Parallel Prefix

CSE 332 Spring 2021

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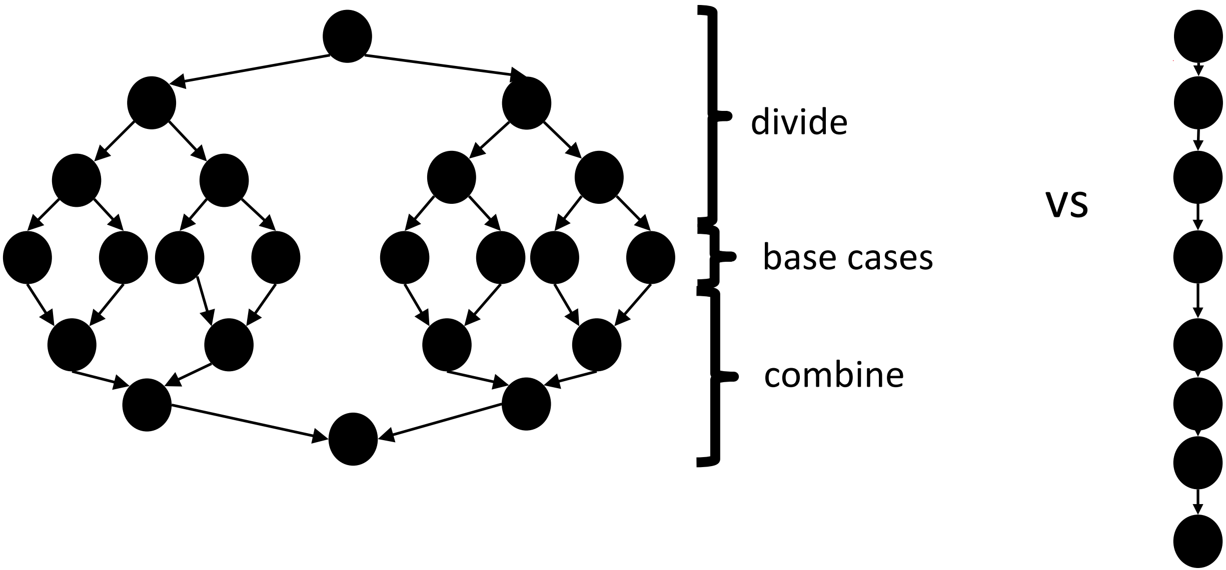
- Define **work** and **span**
- How do we calculate work and span?
- What, if any, effect does adding more processors have on work? On span?



Announcements

- P2 due this week

- Keep up with the readings if you have any questions

Lecture Outline

- **Amdahl's Law: Is the half-empty or half-full?**
- Parallel Prefix-Sum

And Now for the Good / Bad News …

- In practice, it's common that a program has:
  a) Parts that **parallelize** well:
     - E.g. maps/reduces over arrays and trees
  b) … and parts that **don't parallelize** at all:
     - E.g. reading a linked list
     - E.g. waiting on input
     - E.g. computations where each step needs the results of previous step

- These unparallelizable parts turn out to be a big bottleneck, which brings us to Amdahl's Law …

Span = T_∞ = sum of runtime of all nodes in the DAG's *most-expensive path*
Work = T_1 = sum of runtime of all nodes in the DAG
Speed-up = T_1 / T_P
  Perfect linear speedup when $T_1 / T_P = P$
Parallelism = T_1 / T_∞

Amdahl's Law

- Let the work (T_1) be 1 unit of time and **S** be the unparallelizable portion of execution time:

$$T_1 = 1 = S + (1-S)$$

- Suppose *perfect linear speed-up* on the parallelizable portion. Then:

$$T_P = S + (1-S)/P$$

- Amdahl's Law states the speed-up with P processors is:

$$T_1 / T_P = 1 / (S + (1-S)/P)$$

- and the parallelism (maximum possible speed-up) is:

$$T_1 / T_\infty = 1 / S$$

Amdahl's Law Example

- Recall: T_1 **= 1 = S + (1-S)** **and** T_P **= S + (1-S)/P**

- Suppose: T_1 **= 1/3 + 2/3 = 1** *(eg, T_1 = 100s = 33s + 67s)*
- Then: T_P **= 33 sec + (67 sec)/P**
  T_3 **= 33 sec + (67 sec)/3 =** 33 + 22 = 55
  T_6 **= 33 sec + (67 sec)/6 =** 33 + 11 = 44
  T_{67} **= 33 sec + (67 sec)/67 =** 33 + 1 = 34

- If 33% of a program is sequential, a billion processors won't give a speedup over 3!!!
- No matter how many processors you use, your speedup is bounded by the sequential portion of the program

Implications of Amdahl's Law

Speedup:	$T_1 / T_P = 1 / (S + (1-S)/P)$
Max Parallelism:	$T_1 / T_\infty = 1 / S$

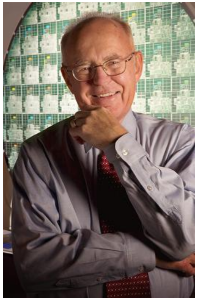
- In "the good old days" (1980-2005), ~12 years = 100x speedup

- Now suppose in 12 years, clock speed is the same but you get 256 processors instead of 1. What portion of the program must be parallelizable to get 100x speedup?
  - *For 256 processors to get at least 100x speedup, we need*

$$100 \leq 1 / (\mathbf{S} + (1-\mathbf{S})/256)$$

  - *Which means* $\mathbf{S} \leq .0061$ *(i.e., 99.4% must be parallelizable)*

Moore and Amdahl



- Moore's "Law" is an **observation** about the progress of the semiconductor industry
  - Transistor density doubles roughly every 18 months
- Amdahl's Law is a **mathematical theorem**
  - Diminishing returns of adding more processors
- Both are incredibly important in designing computer systems

The Challenge Posed by Amdahl's Law

- Amdahl's Law tells us unparallelized parts become a bottleneck very quickly
  - But it *doesn't* tell us additional processors are worthless

- ... because we can find new parallel algorithms
  - Some things that seem sequential turn out to be parallelizable
  - Eg: How can we parallelize a 'running sum' array?

input	6	4	16	10	16	15	2	8
output	6	10	26	36	52	67	69	77

- We can also change the problem we're solving
  - Eg: Video games use tons of parallel processors; they are not rendering 10-year-old graphics faster

Lecture Outline

- Amdahl's Law: Is the  half-empty or half-full?

- **Parallel Prefix-Sum**

The Prefix-Sum Problem (1 of 2)

- Given `int[] input`, produce `int[] output` where:

  `output[i] = input[0]+input[1]+…+input[i]`

input	6	4	16	10	16	15	2	8
output	6	10	26	36	52	67	69	77

- Problem is "inherently sequential" because each value depends on the values before it

The Prefix-Sum Problem (2 of 2)

input	6	4	16	10	16	15	2	8
output	6	10	26	36	52	67	69	77

- Sequential solution feels like a CSE142 exam problem:

```
int[] prefix_sum(int[] input) {
  int[] output = new int[input.length];
  output[0] = input[0];
  for (int i=1; i < input.length; i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

- Doesn't seem parallelizable!
  - Work: O(n), Span: O(n)
  - There's a different algorithm with Work: O(n), Span: O(log n) 😮

Parallel Prefix-Sum: Overview (1 of 2)

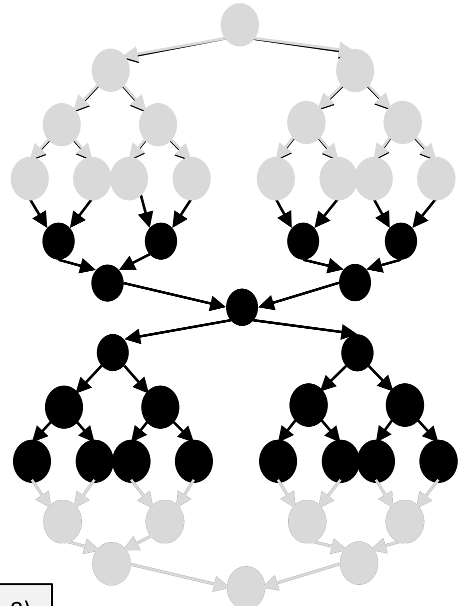
1968? 1973?

Recent

- Local bragging:
  - Algorithm due to R. Ladner and M. Fischer *at UW in 1977*
  - Richard Ladner joined the UW faculty in 1971 and hasn't left

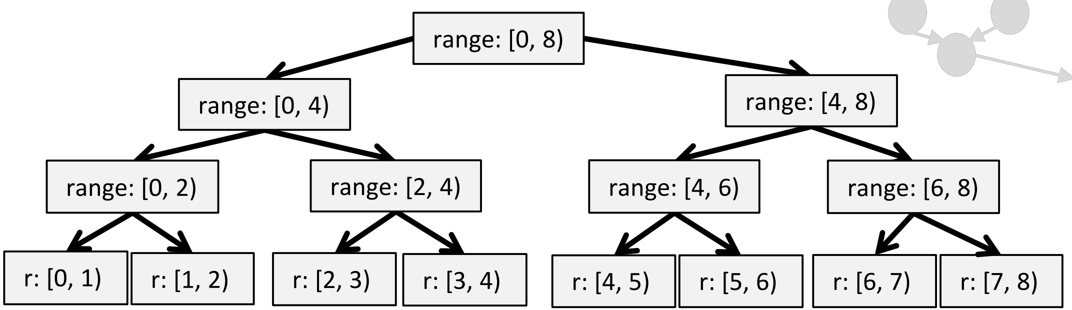
- Parallel-prefix sum algorithm has two passes:
  - Each pass is O(n) work and O(log n) span
  - So – as with array summing – parallelism is n/log n: exponential!

Parallel Prefix-Sum: Overview (2 of 2)



- First pass builds a *binary tree* from the bottom: the "up" pass
- Second pass *processes* the binary tree: the "down" pass

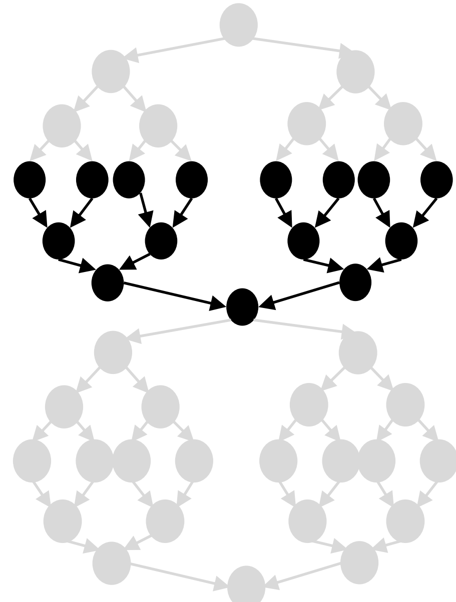
- Sequential algorithm is linear, but this algorithm uses two logarithmic passes



input	6	4	16	10	16	15	2	8

Parallel Prefix-Sum: The "Up" Pass: Overview

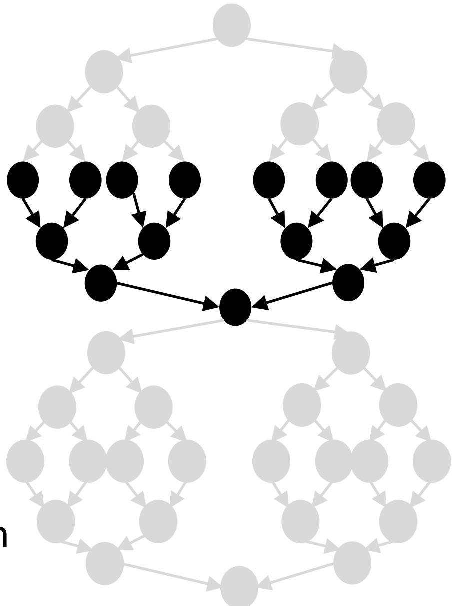
- This first pass builds a *binary tree* from the bottom: the "up" pass



- Parallel Prefix-Sum's binary tree:
  - Internal nodes have a range and sum of [lo, hi)
    - … and the root has [0, n+1)
  - Left child has range and sum of [lo, middle)
  - Right child has range and sum of [middle, hi)
  - A leaf has range and sum of [i,i+1); the sum is simply input[i]

- Unlike parallel-sum, we actually *create the tree;* we need it for the next pass (the "down" pass)
  - Doesn't have to be an actual tree; could use an array (eg, binary heap)

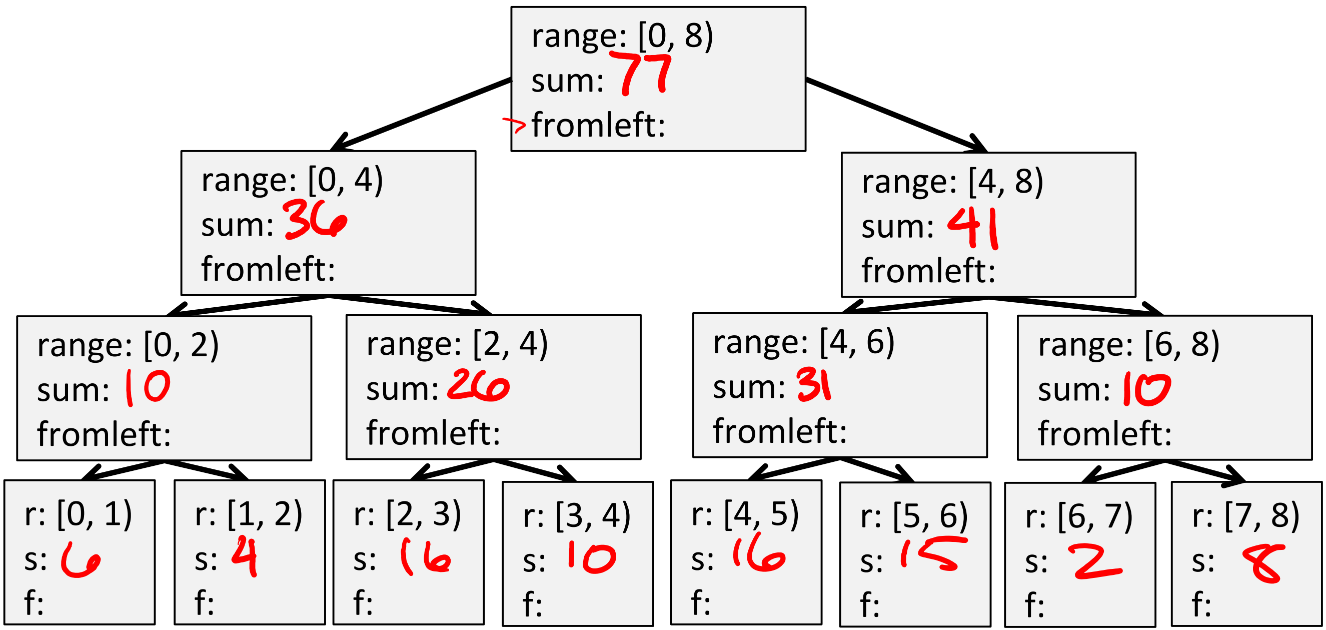
Parallel Prefix-Sum: The "Up" Pass: Details



- Parent has range and sum of [lo, hi)
  - left has [lo, middle), and right has [middle, hi)
- Build sum from the bottom of the tree:
  - A leaf's sum is just its value: input[i]
- Easy fork-join computation!
  - Save partial sums from parallel-sum algorithm
  - Tree is built from bottom-up, in parallel
- Analysis of the up pass:
  - Work: $O(n)$
  - Span: $O(\log n)$

Parallel Prefix-Sum's Tree

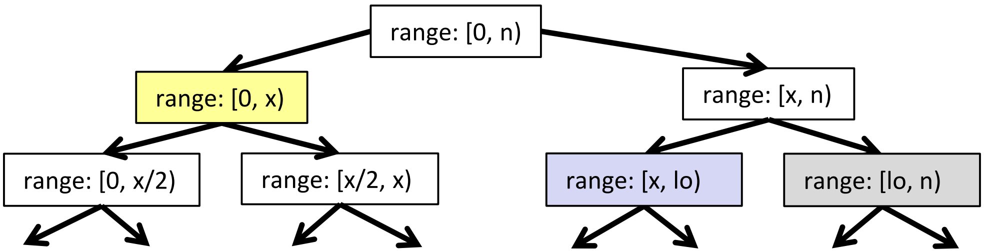
input								
input	6	4	16	10	16	15	2	8



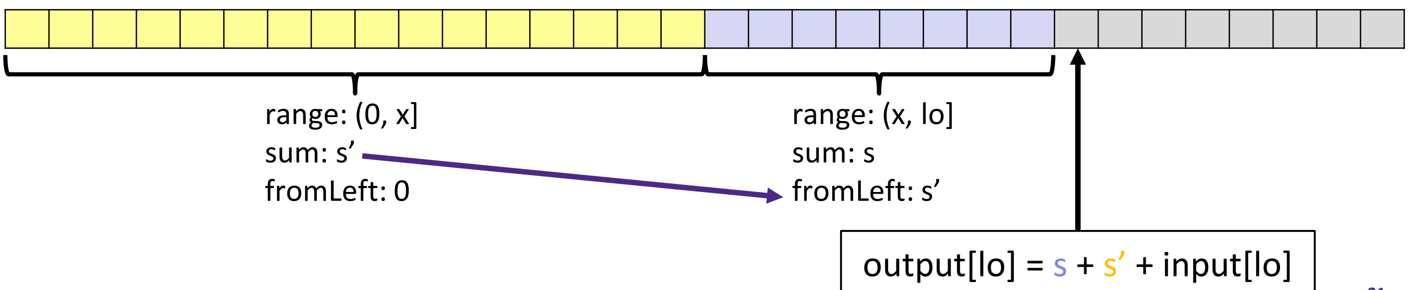
output								

Parallel Prefix-Sum: The “Down” Pass: Overview

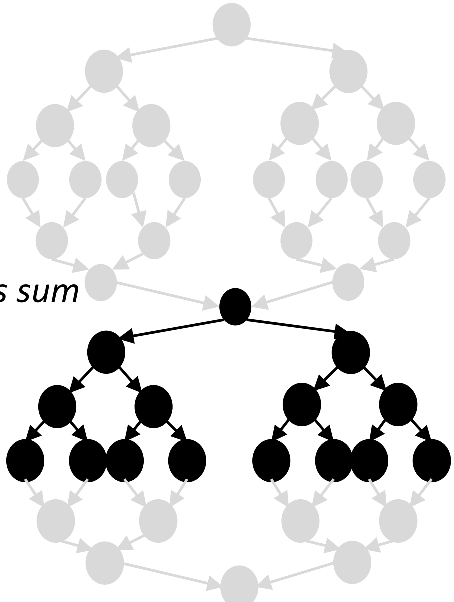
- This second pass *processes* the binary tree: the “down” pass



- All nodes have a range and sum of [lo, hi); now populate fromLeft
  - Invariant: fromLeft is sum of elements left of the node’s range: [0, lo)



Parallel Prefix-Sum: The "Down" Pass: Details

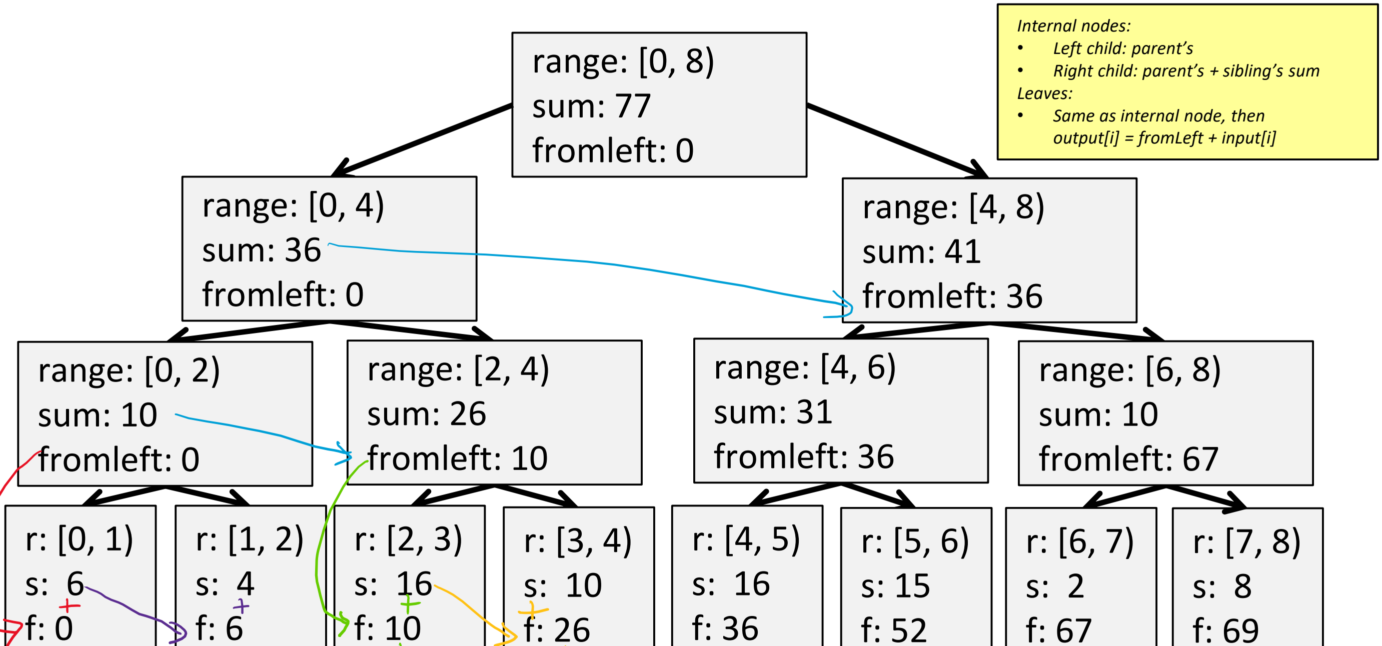


- Propagate fromLeft down:
  - Root starts with a fromLeft of 0 *(why?)*
  - Internal node takes its fromLeft value and
    - Passes its left child *the same* fromLeft
    - Passes its right child *its fromLeft plus its left child's sum*
  - At the leaf, *must also* `output[i] = fromLeft + input[i]`
- Also an easy fork-join computation!
  - Traverse the tree built in step 1
  - Don't produce an explicit result; the leaves will assign to `output`

- Analysis of down pass: Work: O(n), Span: O(log n)
- Total for algorithm: Work: O(n), Span: O(log n)

Parallel Prefix-Sum's Example: The "Down" Pass

input	6	4	16	10	16	15	2	8



output	6	10	26	36	52	67	69	77

Sequential Cutoff for Prefix-Sum

- Adding a sequential cut-off isn't too bad:
  1. Propagating up the sums:
     - Leaf node just holds the sum of a range of values (i.e., sequentially compute sum for that range)
     - The tree itself will be shallower
  2. Propagating down the fromLefts:
     - Have leaf compute prefix sum sequentially over its [lo,hi), then:

```
output[lo] = fromLeft + input[lo];
for(i=lo+1; i < hi; i++)
  output[i] = output[i-1] + input[i]
```

Generalized Parallel-Prefix-Sum = Parallel-Prefix

- Sum-array was an example of a common pattern

- Prefix-sum is also a pattern that arises in many problems:
  - Minimum, maximum of all elements **to the left of i**
  - Is there an element **to the left of i** satisfying some property?
  - Count of elements **to the left of i** satisfying some property

You now know the "one weird trick": parallel-prefix!