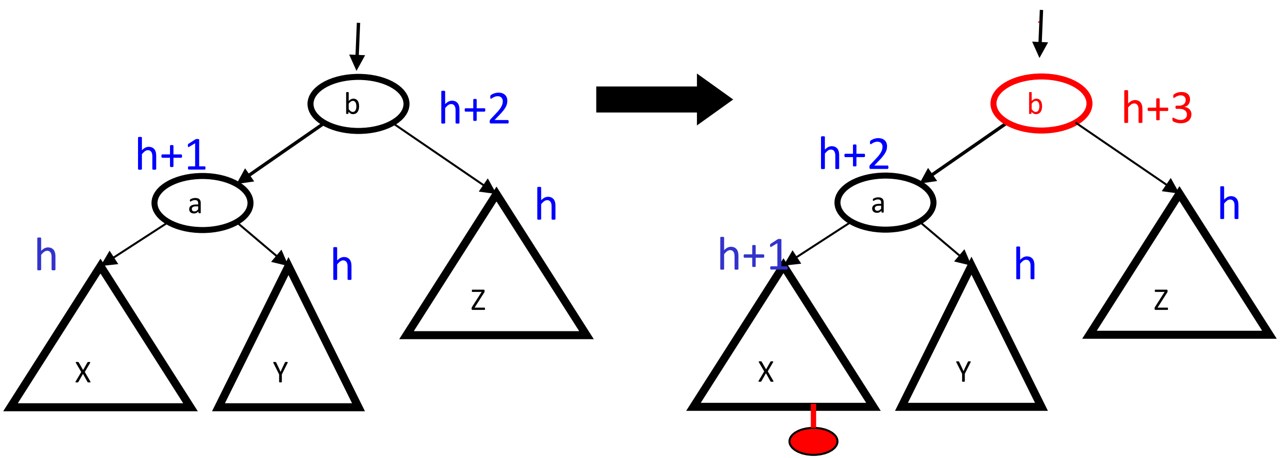
rotateRight rotates the tree clockwise

Case #1: Why It Works (1 of 2)

Oval: a node in the tree
Triangle: a subtree

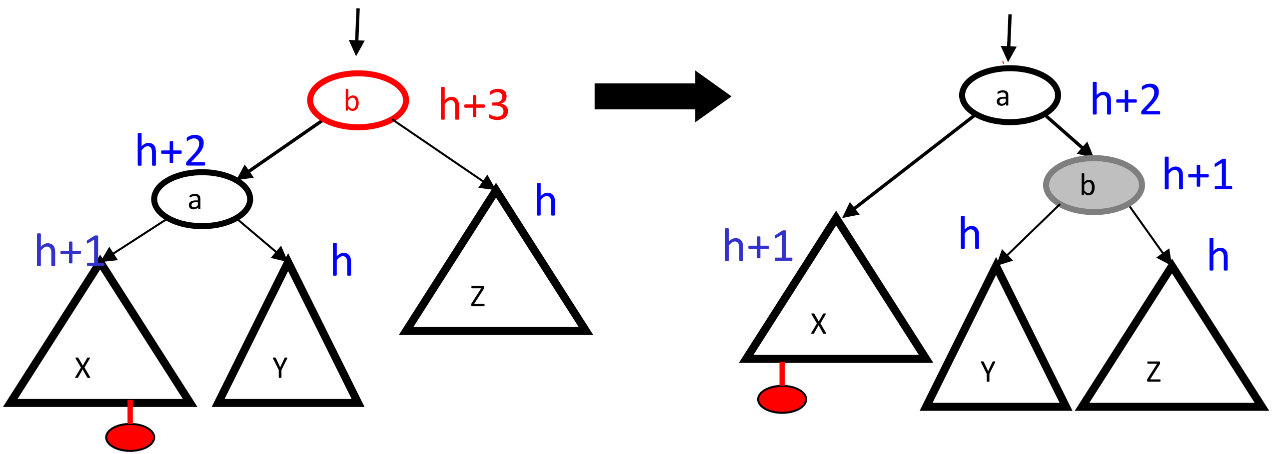
- Node is imbalanced due to insertion *somewhere* in **left-left grandchild**

- First we did the insertion, which would make ***b*** imbalanced



Case #1: Why It Works (2 of 2)

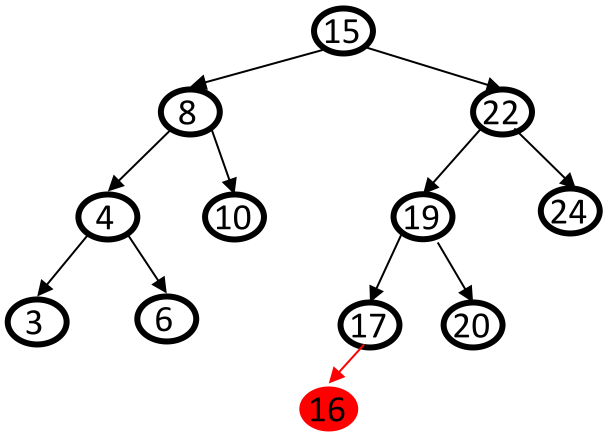
- So we rotate at b, maintaining BST order: X < a < Y < b < Z
- Result:
  - A single rotation restores balance at the formerly-imbalanced node
  - Height is same as before insertion, so ancestors now balanced



Case #1: Another Example: add(16)

The insertion is in the:

1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*



Case #1: Another Example: add(16)

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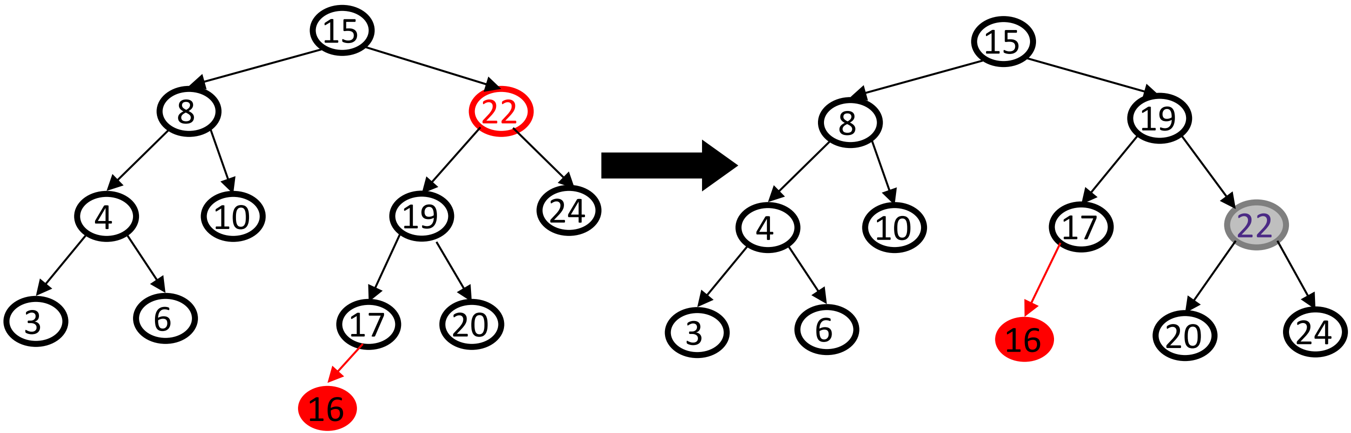
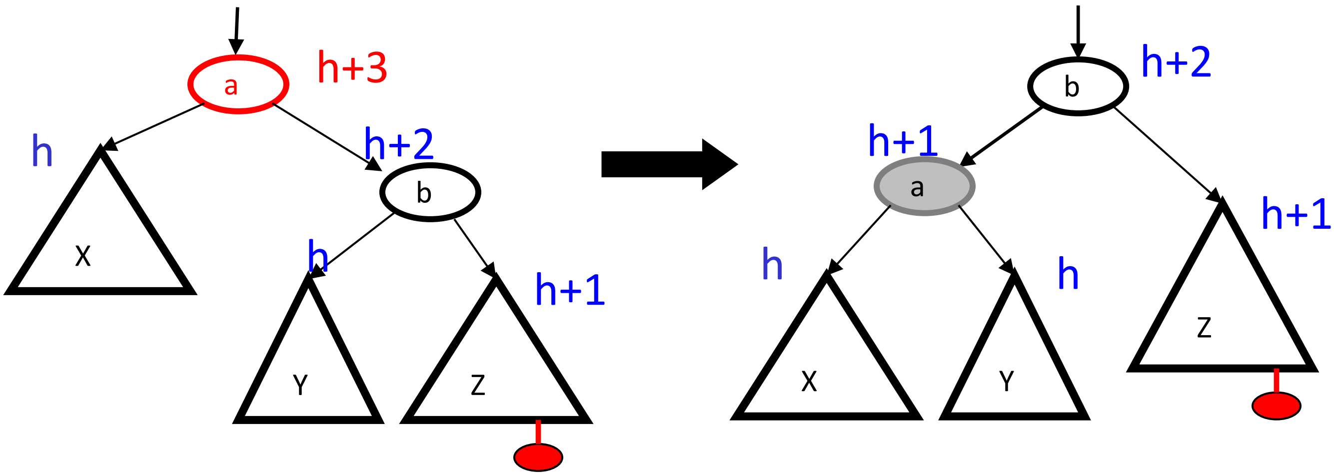


Case #1 ≈ Case #4

The insertion is in the:

1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*

- Mirror image of left-left case, so you rotate the other way
  - Exact same concept, but need different code



RotateWithRightChild rotates the tree counter-clockwise

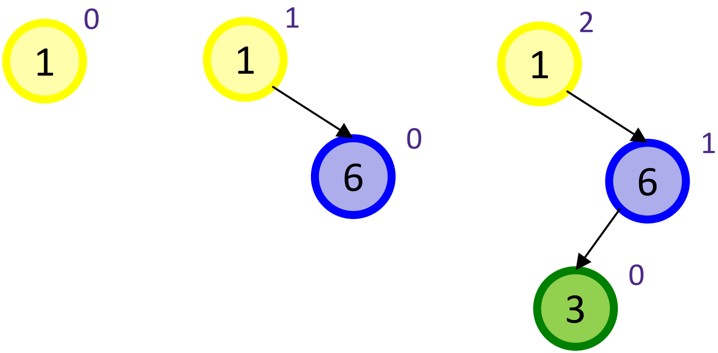
Case #3: Example

Insert(1)

Insert(6)

Insert(3)

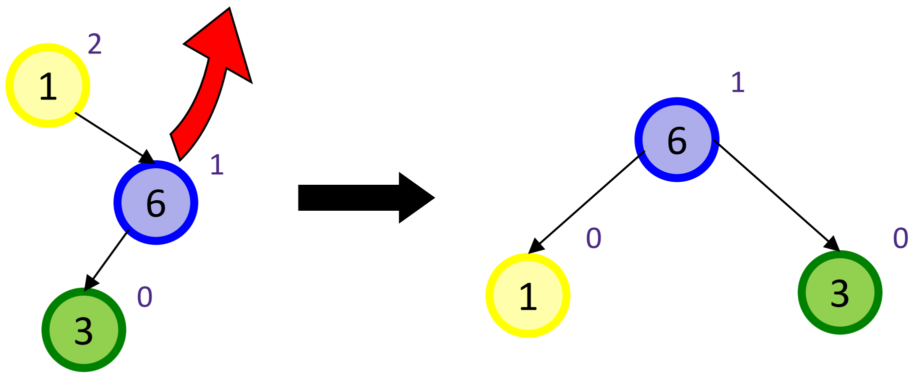
The insertion is in the:
1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*



- Single rotations are not enough for insertions into the left-right subtree (or the right-left subtree; ie, case #2)

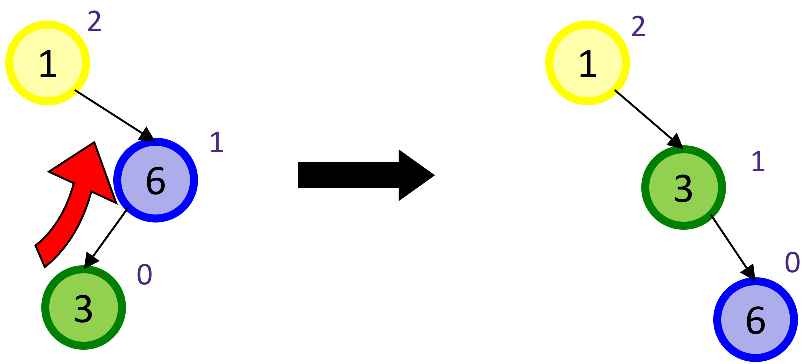
Case #3: Wrong Fix #1

- **First wrong idea**: single left rotation like we did for left-left
  - Violates BST ordering property!



Case #3: Wrong Fix #2

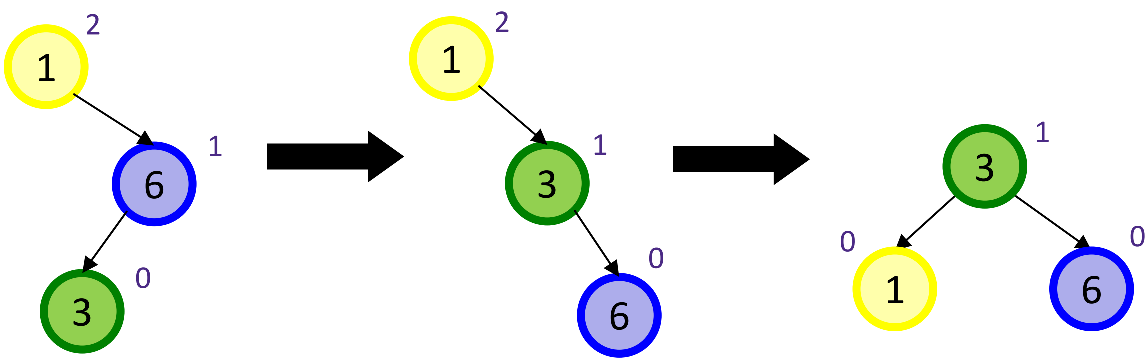
- **Second wrong idea:** single rotation on the child of the unbalanced node
  - Doesn't actually fix anything!



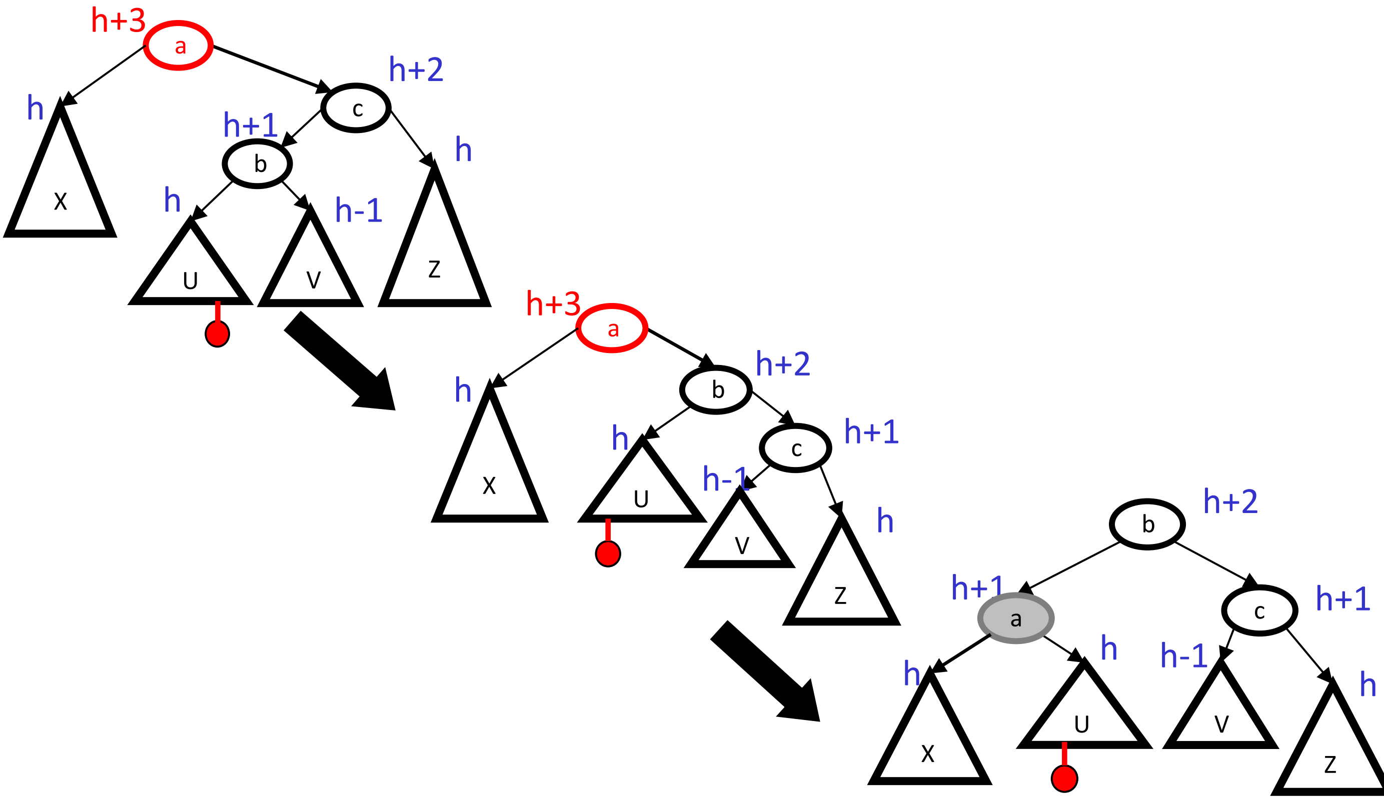
Case #3: Sometimes Two Wrongs Make a Right ☺

- First idea violated the BST ordering
- Second idea didn't fix balance
- … but if we do both single rotations, starting with the second, it works!

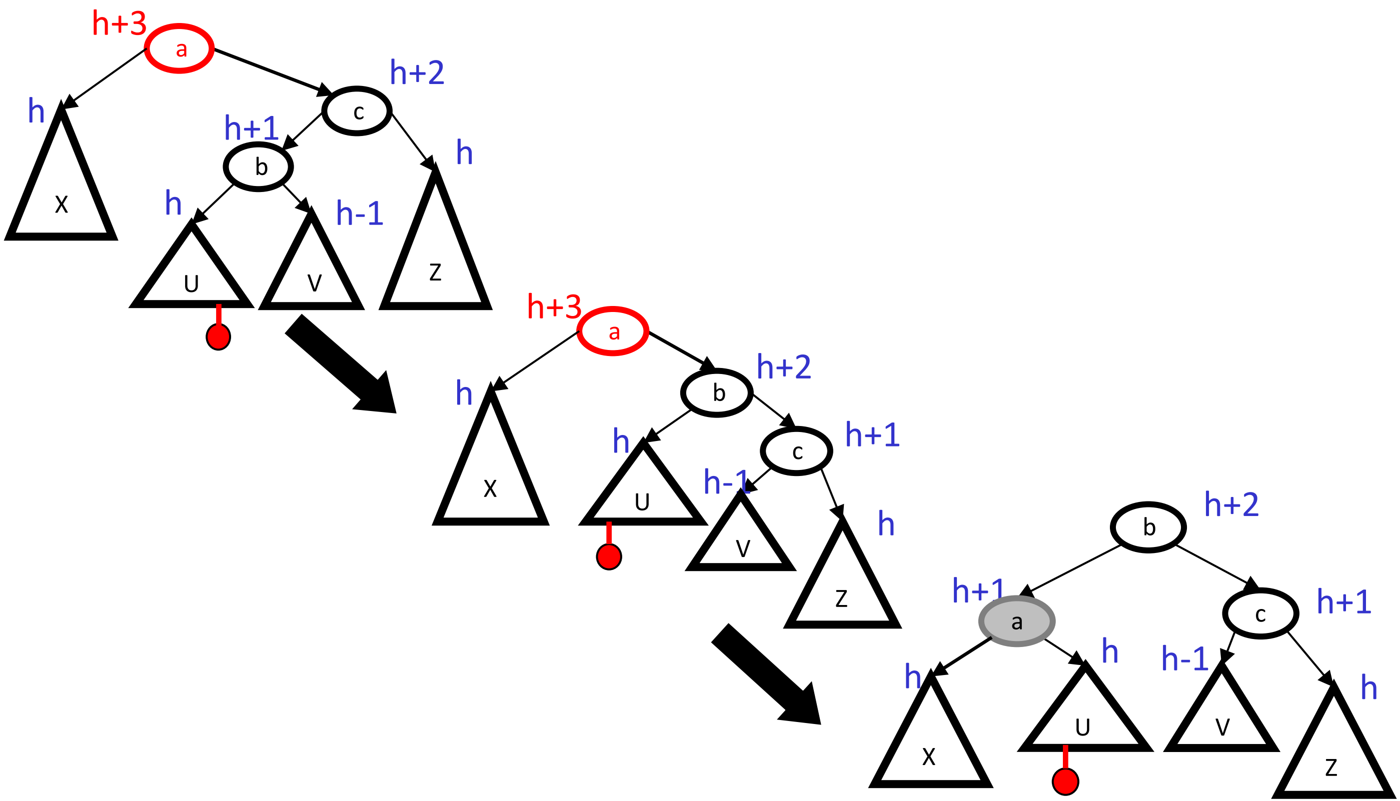
DoubleRotation:
1. Rotate problematic child and grandchild
2. Then rotate between self and new child



Case #3: Adoption



Case #3: Why It Works



Case #3 ≈ Case #2

The insertion is in the:
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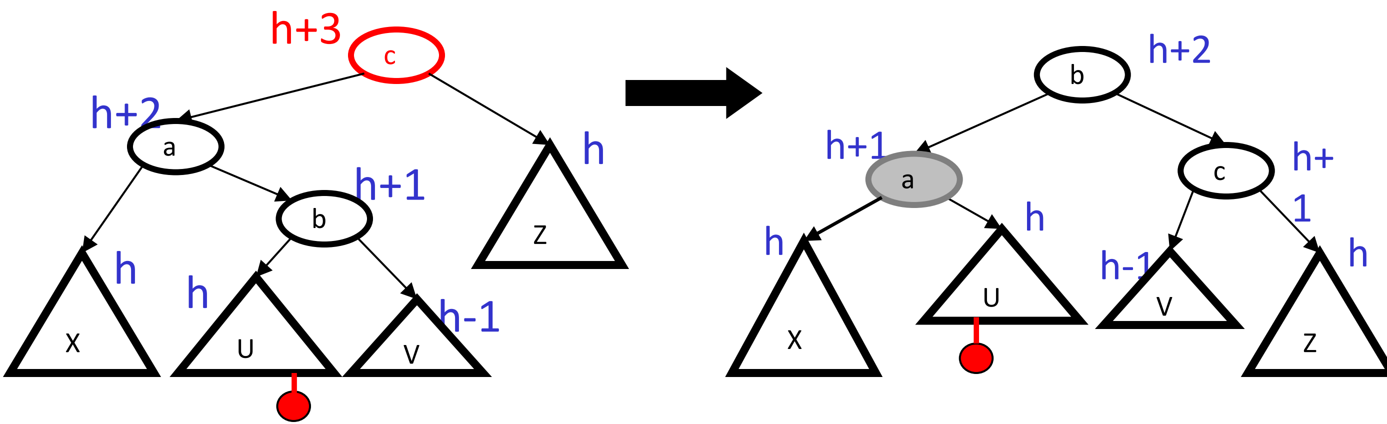
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*

- Mirror image of right-left
  - Again, no new concepts, only new code to write



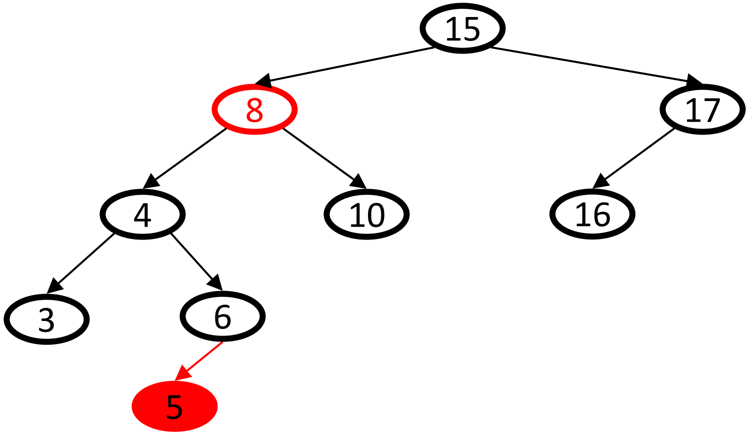
AVL add(): Summary

- Insert as if a BST
- Check back up path for imbalance, which will be 1 of 4 cases:
  1. node's left-left grandchild is too tall
  2. node's left-right grandchild is too tall
  3. node's right-left grandchild is too tall
  4. node's right-right grandchild is too tall
- *<u>Only one case occurs</u>* because tree was balanced before insert
- After the appropriate rotation, the smallest-unbalanced subtree has the same height as before insertion
  - So all ancestors are now balanced

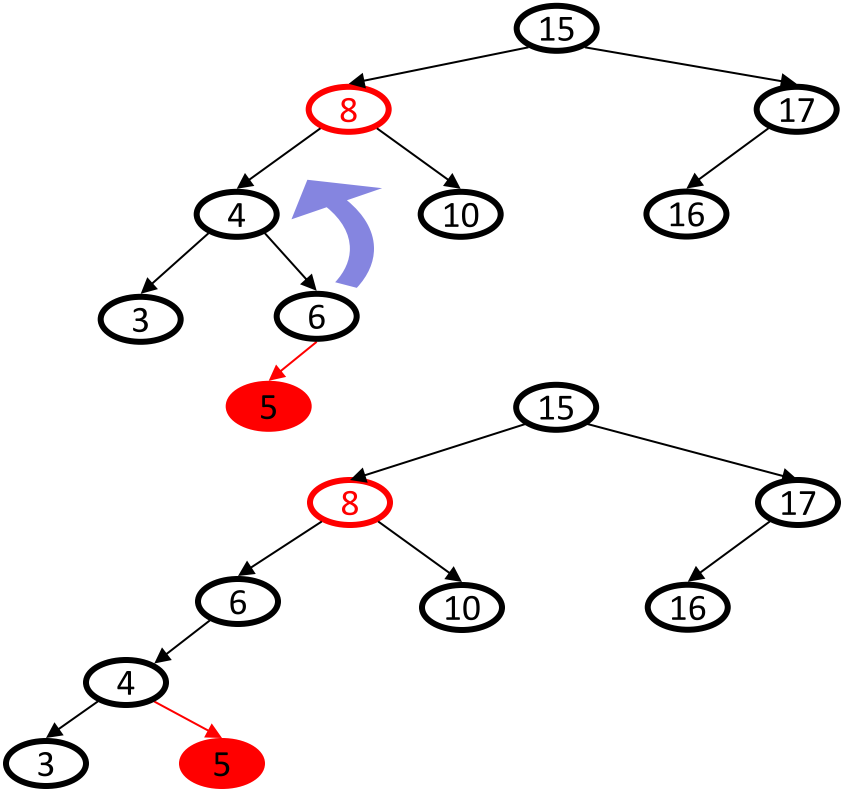
Lecture Outline

- AVL Tree
  - Bounding a BST's height
  - Find
  - Add
    - ***(Add Exercises)***
  - Remove
  - Wrapup

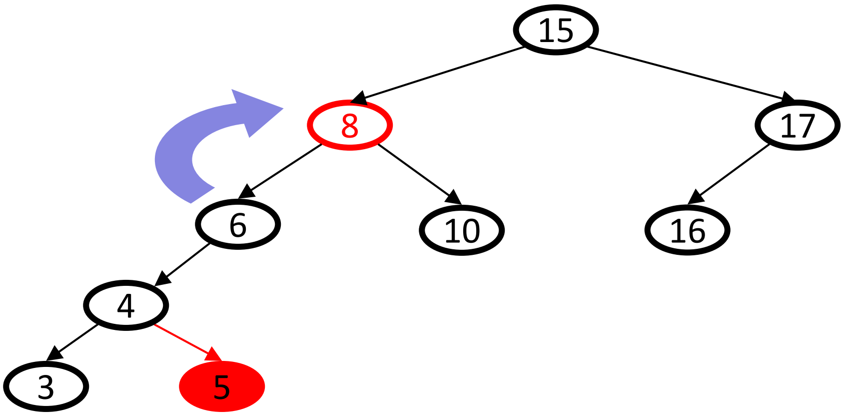
Double Rotation: Example (1 of 3)

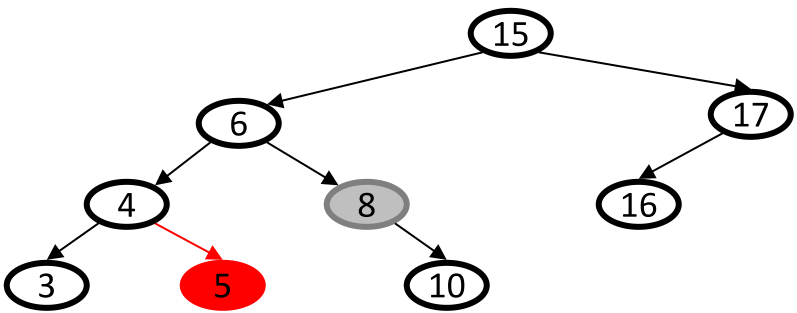


Double Rotation: Example (2 of 3)



Double Rotation: Example (3 of 3)



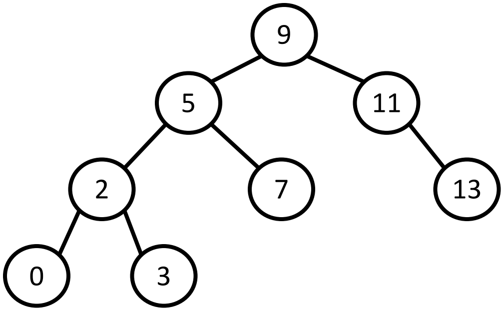


add() into an AVL tree

- add(a)
- add(b)
- add(e)
- add(c)
- add(d)

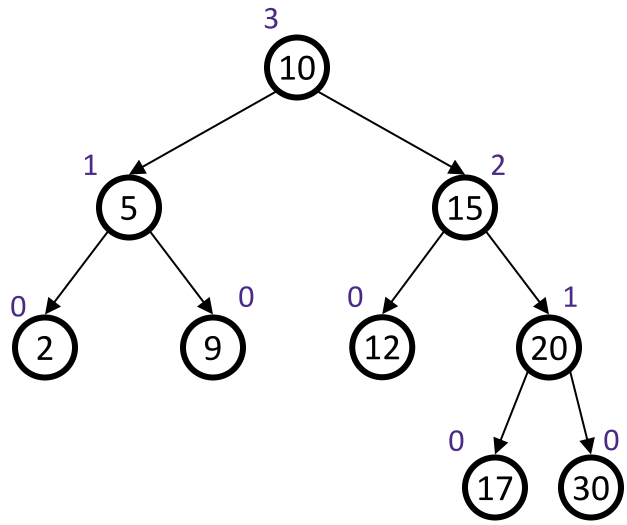
Single and Double Rotations

- Inserting which integer values would cause this tree to need a:
  - Single Rotation?
  - Double Rotation?
  - No Rotation?



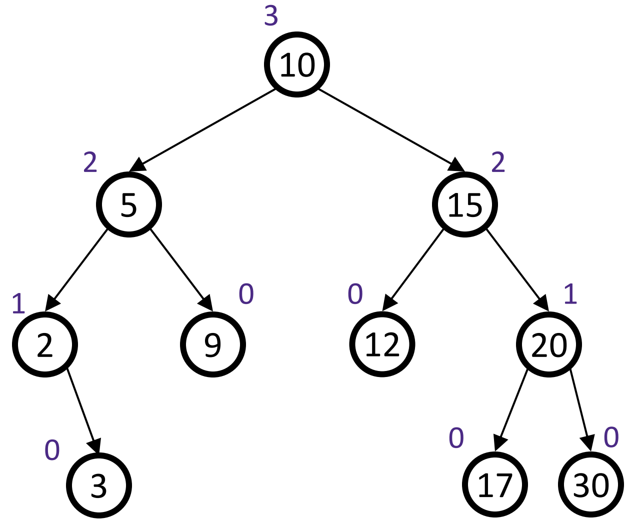
Add Sequence (1 of 2)

- add(3)
  - Is the resultant tree balanced?
  - If not, how would you fix it?



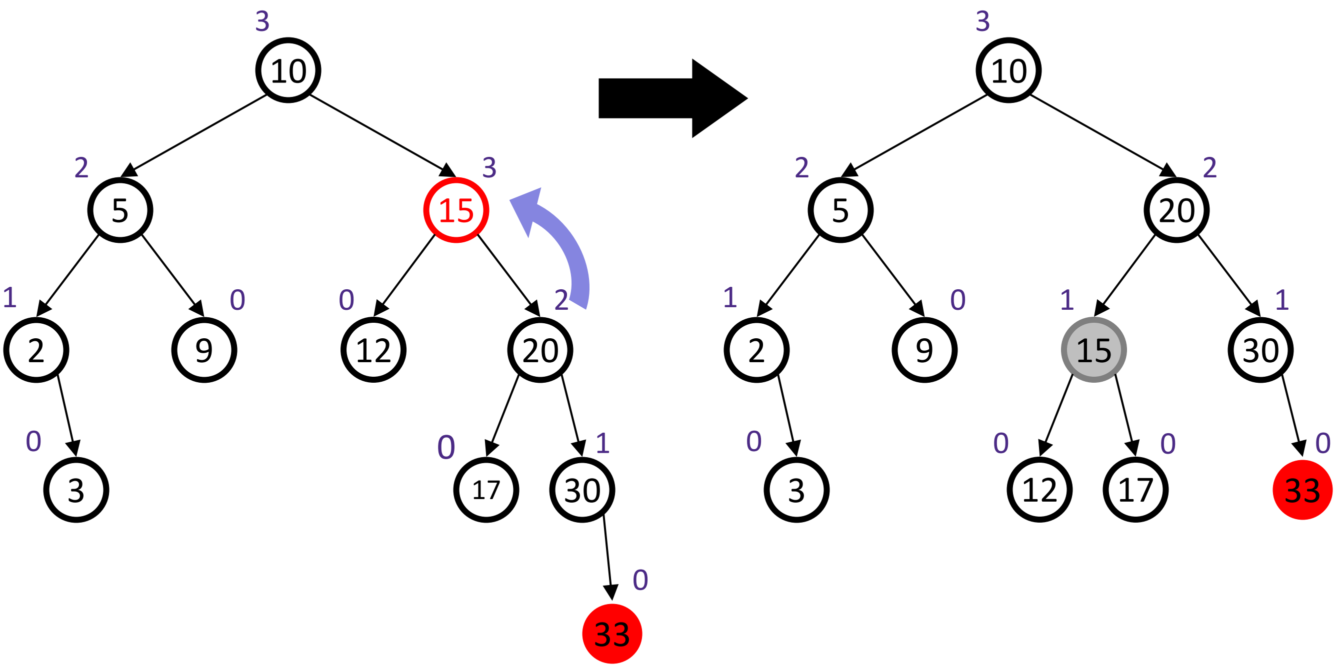
Add Sequence (2 of 2)

- Next, add(33)
  - Is the resultant tree balanced?
  - If not, how would you fix it?



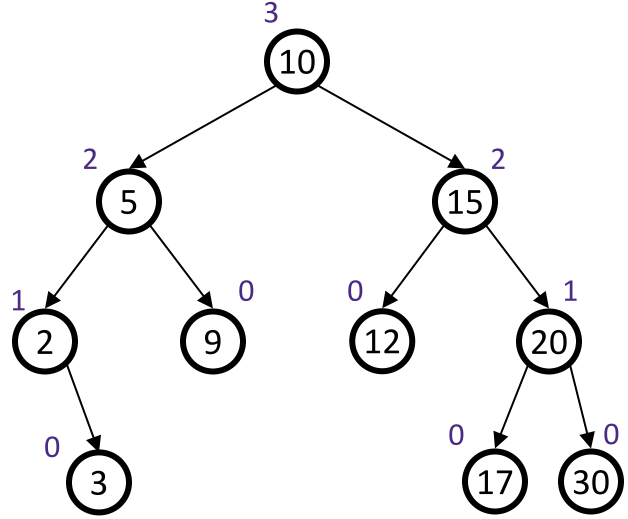
Answer

- Single rotation to the rescue!



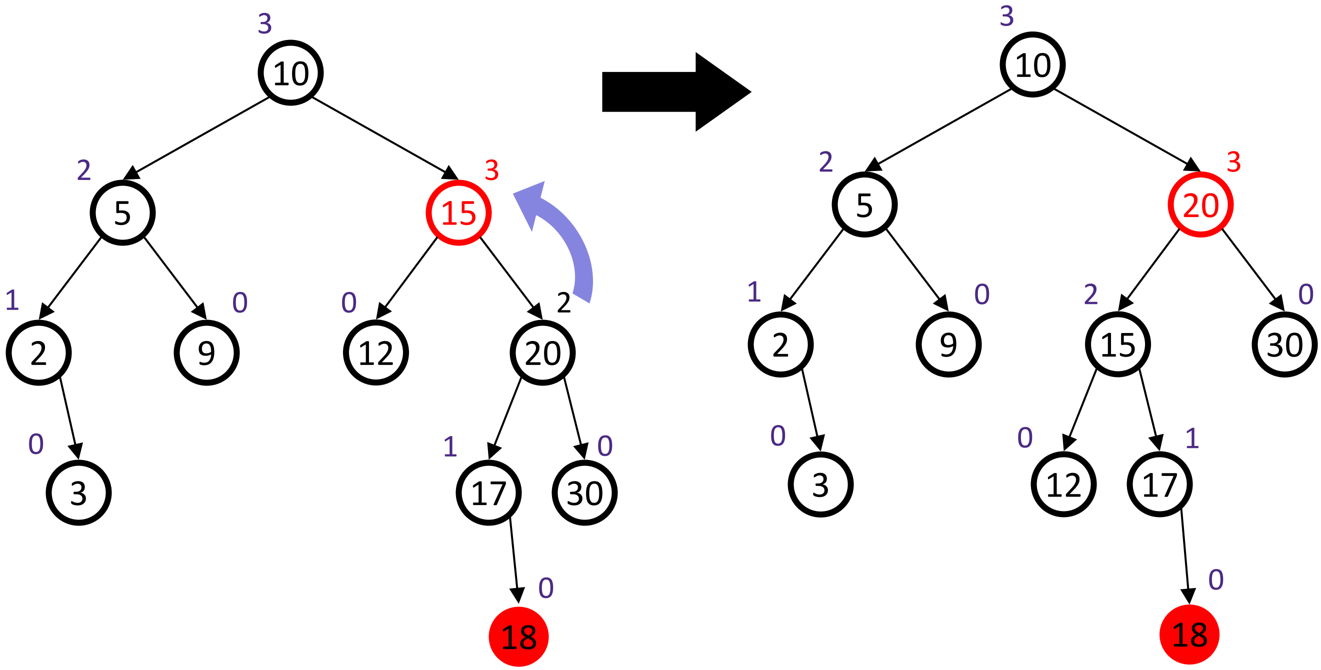
Harder Add Sequence (1 of 2)

- add(18)
  - Is the resultant tree balanced?
  - If not, how would you fix it?



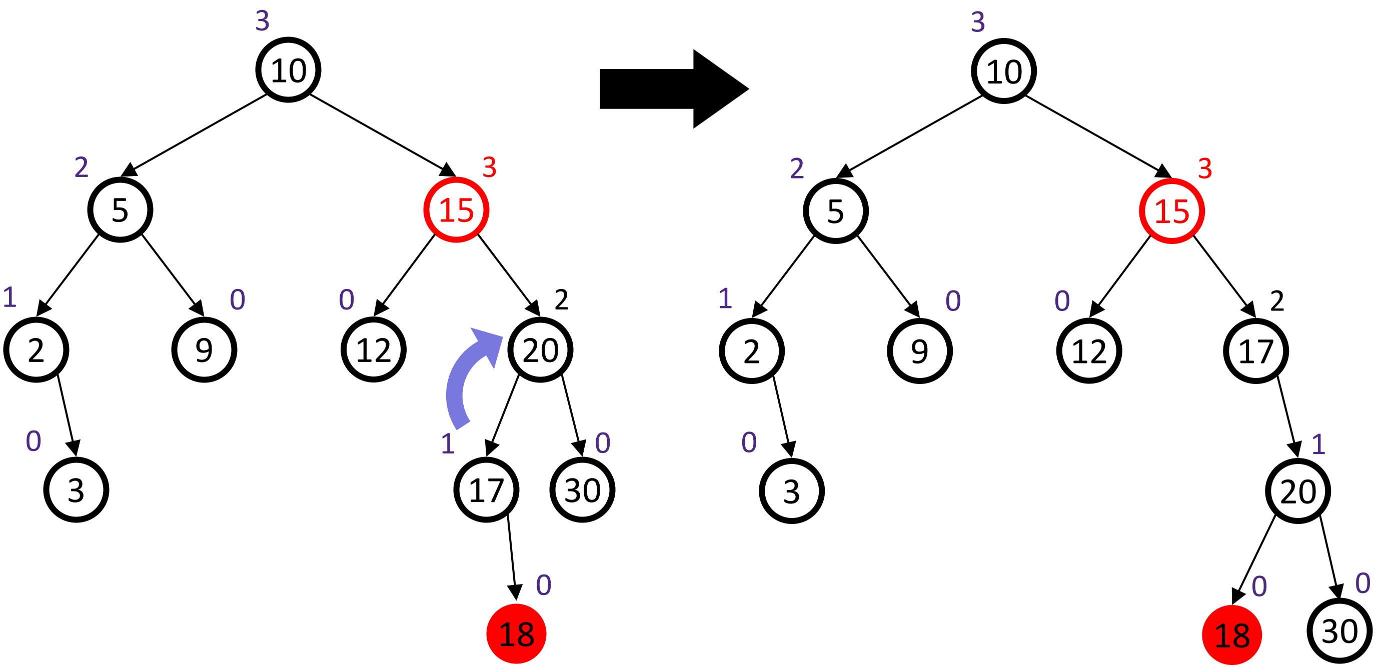
Harder Add Sequence (2 of 2)

- Single Rotation doesn't work



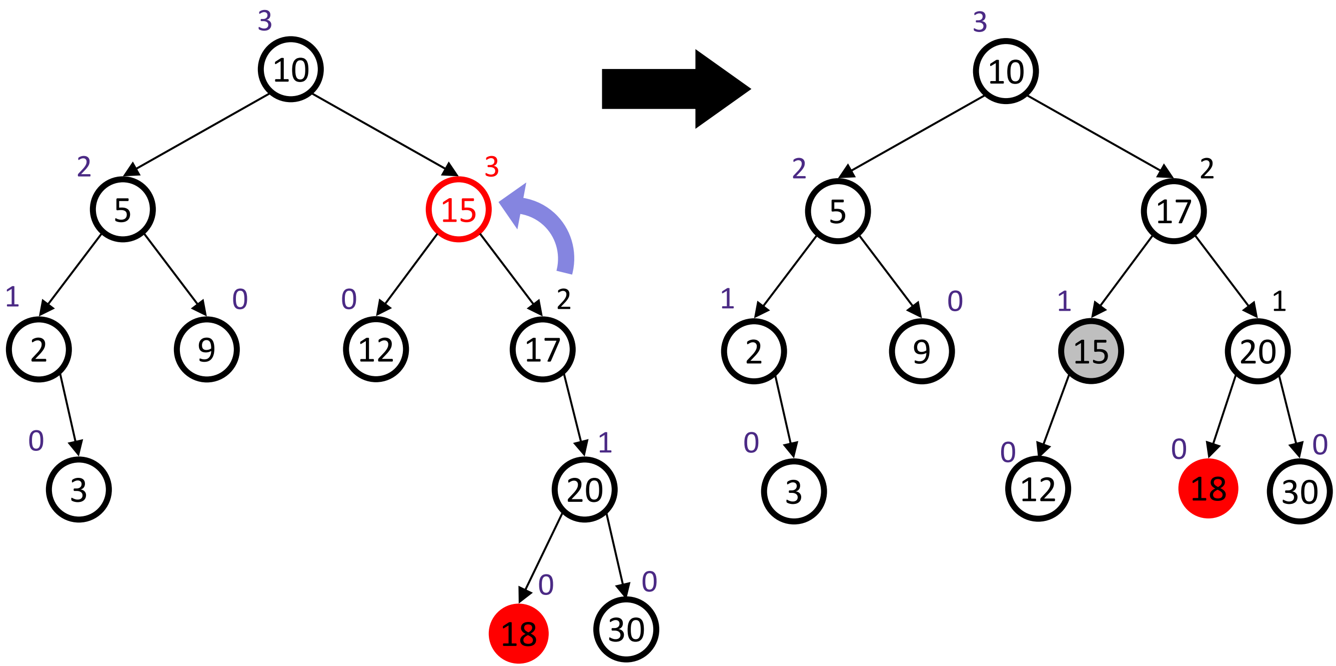
Answer (1 of 2)

- Double rotation, part 1



Answer (2 of 2)

- Double rotation, part 2

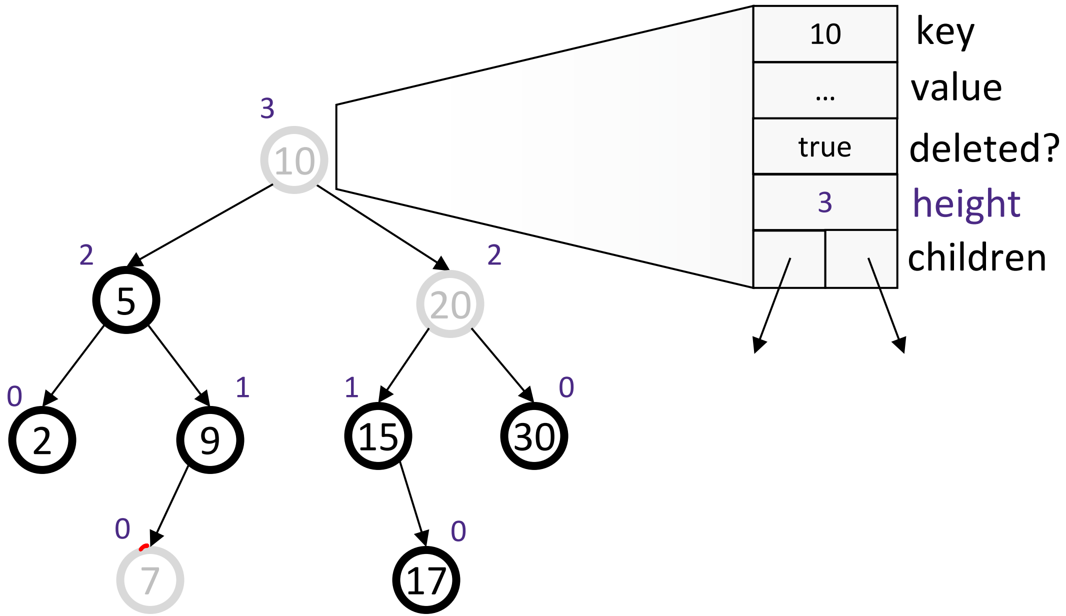


Lecture Outline

- AVL Tree
  - Bounding a BST's height
  - Find
  - Add
  - **Remove**
  - Wrapup

AVL Remove

- The "easy way" is lazy deletion
- The "hard way" will result in many imbalance cases
  - Only do this if you're feeling ambitious



Lecture Outline

- AVL Tree
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  - Add
  - Remove
  - **Wrapup**

AVL Tree Wrapup

- AVL find:
  - Same as BST find
  - Worst-case complexity:
    - Tree is balanced!

- AVL add:
  - First BST add, then check balance and potentially “fix” the AVL tree
  - Four different imbalance cases
  - Worst-case complexity:
    - Tree starts and ends balanced
    - A rotation is O(1) and there’s an O(log n) path to root

AVL Tree Wrapup

- AVL remove
  - We suggest lazy deletion
    - Worst-case complexity:
  - Deletion requires more rotations than insert; but worst-case complexity still O(log n)

Pros and Cons of AVL Trees

- Arguments for AVL trees:
  - All operations are logarithmic worst-case because trees are always balanced
  - Height rebalancing adds no more than a constant factor to the speed of add and remove

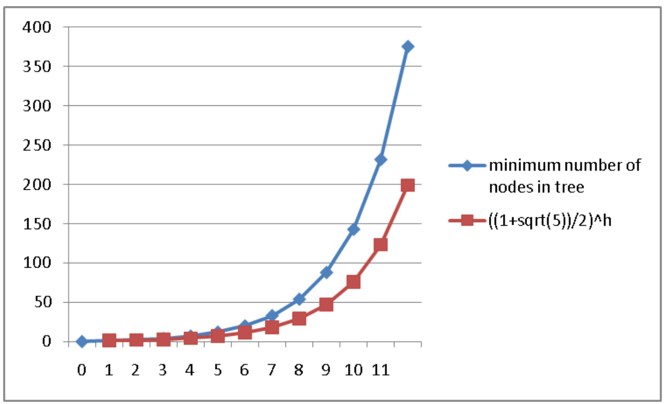
- Arguments against AVL trees:
  - Difficult to program and debug
  - Additional space for the `height` and `deleted?` fields
  - Asymptotically faster, but rebalancing takes time
  - Compared to other balanced BSTs (eg, Red-Black trees), the constants aren't great
  - Most large data sets require database-like systems on disk, and thus use other structures (e.g., B-trees, our next data structure)

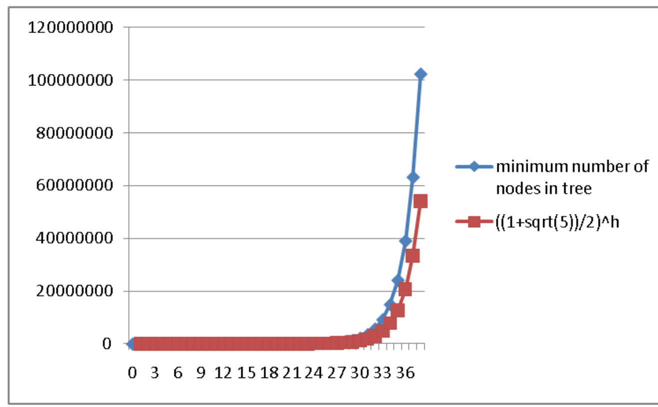
Lecture Outline

- AVL Tree
  - Bounding a BST's height
  - *(Proving the AVL tree's height bound)*
  - Find
  - Add
    - *(Add Exercises)*
  - Remove
  - **Proof of height bound**

Before We Prove It

- Good intuition from plots comparing:
  1. S(h) computed directly from the definition
  2. $((1+\sqrt{5})/2)^h \approx 1.62^h$
- S(h) is always bigger, up to trees with huge # of nodes
  - Graphs aren't proofs, so let's prove it



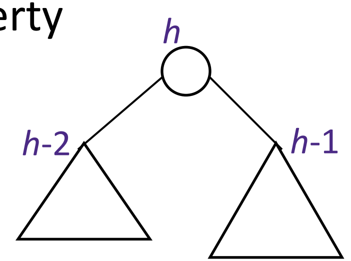


The Proof Outline

Let $S(h)$ = the min # of nodes in an AVL tree of height h

- If we can prove that $S(h)$ grows exponentially in h, then a tree with n nodes has a logarithmic height

- Step 1: Define $S(h)$ inductively using AVL property
  - $S(-1)=0$, $S(0)=1$, $S(1)=2$
  - $S(h) = 1 + S(h-1) + S(h-2)$ for $h \geq 1$



- Step 2: Show this recurrence grows really fast
  - Similar to Fibonacci numbers
  - Can prove for all h, $S(h) > \phi^h - 1$ where ϕ is the golden ratio, $(1+\sqrt{5})/2 \approx 1.62$
  - Growing faster than 1.62^h is "plenty exponential"

Interlude: The Golden Ratio

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$$

This is a special number

- Aside: Since the Renaissance, many artists and architects have proportioned their work (e.g., length:height) to approximate the *golden ratio*: If `(a+b)/a = a/b`, then `a = ϕb`
- We will need one special arithmetic fact about ϕ :

$$\begin{aligned}
\phi^2 &= ((1+5^{1/2})/2)^2 \\
&= (1 + 2*5^{1/2} + 5)/4 \\
&= (6 + 2*5^{1/2})/4 \\
&= (3 + 5^{1/2})/2 \\
&= 1 + (1 + 5^{1/2})/2 \\
&= 1 + \phi
\end{aligned}$$

The Proof (1 of 2)

$S(-1)=0, \quad S(0)=1, \quad S(1)=2$
$S(h)=1 + S(h-1) + S(h-2) \quad \text{for } h \geq 1$

Theorem: For all $h \geq 0$, $S(h) > \phi^h - 1$

Proof: By induction on h

Base cases:

$S(0) = 1 > \phi^0 - 1 = 0$

$S(1) = 2 > \phi^1 - 1 \approx 0.62$

The Proof (2 of 2)

```
S(-1)=0,   S(0)=1,   S(1)=2
S(h)=1 + S(h-1) + S(h-2)   for h≥ 1
```

Theorem: For all $h \geq 0$, $S(h) > \phi^h - 1$

Proof: By induction on h

Inductive case ($k > 1$):

Show that $S(k+1) > \phi^{k+1}-1$, assuming $S(k) > \phi^k-1$ and $S(k-1) > \phi^{k-1} - 1$

S(*k*+1)	$= 1 + S(k) + S(k-1)$	*by definition of S*
	> $1 + (\phi^k - 1) + (\phi^{k-1} - 1)$	*by induction*
	$= \phi^k + \phi^{k-1} - 1$	*by arithmetic (1-1=0)*
	$= \phi^{k-1} (\phi + 1) - 1$	*by arithmetic (factor ϕ^{k-1})*
	$= \phi^{k-1} \phi^2 - 1$	*by special property of ϕ*
	$=$ **$\phi^{k+1} - 1$**	*by arithmetic (add exponents)*