Kruskal's Algorithm; Disjoint Sets

CSE 332 Spring 2021

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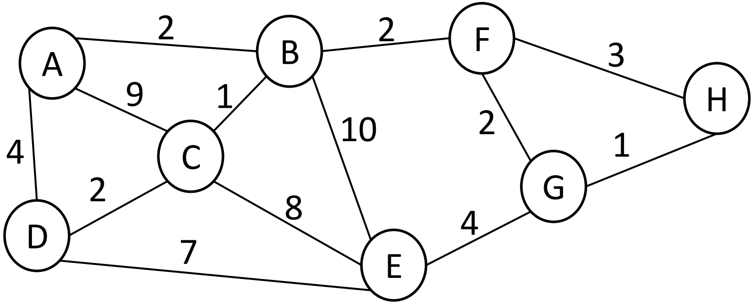
- How many times do you need to take the log of 20,035,299,304,068,464,649,790,723,515,602,557,504,478,254,755,697,514,192,650,169,737 to get to a value <= 1?
  - Hint: that value is equal to 2^{65536}, and 65536 = 2^{16}

- *Bonus*: find an MST using Prim's Algorithm on this graph:



Lecture Outline

- **Disjoint Sets ADT (aka Union/Find ADT)**

- Kruskal’s Algorithm, for realz
    - Review and Example
    - Correctness Proof

- Up-Trees Data Structure
    - Representation
    - Optimization: Weighted Union
    - Optimization: Path Compression

Disjoint Sets ADT (1 of 2)

Disjoint Sets ADT. A collection of elements and sets of those elements.

- An element can only belong to a single set.
- Each set is identified by a unique id.
- Sets can be combined/ connected/ unioned.

- The Disjoint Sets ADT has two operations:
  - find(e): gets the id of the element's set
  - union(e1, e2): combines the set containing e1 with the set containing e2
- Example: ability to travel to drive to a country
  - union(france, germany)
  - union(spain, france)
  - find(spain) == find(germany)?
  - union(england, france)

Disjoint Sets ADT (2 of 2)

- Applications include percolation theory (computational chemistry) and …. Kruskal’s algorithm

- Simplifying assumptions
  - We can map elements to indices quickly
  - We know all the items in advance; they’re all disconnected initially

- Later this lecture, we’ll see:
  - We can do union() in constant time
  - We can get find() to be ***amortized*** constant time
    - Worst case O(log n) for an individual find operation

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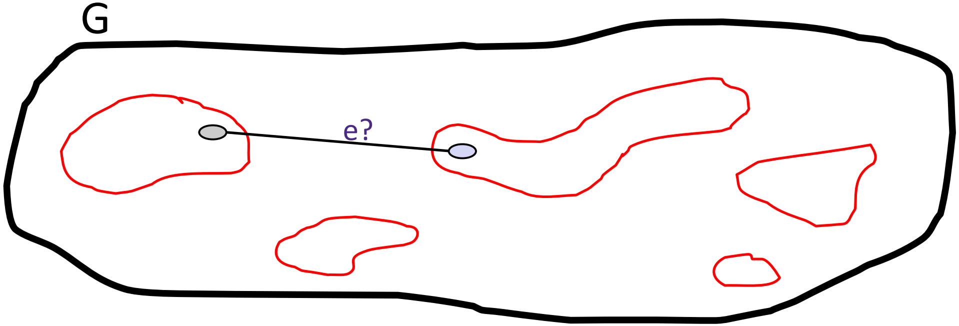
Kruskal's Algorithm

- Kruskal's thinks edge by edge
  - Eg, start from lightest edge and consider by increasing weight
  - Compare against Dijkstra's and Prim's, which think vertex by vertex

- *Outline*:
  - Start with a ***forest*** of |V| MSTs
  - Successively connect them ((ie, eliminate a tree) by adding edges
  - Do not add an edge if it creates a cycle



Kruskal’s Algorithm: Pseudocode

```
kruskals(Graph g) {
  mst = {}
  forests = buildDisjointSets(g.vertices)
  numforests = g.vertices
  edges = buildHeap(g.edges)

  while (numForests 1):
    e = edges.deleteMin()
    u_id = forests.find(e.u)
    v_id = forests.find(e.v)
    if (u_id != v_id):
      mst.addEdge(e)
      forests.union(e.u, e.v)
      numforests--
}
```

|E| deleteMin()s

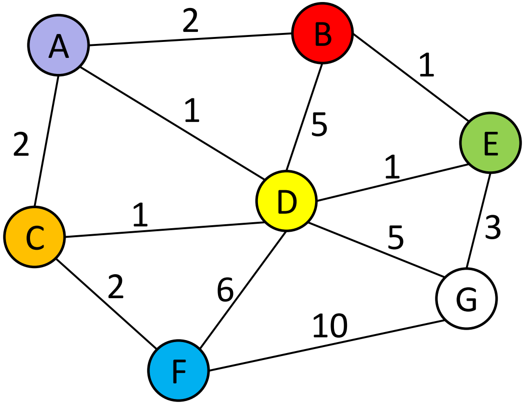
2|E| find()s

|V| union()s

Runtime: $|E|(\log|E| + 2\log|V| + 1) + |V|(1 + 1 + 1) \in O(|E|\log|V| + |V|\log|V|)$

However, since we know $E \in O(|V|^2)$, *runtime* $\in O(|E|\log|V|)$

Kruskal's Algorithm: Example

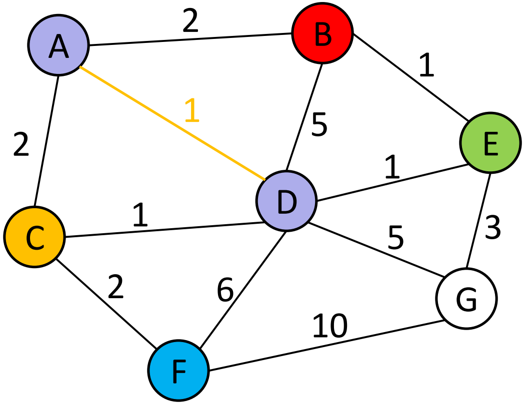


Weight	Edges
1	(A,D), (C,D), (B,E), (D,E)
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:

Num Trees: 7

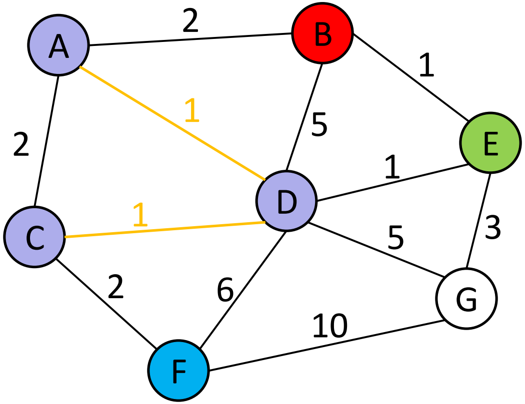
Kruskal's Algorithm: Example



Weight	Edges
1	*(A,D),* (C,D), (B,E), (D,E)
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D)
Num Trees: 6

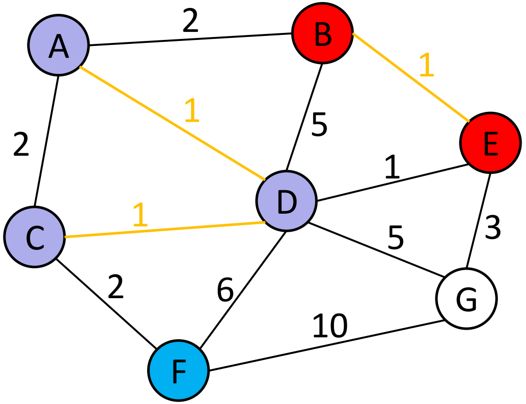
Kruskal's Algorithm: Example



Weight	Edges
1	*(A,D), (C,D),* (B,E), (D,E)
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D), (C, D)
Num Trees: 5

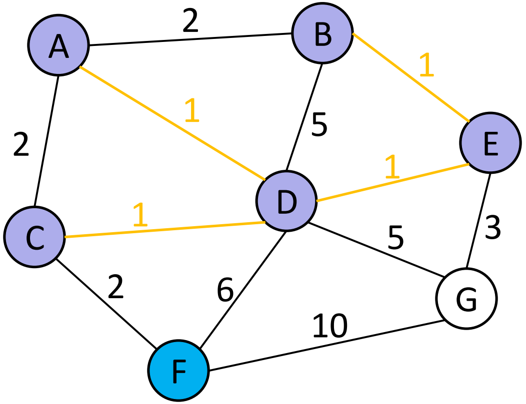
Kruskal's Algorithm: Example



Weight	Edges
1	*(A,D), (C,D), (B,E),* (D,E)
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D), (C, D), (B, E)
Num Trees: 4

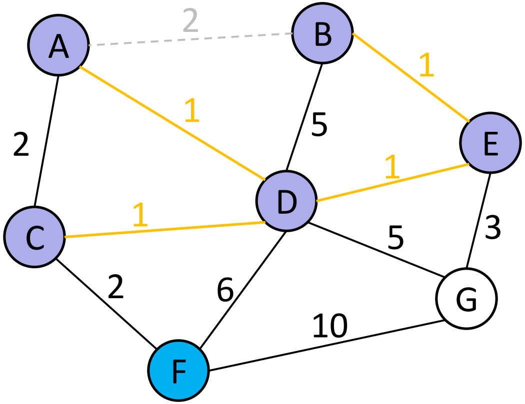
Kruskal’s Algorithm: Example



Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D), (C, D), (B, E), (D, E)
Num Trees: 3

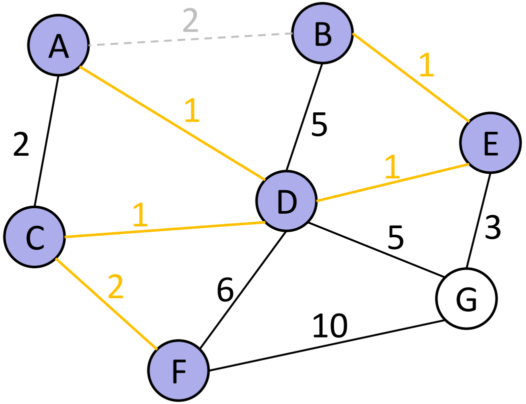
Kruskal's Algorithm: Example



Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	*(A,B),* (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D), (C, D), (B, E), (D, E)
Num Trees: 3

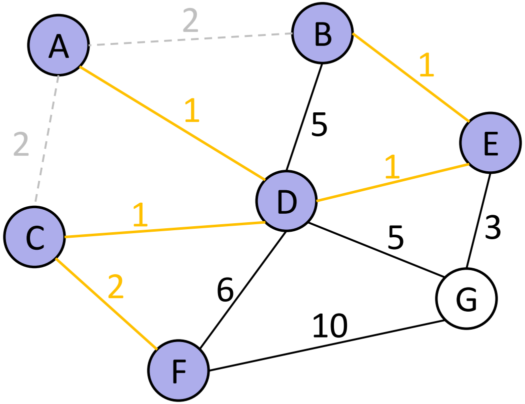
Kruskal's Algorithm: Example



Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	*(A,B), (C,F),* (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D), (C, D), (B, E), (D, E), (C, F)
Num Trees: 2

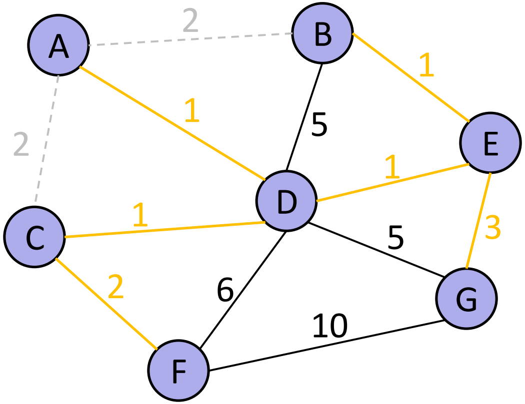
Kruskal’s Algorithm: Example



Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	*(A,B), (C,F), (A,C)*
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D), (C, D), (B, E), (D, E), (C, F)
Num Trees: 2

Kruskal’s Algorithm: Example



Hark, an MST!!!

Total Cost: 9

Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	*(A,B), (C,F), (A,C)*
3	*(E,G)*
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D), (C, D), (B, E), (D, E), (C, F), (E, G)
Num Trees: 1

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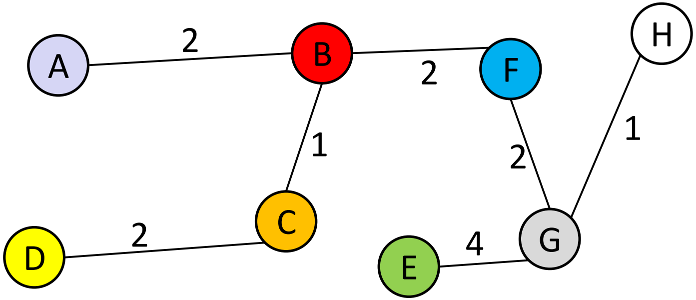
- Find an MST in this graph using Kruskal's Algorithm:



Weight	Edges
1	(B, C), (G, H)
2	(A,B), (B, F), (C,D), (F,G)
4	(E,G)

MST:

Kruskal’s Algorithm: Demos and Visualizations

- Prim’s Visualization
    - https://www.youtube.com/watch?v=6uq0cQZOyoY
    - Prim’s jumps around the fringe, adding edges by edge weight

- Kruskal’s Visualization:
    - https://www.youtube.com/watch?v=ggLyKfBTABo
    - Kruskal’s jumps around the graph – not just the fringe – because it chooses edges by edge weight independent of the “tree under construction”

- Conceptual demo:
    - https://docs.google.com/presentation/d/1RhRSYs9Jbc335P24p7vR-6PLXZUl-1EmeDtqieL9ad8/present?ueb=true&slide=id.g375bbf9ace_0_645

Lecture Outline

- Disjoint Sets ADT (aka Union/Find ADT)
- Kruskal’s Algorithm, for realz
  - Review and Example
  - **Correctness Proof**
- Up-Trees Data Structure
  - Representation
  - Optimization: Weighted Union
  - Optimization: Path Compression

Kruskal’s Algorithm: Correctness

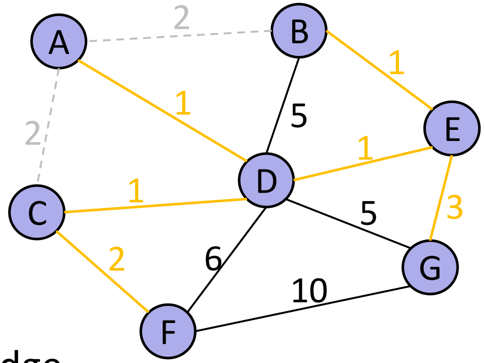
- Kruskal’s algorithm is clever, simple, and efficient
    - But does it generate a minimum spanning tree?

- *First*: it generates a spanning tree
    - To show treeness, need to show lack of cycles
    - To show that it’s a single tree, need to show it’s connected
    - To show spanningness, need to show that all vertices are included

- *Second*: there is no spanning tree with lower total cost …

Kruskal’s Output is a Spanning Tree (1 of 2)

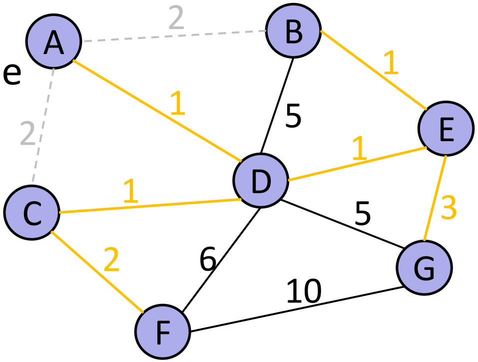
- To show treeness, need to show lack of cycles
  - *By definition*: Kruskal’s doesn’t add an edge if it creates a cycle



- To show that it’s a single tree, need to show it’s connected
  - *By contradiction*: suppose Kruskal’s generates >1 tree. Since the original graph **G** was connected, there exists an edge in **G** that connects Kruskal’s trees. Adding this edge would not create a cycle, so Kruskal’s would have included it. **CONTRADICTION**

Kruskal’s Output is a Spanning Tree (2 of 2)

- To show spanningness, need to show that all vertices are included
  - *By contradiction*: suppose Kruskal’s tree **T** does not include *any* edges adjacent to some vertex **v**. Since the original graph **G** was connected, there exists at least one edge in **G** that is adjacent to **v**. The minimum of these edges would not have created a cycle with T, so Kruskal’s would have included it. **CONTRADICTION**



Kruskal’s Optimality: Inductive Proof Setup

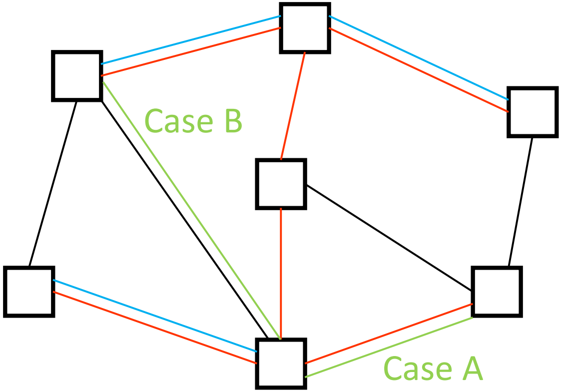
- Let **F** (stands for “forest”) be the set of edges Kruskal has added at some point during its execution.

- *Claim*: **F** is a subset of *one or more* MSTs for the graph
  - *(Therefore, once |F|=|V|-1, we have a single MST)*

- *Proof*: By induction on |**F**|
  - *Base case*: |**F**|=0. The empty set is a subset of all MSTs
  - *Inductive case*: |**F**|=k+1. By induction, before adding the (k+1)th edge (call it **e**), there was some MST **T** such that **F**-{**e**} $\subseteq$ **T** …

Staying a Subset of Some MST

T is "the real" MST
F is Kruskal's output at the k+1th step
e is the the k+1th edge Kruskal's will add

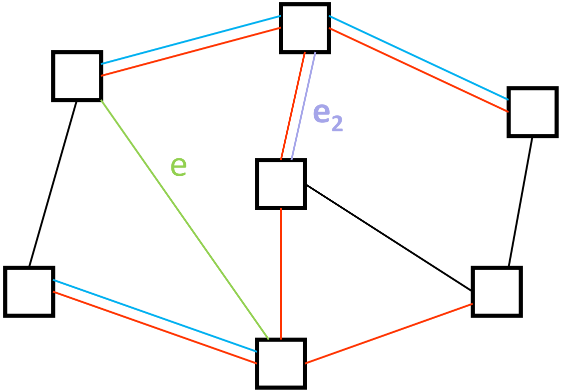


- *Claim*: **F** is a subset of *one or more* MSTs for the graph
- *Things we know so far*:
  - **F-{e}** $\subseteq$ **T**
- *Proof*: Two disjoint cases:
  A. If **{e}** $\subseteq$ **T**, then **F** $\subseteq$ **T** and proof is done
  B. Else, **e** forms a cycle with some simple path (call it **p**) in **T**
    - Must be a cycle since **T** is a spanning tree

Staying a Subset of Some MST

T is "the real" MST
F is Kruskal's output at the k+1th step
e is the "wrong" edge Kruskal's will add
e_2 is an edge in **T** (but not **F**) along a cycle

- *Claim*: **F** is a subset of *one or more* MSTs for the graph
- *Things we know so far*:
  - **F**-{e} ⊆ **T**
  - e forms a cycle with **T**

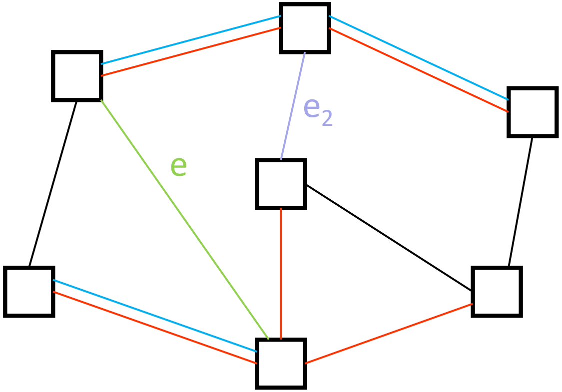


- *New claim*: There is an edge e_2 on **p** such that e_2 is not in **F**
  - Otherwise, Kruskal's would not have added e

T is "the real" MST
F is Kruskal's output at the k+1th step
e is the "wrong" edge Kruskal's will add
e$_2$ is the "right" edge Kruskal's missed/will miss

Staying a Subset of Some MST

- *Claim*: **F** is a subset of *one or more* MSTs for the graph
- *Things we know so far*:
  - **F**-{**e**} $\subseteq$ **T**
  - e forms a cycle with **T**
  - e$_2$ (on **p**) is not in **F**

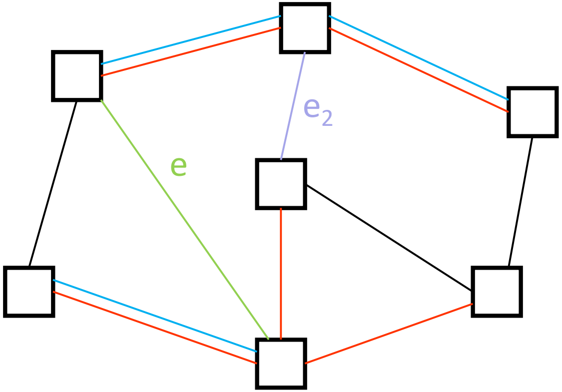


- *New claim*: e$_2$.weight == e.weight
  - If e$_2$.weight > e.weight, then **T** is not an MST
    - **T**-{e$_2$}+{e} is a spanning tree with lower cost. **Contradiction!!**
  - If e$_2$.weight **<** e.weight, then Kruskal's would have already considered e$_2$
    - Would have added it since **F**-{e} has no cycles (**T** has no cycles and **F**-{e} $\subseteq$ **T**)
    - But e$_2$ is not in **F**. **Contradiction!!**

Staying a Subset of Some MST

T is "the real" MST
F is Kruskal's output at the k+1th step
e is the "wrong" edge Kruskal's will add
e_2 is the "right" edge Kruskal's missed/will miss

- *Claim*: **F** is a subset of *one or more* MSTs for the graph
- *Things we know so far*:
  - **F**-{**e**} $\subseteq$ **T**
  - **e** forms a cycle with **T**
  - e_2 (on **p**) is not in **F**
  - e_2.weight == **e**.weight



- *New claim*: **T**-{e_2}+{**e**} is (also) an MST
  - It's a spanning tree because **p-{e_2}+{e}** connects the same nodes as **p**
  - It's minimal because its cost equals cost of **T**, an MST

Done!

- Since **F** $\subseteq$ **T**-{e_2}+{**e**}, **F** is a subset of one or more MSTs

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  - Correctness Proof

- Up-Trees Data Structure
  - **Representation**
  - Optimization: Weighted Union
  - Optimization: Path Compression

Implementing the Disjoint Sets ADT (1 of 2)

- If we have *n* elements, what is the total cost of *m* find()s + ≤ *n-1* union()s?
  - Can we have >n union()s?

- Goal: O(*m*+*n*) total for these operations
  - i.e. O(1) amortized for all operations!

- Is our goal possible?
  - Can get O(1) worst-case union()
  - Would be nice if we could also get O(1) worst-case find(), but…
  - *Known result*: both find() and union() can’t have worst-case O(1)

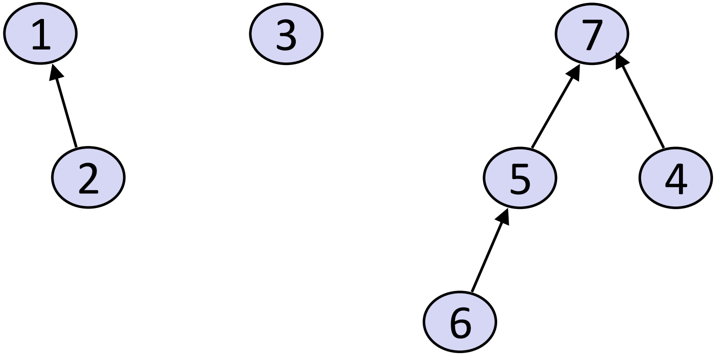
Implementing the Disjoint Sets ADT (2 of 2)

- *Observation*:
  - Trees let us find many elements given a single root
- *Idea*:
  - If we reverse the pointers (ie, point up from child to parent), we can find a single root from many elements
- *Decision*:
  - One up-tree for each set
  - The ID of the set is (hash of) the tree root
  - *(as before, we will use integer elements for in-lecture examples)*

Up-Trees Data Structure for Disjoint Sets ADT

- Initial State:

- After several union()s:

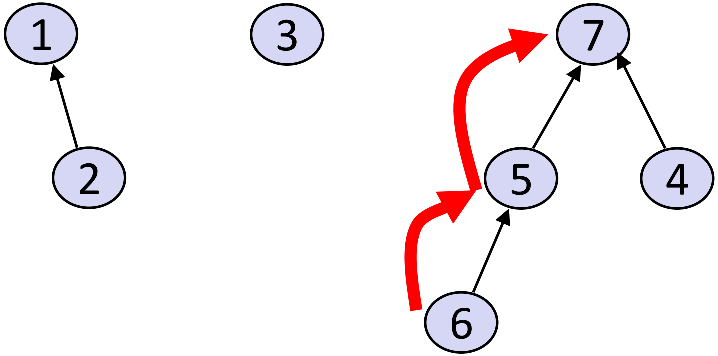


- Roots are the IDs for each set:

1,3,7

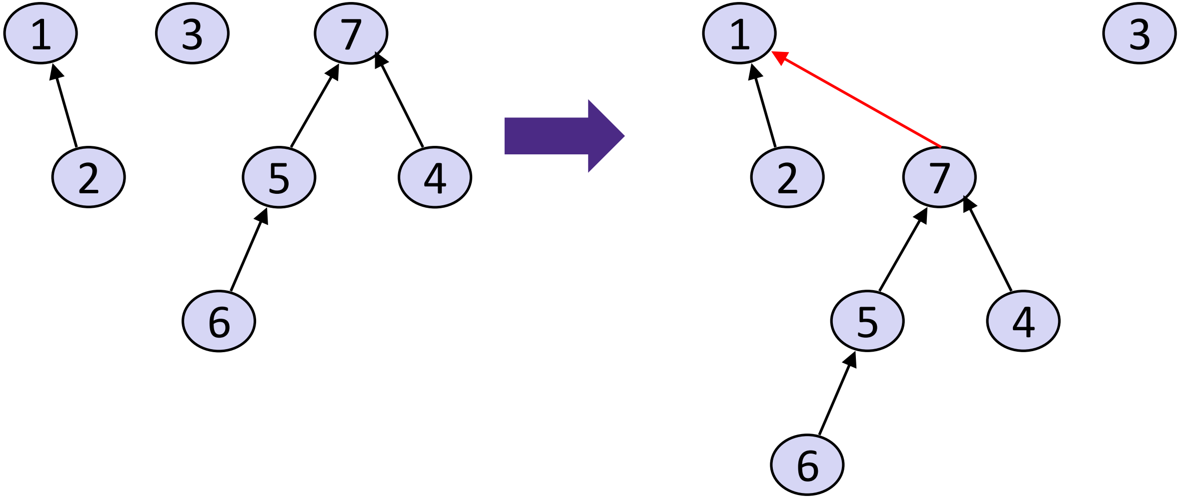
Up-Trees Find

- `find(x)`: follow x to the root and return the root ID
    - Eg: find(6) = 7



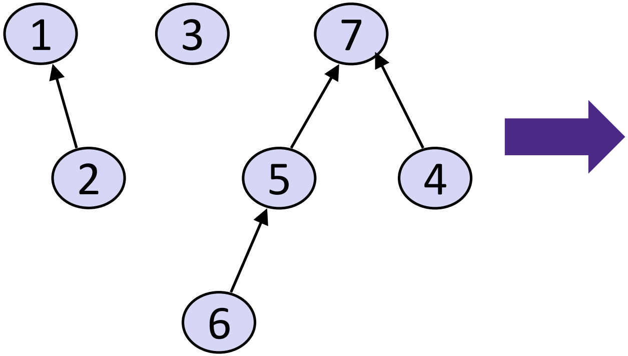
Up-Trees Union

- `union(x, y)`: assuming x and y are roots, point y to x
  - If x or y are not roots, can require caller to call find() first or do a find() internally
  - Eg: union(1, 7) vs union(2, 5)



Up-Trees Representation (1 of 2)

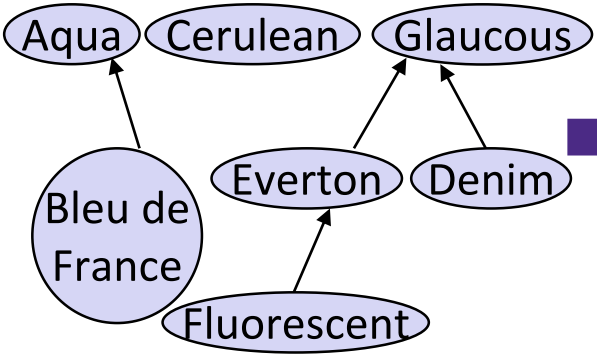
- Up-trees can be represented as an array of indices, where the element is the index of the parent
  - up[x] = 0 means x is a root
  - *Note: in these slides, array is 1-indexed; 0-indexed is also fine*



int[] up	0	1	0	7	7	5	0
	1	2	3	4	5	6	7

Up-Trees Representation (2 of 2)

- Up-trees can be represented as an array of indices, where the element is the index of the parent
    - Can contain non-integer values if we use a hash table to map values to indices



int[] up	0	1	0	7	7	5	0
	1	2	3	4	5	6	7

Aqua	1
Bleu de France	2
Cerulean	3
Denim	4
Everton	5
Fluorescent	6
Glaucous	7

Up-Trees Implementation

```
void union(int x, int y) {
  up[y] = x;
}
```

```
int find(int x) {
  while (up[x] != 0) {
    x = up[x];
  }
  return x;
}
```

- Worst-case runtime for union():

- Worst-case runtime for find():

- Total runtime for n-1 union()s and m find()s:

Remember: we can't have ≥n calls to union()

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- What is the runtime for …
  - union(), worst-case
  - find(), worst-case
  - n-1 union()s + m find()s

```
void union(int x, int y) {
  up[y] = x;
}
```

```
int find(int x) {
  while (up[x] != 0) {
    x = up[x];
  }
  return x;
}
```

A. $\Theta(1)$ / $O(1)$ / $O(n + m)$

[✓ B.] $\Theta(1)$ / $O(h)$ / $O(n + mh)$

- *h is the height of the up-tree*

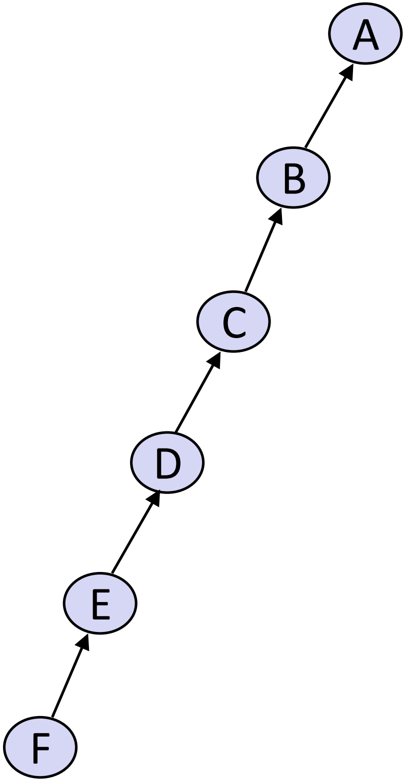
C. $\Theta(1)$ / $O(n)$ / $O(n^2)$

[✓ D.] $\Theta(1)$ / $O(n)$ / $O(n + mn)$

E. $\Theta(1)$ / $O(n)$ / $O(n + m^2)$

Worst-case Union

```
union(A, B)
union(B, C)
union(C, D)
union(D, E)
union(E, F)
```



🤔 *If only I could keep these trees (semi-?)balanced*

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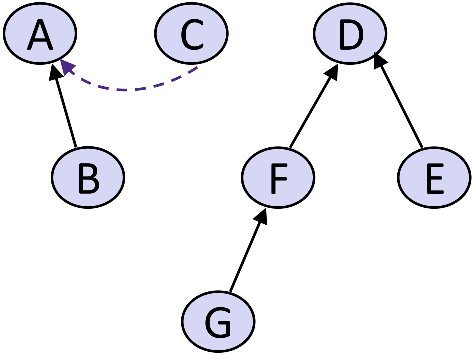
  - Optimization: Path Compression

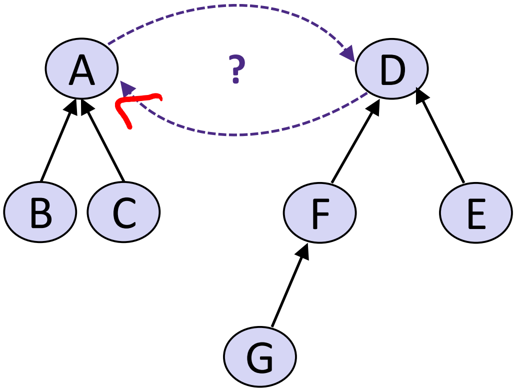
Weighted Union (1 of 3)

- Our naïve union() always picked the same argument (the second one) to become the child in the unioned result



```
union(A, B)
union(A, C)
union(A, D)
…
```

Weighted Union (2 of 3)



- Our naïve union() always picked the same argument (the second one) to become the child in the unioned result
- Let's make it smarter:
  - Pick the smaller tree (ie, tree with fewer nodes) to be the new child
    - i.e., "weight" = "num nodes"
  - Add the new child to the heavier-tree's root

```
union(A, B)
union(A, C)
union(A, D)
…
```

Weighted Union (3 of 3)

- Our naïve union() always picked the same argument (the second one) to become the child in the unioned result
- Weighted union:
  - Pick the smaller tree (ie, tree with fewer nodes) to be the new child
    - i.e., “weight” = “num nodes”
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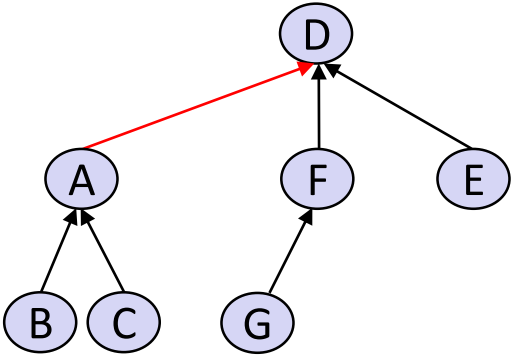
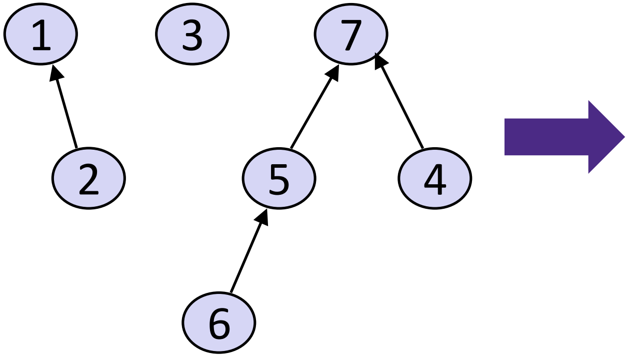


```
union(A, B)
union(A, C)
union(A, D)
…
```

Weighted Union: Representation

- Need to store *number of nodes* (or “weight”) of each tree
- Instead of ‘0’, we can store the root’s weight instead!
  - Use negative values to indicate they’re not indices
  - See Weiss, 8.4



int[] up	**-2**	1	**-1**	7	7	5	**-4**
	1	2	3	4	5	6	7

Weighted Union: Implementation

```
void union(int x, int y) {
  up[y] = x;
}
```

```
weightedUnion(int x, int y) {
  wx = weight[x];
  wy = weight[y];
  if (wx < wy) {
    up[x] = y;
    weight[y] = wx + wy;
  } else {
    up[y] = x;
    weight[x] = wx +wy;
  }
}
```

union()'s runtime is still O(1)!

Does this (slightly) added complexity help us balance the up-trees and improve find()?

Weighted Union: Performance

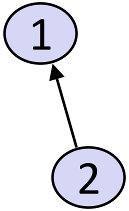
- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are “spindly”

N	H
1	0

Weighted Union: Performance

- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are “spindly”

N	H
1	0
2	1



Weighted Union: Performance

- Consider the worst case: tree height grows as fast as possible
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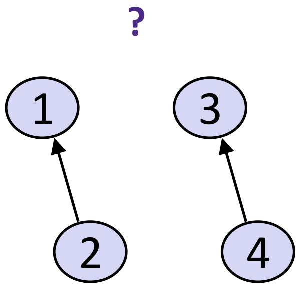
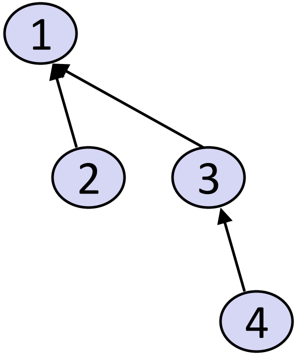


N	H
1	0
2	1
4	?

Weighted Union: Performance

- Consider the worst case: tree height grows as fast as possible

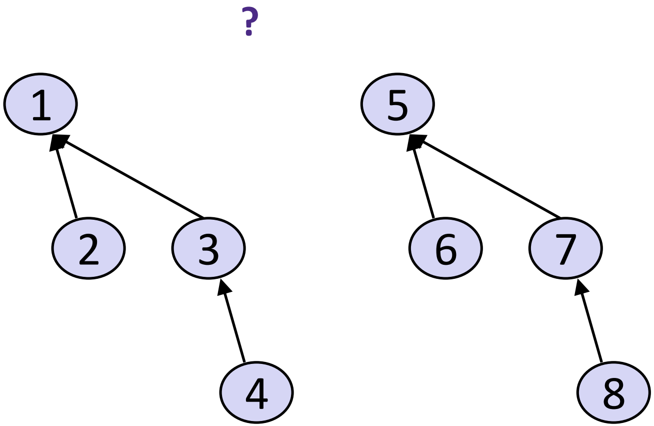


N	H
1	0
2	1
4	2

Weighted Union: Performance

- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are "spindly"

N	H
1	0
2	1
4	2
8	?

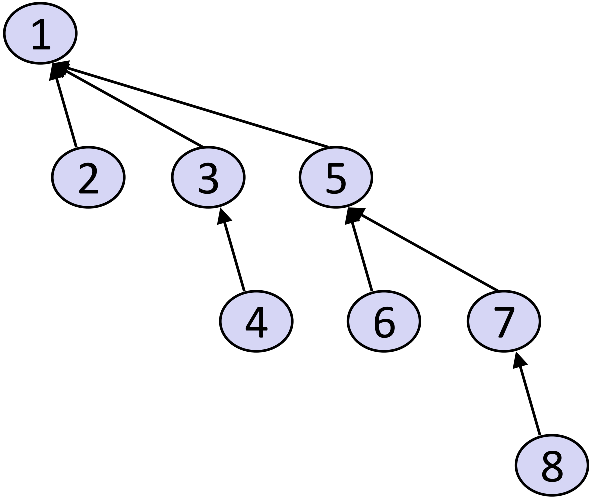


Weighted Union: Performance

- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are “spindly”

N	H
1	0
2	1
4	2
8	3
2^n	n

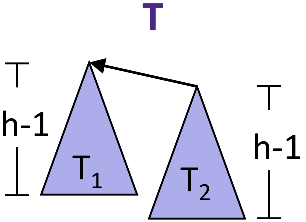
- Worst-case height and worst-case find() is Θ(log N)



Weighted Union Performance: Proof

- An up-tree with height **h** using weighted union has weight at least 2^h

- Proof by induction
  - *Base-case*: h = 0. The up-tree has one node and $2^0 = 1$
  - *Inductive step*: Assume true for all h’ < h



Minimum weight up-tree of height h formed by weighted unions

We know:

$$\left.\begin{aligned} W(T_1) &\geq 2^{h-1} \\ W(T_2) &\geq 2^{h-1} \end{aligned}\right\} \textit{Induction hypothesis}$$

$$\left. W(T_1) \geq W(T_2) \right\} \textit{Definition of weighted union}$$

Since $W(\mathbf{T}) = W(T_1) + W(T_2)$, we know that

$$\begin{aligned} W(\mathbf{T}) &\geq W(T_1) + W(T_2) \\ &= 2^{h-1} + 2^{h-1} \\ &= 2^h \end{aligned}$$

Therefore $W(\mathbf{T}) \geq 2^h$

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- What is the runtime for …
  - weighted union(), worst-case
  - find(), worst-case
  - n-1 union()s + m find()s

A. $\Theta(1)$ / $\Theta(1)$ / $O(n + m)$

B. $\Theta(1)$ / $\Theta(n)$ / $O(n + m^2)$

[✓ C.] $\Theta(1)$ / $\Theta(\log n)$ / $O(n + m \log n)$

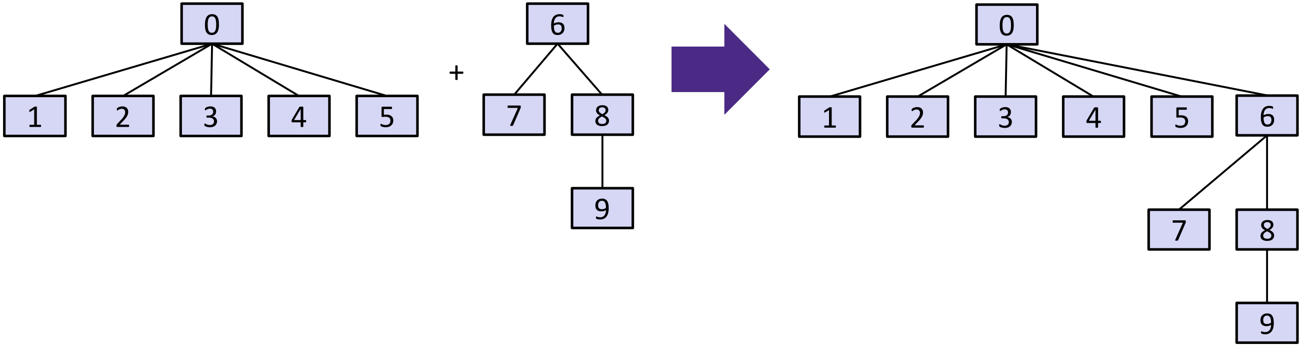
D. $\Theta(1)$ / $\Theta(\log n)$ / $O(n + m^2)$

```
weightedUnion(int x, int y) {
  wx = weight[x];
  wy = weight[y];
  if (wx < wy) {
    up[x] = y;
    weight[y] = wx + wy;
  } else {
    up[y] = x;
    weight[x] = wx +wy;
  }
}
```

```
int find(int x) {
  while (up[x] > 0) {
    x = up[x];
  }
  return x;
}
```

Why Weights Instead of Heights?

- We used the *number of items* in a tree to decide upon the root



- Why not use the *height* of the tree?
  - Heighted Union's runtime is asymptotically the same: Θ(log N)
    - Proof is left as an exercise to the reader ;)
  - Easier to track weights than heights, and heighted union doesn't combine very well with the next optimization technique for find()

Lecture Outline

- Disjoint Sets ADT (aka Union/Find ADT)

- Kruskal's Algorithm, for realz
  - Review and Example
  - Correctness Proof

- Up-Trees Data Structure
  - Representation
  - Optimization: Weighted Union
  - **Optimization: Path Compression**

Modifying Data Structures To Preserve Invariants

- Thus far, the modifications we've studied are designed to *preserve invariants* (aka "repair the data structure")
  - **Tree rotations**: preserve AVL tree balance
  - **Promoting keys / splitting leaves**: preserve B-tree node sizes (eg, L+1 keys stored in a leaf node)

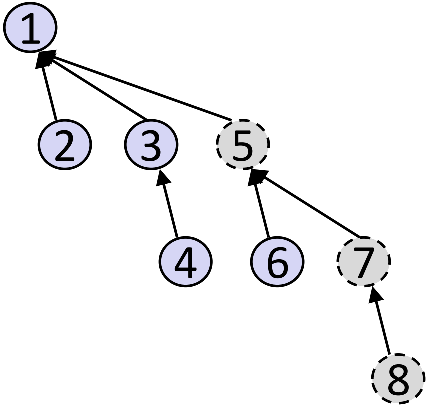
- Notably, the modifications don't improve runtime *between identical method calls*
  - If avl.find(x) takes 2 µs, we expect future calls to take ~2 µs
  - If we call avl.find(x) *m* times, the total runtime should be ~2*m* µs

Modifying Data Structures for Future Gains

- Path compression is entirely different: we are modifying the up-tree to *improve future performance*
  - If uptree.find(x) takes 2 µs, we expect future calls to take *<2 µs*
  - If we call uptree.find(x) *m* times, the total runtime should be <2*m* µs
    - … and possibly even << 2m µs

Path Compression: Idea

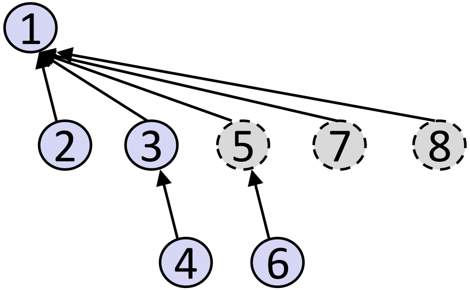
- Recall the worst-case structure if we use weighted union:



- *Idea*: When we find(8), move all visited nodes under the root
  - Additional cost is insignificant (same order of growth) , so run path compression on every find()

Path Compression: Example

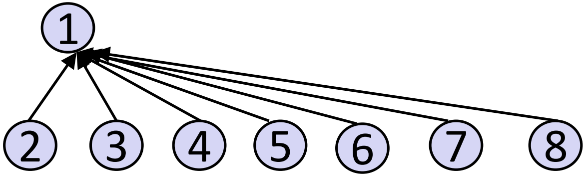
- Recall the worst-case structure if we use weighted union



- *Idea*: When we find(8), move all visited nodes under the root
  - Additional cost is insignificant (same order of growth), so run path compression on every find()
  - Doesn't meaningfully change runtime for *this* invocation of find(8), but *subsequent* find(8)s (and subsequent find(7)s and find(5)s and …) will be faster!

Path Compression: Details and Runtime

- With “enough” find()s, we end up with a very shallow tree:



- How much is “enough”? Probably m>n

(hopefully we're finding unique values. . . .)

Path Compression: Implementation

```
int find(int x) {
  while (up[x] != 0) {
    x = up[x];
  }
  return x;
}
```

```
int pathCompressionFind(int x) {
  while (up[x] > 0) {
    x = up[x];
  }
  int root = x;

  // Change the parent for all
  // nodes along this path
  while (up[x] > 0) {
    x = up[x];
    up[x] = root;
  }
  return root;
}
```

find()'s worst-case runtime is still O(log n)!

Does this (slightly) added complexity help us make the up-trees shallower and improve sequences of find()?

Path Compression: Runtime

- A sequence of *m* find()s on *n* elements has total O(*m* log**n*) time
  - Assumes weighted union and path compression
  - See Weiss for proof

- log**n* is *really* cheap!
  - log**n* is the “iterated log”: the number of times you need to apply log to *n* before the result is <=1
  - For all practical purposes, log**n* < 5 🤯
  - So O(*m* · 5) for *m* operations!

- So find() is <u>amortized</u> O(1)
  - And union() is still <u>worst case</u> O(1)

n	log* n
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

2^{16}

Number of atoms in the known universe is 2^{256}ish

Interlude: A Really Slow Function

- Ackermann’s function is a really big function A(x, y) with inverse α(x, y) which is really small

- α shows up in:
  - Computation Geometry (surface complexity)
  - Combinatorics of sequences

- How fast does α(x, y) grow?
  - Even slower than iterated log!
  - For all practical purposes, $\alpha(x, y) < 4$

Path Compression: Tighter Runtime

- A sequence of m union()s + find()s on a set of n elements has worst-case total $O(m \cdot \alpha(m, n))$ time
  - Assumes weighted union and path compression
  - Proved by Robert Tarjan in 1984
    - (Tarjan is also known for Fibonacci heaps and splay trees)
  - Complex analysis, but inverse-Ackermann's is a tighter bound than iterated-log

- So find() is still amortized O(1)
  - Since $O(m \cdot 4)$ for m operations!
  - And union() is still worst case O(1)