CSE 332: Data Structures and Parallelism

Section 4: Balanced Trees Solutions

0. The $\mathcal{A}$BC's of $\mathcal{A}$VL Trees

What are the constraints on the data types you can store in an AVL tree? When is an AVL tree preferred over another dictionary implementation, such as a HashMap?

Solution:

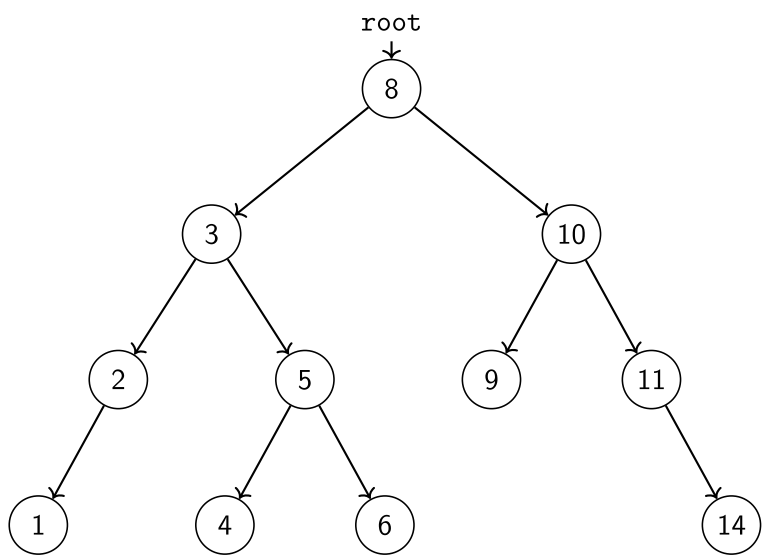
AVL trees are similar to TreeMaps. The constraint is that they require that keys be comparable. The value type can be anything, just like any other dictionary.

A perk over HashMaps is that keys can be iterated over in sorted order. AVL trees are also preferred over BSTs when there's a possibility of sorted input because the balancing prevents the worst case of a degenerate tree.

1. Let's Plant an AVL Tree.

Insert 10, 4, 5, 8, 9, 6, 11, 3, 2, 1, 14 into an initially empty AVL Tree.

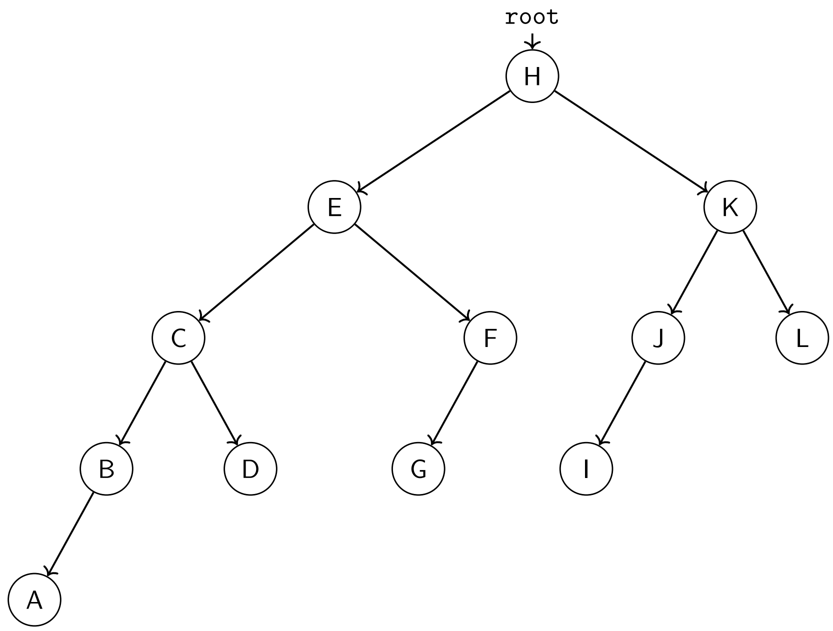
Solution:



2. MinVL Trees

Draw an AVL tree of height 4 that contains the minimum possible number of nodes.

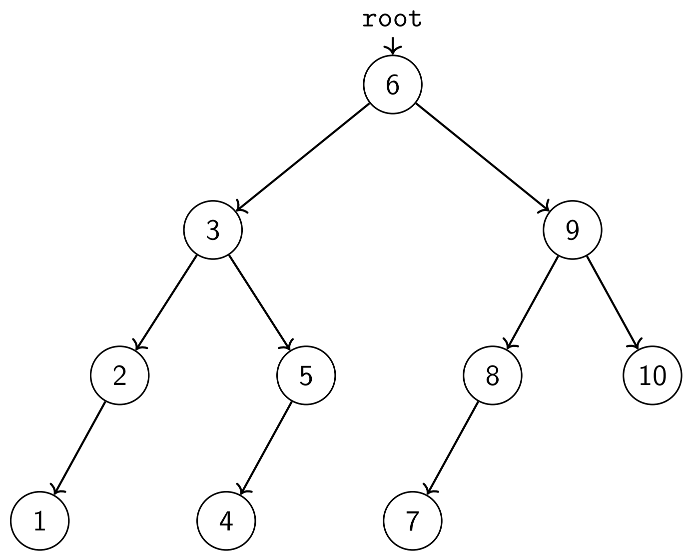
Solution:



3. AVL Trees

Insert 6, 5, 4, 3, 2, 1, 10, 9, 8, 7 into an initially empty AVL Tree.

Solution:



4. The ABC's of B-Trees

(a) What properties must a B-tree of n values have with given values for M and L?

Solution:

(a) B-Tree order property:

i. Every subtree between keys a and b contains all data x where $a \leq x < b$

ii. The values in the leaves are in key sorted order

iii. The keys in the internal nodes are stored in sorted order

(b) B-Tree structure property:

i. If $n \leq L$, the root is a leaf with n values, otherwise the root is an internal node that must have between 2 and M children

ii. All internal nodes must have between $\left\lceil \frac{M}{2} \right\rceil$ and M children (i.e., half-full)

iii. All leaf nodes must have between $\left\lceil \frac{L}{2} \right\rceil$ and L key-value pairs (i.e., half-full)

iv. All leaf nodes must be at the same depth

(b) Give an example of a situation that would be a good job for a B-tree. Furthermore, are there any constraints on the data that B-trees can store?

Solution:

B-trees are most appropriate for very, very large data stores, like databases, where the majority of the data lives on disk and cannot possibly fit into RAM all at once.

B-trees require orderable keys. B-trees are typically not implemented in Java because because what makes them worthwhile is their precise management of memory.

5. Implement a B-Tree? Nah, Let's Analyze!

Given the following parameters for a B-Tree with a page size of 256 bytes:

- Key Size = 8 bytes
- Pointer Size = 2 bytes
- Data Size = 14 bytes per record (includes the key)

Assuming that M and L were chosen appropriately, what are M and L? Recall that M is defined as the maximum number of pointers in an internal node, and L is defined as the maximum number of values in a leaf node. Give a numeric answer and a short justification based on two equations using the parameter values above.

Solution:

We start by defining the following variables.

- 1 page on disk is b bytes
- Keys are k bytes
- Pointers are t bytes
- Key/Value pairs are v bytes

We know that the amount of memory used by one leaf node is vL and the amount of memory used by one internal node is $tM + k(M-1)$. We want select values for M and L such that both equations are $\leq b$.

If we solve both equations for M and L, we obtain $M = \left\lfloor \frac{b+k}{t+k} \right\rfloor$ and $L = \left\lfloor \frac{b}{v} \right\rfloor$

Plugging in the given values, we get $M = \left\lfloor \frac{256+8}{2+8} \right\rfloor = 26$ and $L = \left\lfloor \frac{256}{14} \right\rfloor = 18$

6. Oh, B-Trees

Find a tight upper bound on the *worst case runtime* of these operations on a B-tree. Your answers should be in terms of L, M, and n.

(a) Insert a key-value pair

(b) Look up the value of a key

(c) Delete a key-value pair

Solution:

Insertion, Deletion The steps for insert and delete are similar and have the same worst case runtime.

(a) Find the leaf: $\mathcal{O}(\lg(M)\log_M(n))$. (For more details, see the next solution.)

(b) Insert/remove in the leaf – there are L elements, essentially stored in an array: $\mathcal{O}(L)$

(c) Split a leaf/merge neighbors: $\mathcal{O}(L)$

(d) Split/merge parents, in the worst case going up to the root: $\mathcal{O}(M\log_M(n))$

The total cost is then $\lg(M)\log_M(n) + 2L + M\log_M(n)$.

We can simplify this to a worst-case runtime $\mathcal{O}(L + M\log_M(n))$ by combining constants and observing that $M\log_M(n)$ dominates $\lg(M)\log_M(n)$. Note that in the average case, splits for any reasonably-sized B-tree are rare, so we can amortize the work of splitting over many operations.

However, if we're using a B-tree, it's because what concerns us the most is the penalty of disk accesses. In that case, we might find it more useful to look at the worst-case number of disk lookup operations in the B-tree, which is $\mathcal{O}(\log_M(n))$.

Look up (a) We must do a binary search on a node containing M pointers, which takes $\mathcal{O}(\lg(M))$ time, once at each level of the tree.

(b) There are $\mathcal{O}(\log_M(n))$ levels.

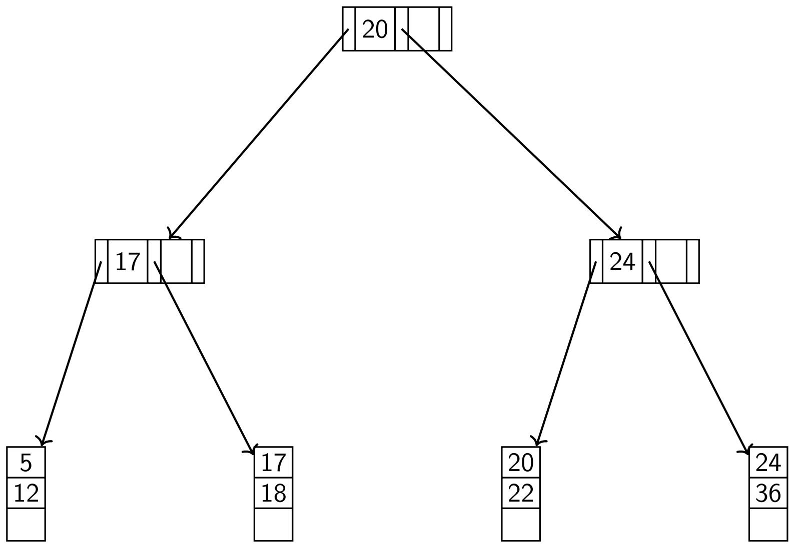
(c) We must do a binary search on a leaf of L elements, which takes $\mathcal{O}(\lg(L))$ time.

(d) Putting it all together, a tight bound on the runtime is $\mathcal{O}(\lg(M)\log_M(n) + \lg(L))$.

7. B-Trees

(a) Insert the following into an empty B-Tree with $M = 3$ and $L = 3$: $12, 24, 36, 17, 18, 5, 22, 20$.

Solution:



(b) Delete $17, 12, 22, 5, 36$

Solution:

18
20
24

(c) Given the following parameters for a B-Tree with $M = 11$ and $L = 8$

- Key Size = 10 bytes
- Pointer Size = 2 bytes
- Data Size = 16 bytes per record (includes the key)

Assuming that M and L were chosen appropriately, what is the likely page size on the machine where this implementation will be deployed? Give a numeric answer and a short justification based on two equations using the parameter values above.

Solution:

We use the following two equations to find M and L to fit as best as possible in the page size, where:

- 1 page on disk is p bytes
- Keys are k bytes
- Pointers are t bytes
- Key/Value pairs are v bytes

$M = \left\lfloor \frac{p+k}{t+k} \right\rfloor$ and $L = \left\lfloor \frac{p}{v} \right\rfloor$

Plugging in the given values, we get:

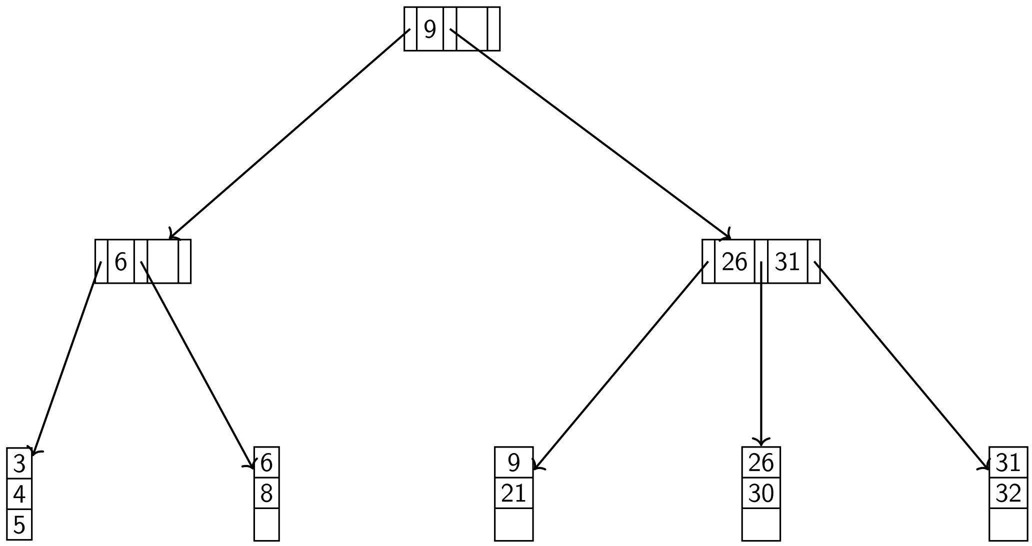
$M = \left\lfloor \frac{p+10}{2+10} \right\rfloor$ and $L = \left\lfloor \frac{p}{16} \right\rfloor$

And solving for p gives us an answer of 128 bytes.

8. It's Fun to B-Trees!

(a) Insert the following into an empty B-Tree with $M = 3$ and $L = 3$: $3, 32, 9, 26, 6, 21, 8, 4, 5, 30, 31$.

Solution:



(b) Delete $4, 5, 21, 9, 31, 3, 26, 8$

Solution: