Minimum Spanning Trees

CSE 332 Summer 2020

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Lecture Q&A: pollev.com/332summer

Lecture clarifications: tinyurl.com/332-08-17A

Announcements

- Welcome to the last week of 332 😭😭😭
- Important Due Dates this Week:
  - Monday, 11:59pm: Ex 14 due
  - Tuesday, 11:59pm: P3 due – Thursday 11:59pm : With 2 late days
  - Thursday, 12:00am: Quiz 5 out
  - Friday, 11:59pm: Ex 15 due
  - Saturday, 3:00am: Quiz 5 due
- Mock interviews running from Tuesday to Thursday
  - No normal section on Thursday

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Lecture Outline

- **Minimum Spanning Tree**
  - Prim's Algorithm
  - Kruskal's Algorithm
- Disjoint Sets ADT
- Up-Tree Data Structure
  - Representation
  - Optimization: Weighted Union
  - Optimization: Path Compression

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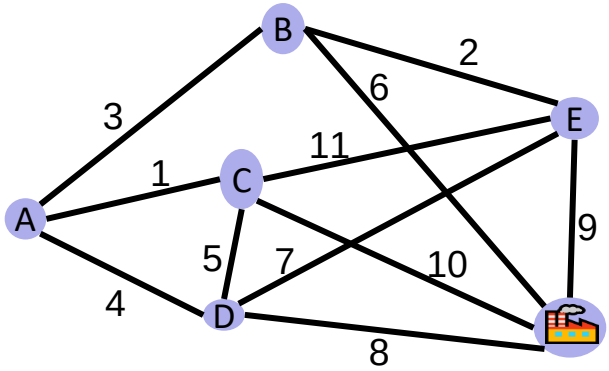
Problem Statement

- Your friend at the electric company needs to connect all these cities to the power plant
- She knows the cost to lay wires between any pair of cities and wants the cheapest way to ensure electricity gets to every city



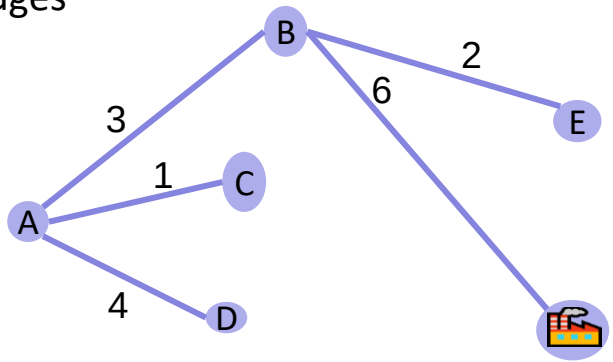
- Assume:
  - The graph is connected and undirected
  - *(In general, edge weights can be negative; just not in this example)*

Solution Statement

- We need a set of edges such that:
  - Every vertex touches at least one edge ("the edges **span** the graph")
  - The graph using just those edges is **connected**
  - The total weight of these edges is **minimized**

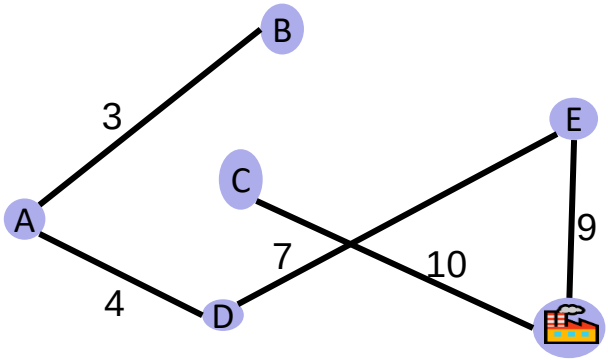
|V| = # vertex
|E| = # edges

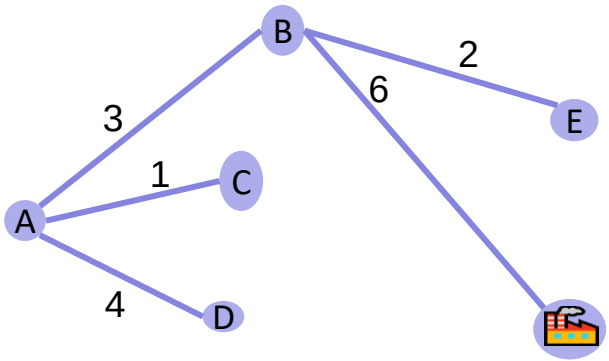
- *Claim*: The set of edges we pick never forms a cycle. Why?
  - V-1 edges is the exact number of edges to connect all vertices
  - Taking away 1 edge breaks connectiveness
  - Adding 1 edge makes a cycle



Solution Statement (v2)

- We need a ~~set of edges such that~~ Minimum Spanning Tree:
  - Every vertex touches at least one edge ("the edges **span** the graph")
  - The graph using just those edges is **connected**
  - The total weight of these edges is **minimized**





Minimum Spanning Trees

- Given an undirected graph G = (V,E), a minimum spanning tree is a graph G' = (V, E') such that:
  - E' is a subset of E
  - |E'| = |V| - 1
  - G' is connected
  - $\sum_{(u,v)\in E'} c_{uv}$ is minimal

cost of (u,v)

Applications of MSTs

- Handwriting recognition
  - http://dspace.mit.edu/bitstream/handle/1721.1/16727/43551593-MIT.pdf;sequence=2

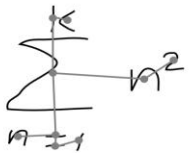
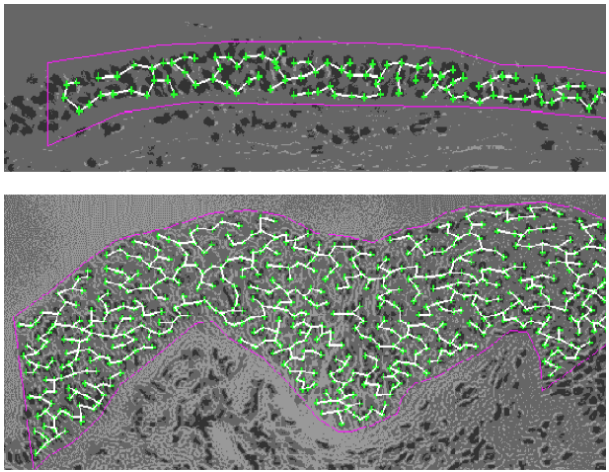


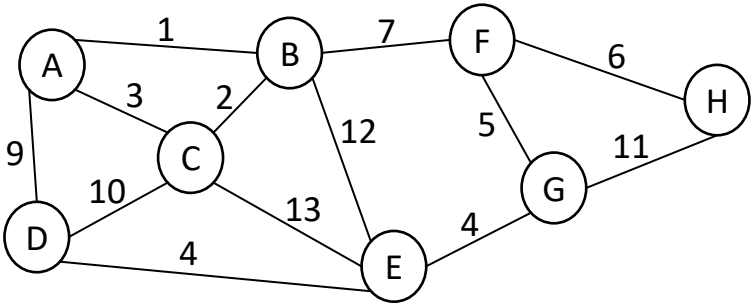
Figure 4-3: A typical minimum spanning tree

- Medical imaging
  - e.g. arrangement of nuclei in cancer cells

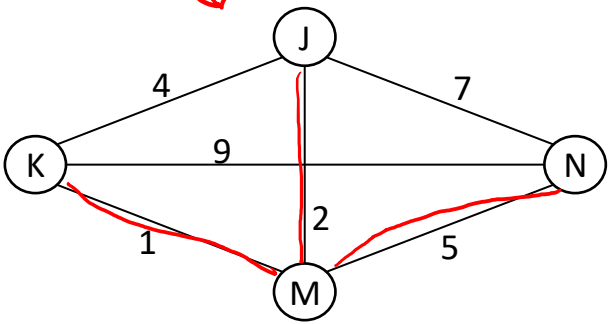


For more, see: http://www.ics.uci.edu/~eppstein/gina/mst.html

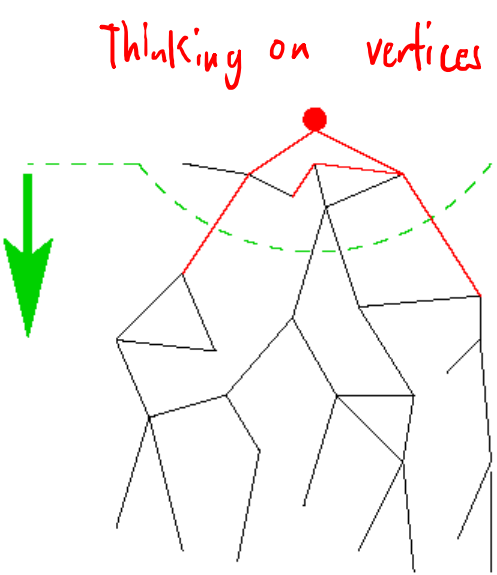
Student Activity: Find the MSTs

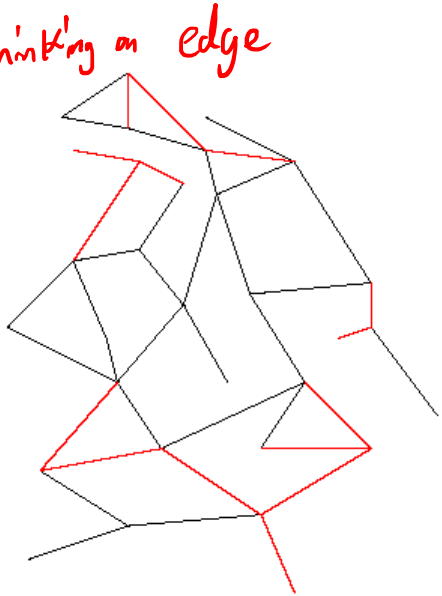


Find MST



MST Algorithms: Two Different Approaches





Prim's Algorithm

Almost identical to Dijkstra's
Start with one node, grow greedily

Kruskals's Algorithm

Completely different!
Start with a *forest* of MSTs, union them together
(Need a new data structure for this)

Lecture Outline

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  - **Prim's Algorithm**
  - Kruskal's Algorithm
- Disjoint Sets ADT
- Up-Tree Data Structure
  - Representation
  - Optimization: Weighted Union
  - Optimization: Path Compression

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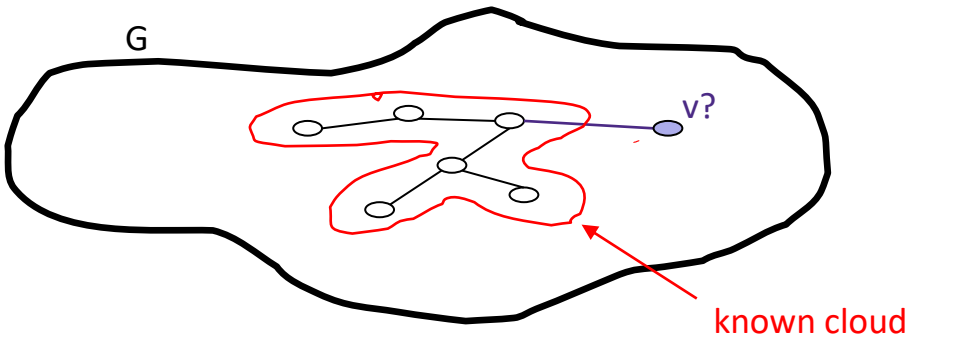
Prim's Algorithm**

- *Intuition*: a vertex-based greedy algorithm
  - Builds MST by greedily adding vertices

- *Summary*: Grow a single tree by picking a vertex from the fringe that has the smallest cost
  - Unlike Dijkstra's, cost is the *edge weight* into the known set



*** This algorithm was developed in 1930 by Votěch Jarník, then independently rediscovered by Robert Prim in 1957 and then Dijkstra in 1959. It's also known as Jarník's, Prim-Jarník, or DJP*

Prim's Algorithm: Pseudocode

```
prims(Graph g) {
  foreach vertex v in g:
    v.distance = ∞
  start = g.getSomeArbitraryVertex()
  start.distance = 0

  mst = {}
  heap = buildHeap(g.vertices - {start})
  foreach vertex v in start.neighbors():
    v.distance = g.weight(start, v)
    v.previous = start
    heap.decreaseKey(v, v.distance)

  while (! heap.empty()):
    v = heap.deleteMin()
    mst.addEdge(v, v.previous)
    foreach edge (v, u) in g:
      d1 = v.distance
      d2 = u.distance
      if (d1 < d2):
        u.previous = v
}
```

Remember our 5-step pattern for a graph traversal?

Prim's Algorithm vs. Dijkstra's Algorithm (1 of 2)

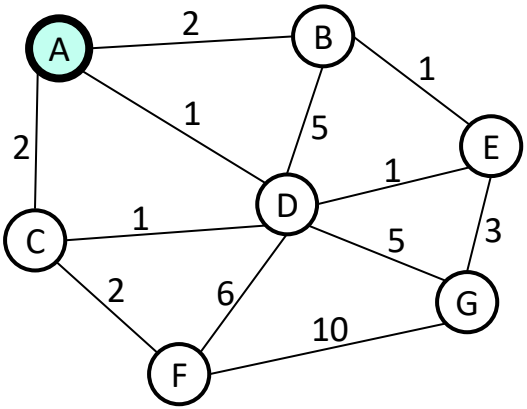
- Dijkstra's picks an unknown vertex with smallest *distance to the source*
  - ie, path weights
- Prim's picks an unknown vertex with smallest *distance to the known set*
  - i.e., edge weights
- Some differences in the initialization, but otherwise identical

Prim's Algorithm: Pseudocode

```
prims(Graph g) {
  foreach vertex v in g:
    v.distance = ∞
  start = g.getSomeArbitraryVertex()
  start.distance = 0

  mst = {}
  heap = buildHeap(g.vertices - {start})
  foreach vertex v in start.neighbors():
    v.distance = g.weight(start, v)
    v.previous = start
    heap.decreaseKey(v, v.distance)

  while (! heap.empty()):
    v = heap.deleteMin()
    mst.addEdge(v, v.previous)
    foreach edge (v, u) in g:
      d1 = v.distance
      d2 = u.distance
      if (d1 < d2):

        u.previous = v
}
```

```
dijkstra(Graph g, Vertex start) {
  foreach vertex v in g:
    v.distance = ∞

  start.distance = 0


  heap = buildHeap(g.vertices)









  while (! heap.empty()):
    v = heap.deleteMin()

    foreach edge (v, u) in g:
      d1 = v.dist + g.weight(v, u)
      d2 = u.dist
      if (d1 < d2):
        heap.decreaseKey(u, d1)
        u.previous = v
}
```

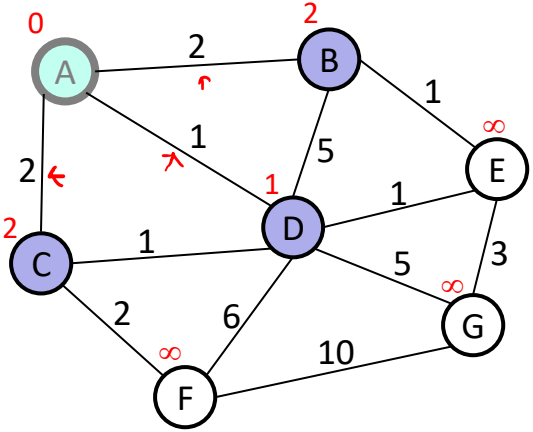
Prim's Algorithm: Example



Order Added to Known Set:

Vertex	Known?	Distance	Previous
A		∞	
B		∞	
C		∞	
D		∞	
E		∞	
F		∞	
G		∞	

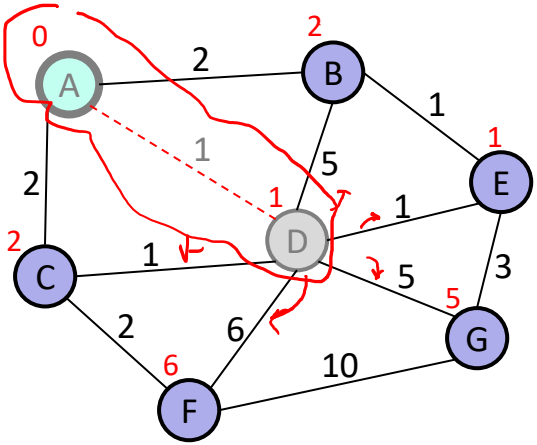
Prim's Algorithm: Example



Order Added to Known Set:
A

Vertex	Known?	Distance	Previous
A	Y	0	\
B		2	A
C		2	A
D		1	A
E		∞	
F		∞	
G		∞	

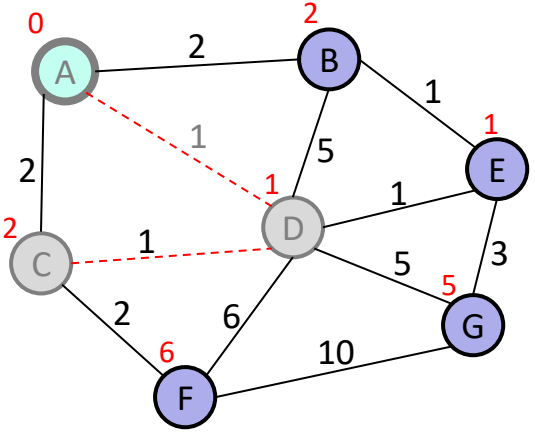
Prim's Algorithm: Example



Order Added to Known Set:
A, D

Vertex	Known?	Distance	Previous
A	Y	0	\
B		2	A
C		1	D
D	Y	1	A
E		1	D
F		6	D
G		5	D

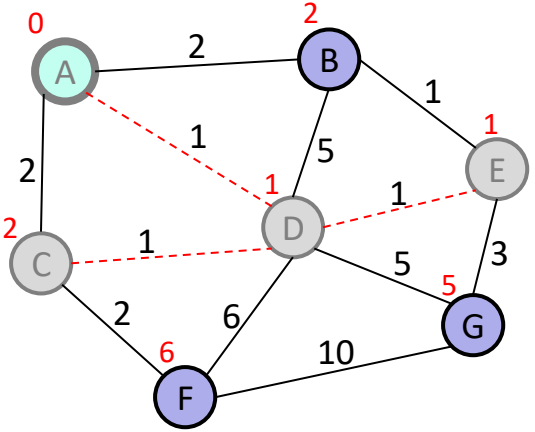
Prim's Algorithm: Example



Order Added to Known Set:
A, D, C

Vertex	Known?	Distance	Previous
A	Y	0	\
B		2	A
C	Y	1	D
D	Y	1	A
E		1	D
F		**2**	**C**
G		5	D

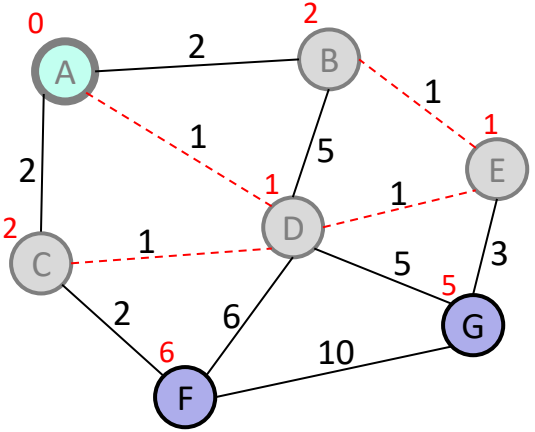
Prim's Algorithm: Example



Order Added to Known Set:
A, D, C, E

Vertex	Known?	Distance	Previous
A	Y	0	\
B		**1**	**E**
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		**3**	**E**

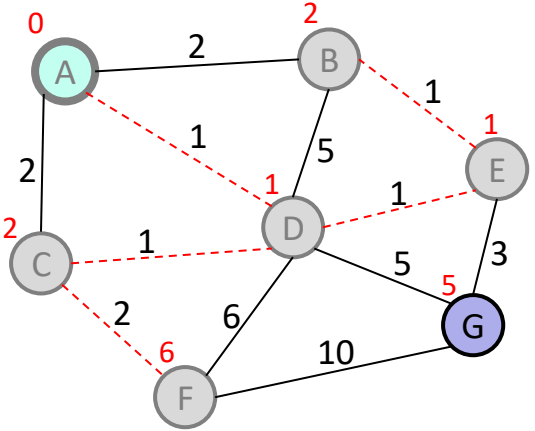
Prim's Algorithm: Example



Order Added to Known Set:
A, D, C, E, B

Vertex	Known?	Distance	Previous
A	Y	0	\
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F		2	C
G		3	E

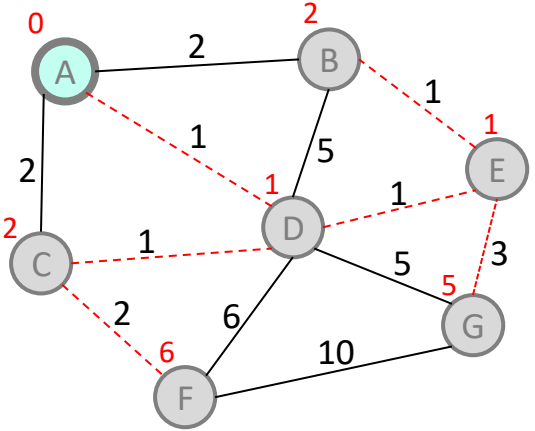
Prim's Algorithm: Example



Order Added to Known Set:
A, D, C, E, B, F

Vertex	Known?	Distance	Previous
A	Y	0	\
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F	Y	2	C
G		3	E

Prim's Algorithm: Example



Order Added to Known Set:
A, D, C, E, B, F

All Done!!!

Total Cost: 9

Vertex	Known?	Distance	Previous
A	Y	0	\
B	Y	1	E
C	Y	1	D
D	Y	1	A
E	Y	1	D
F	Y	2	C
G	Y	3	E

Prim's Algorithm: Demos and Visualizations

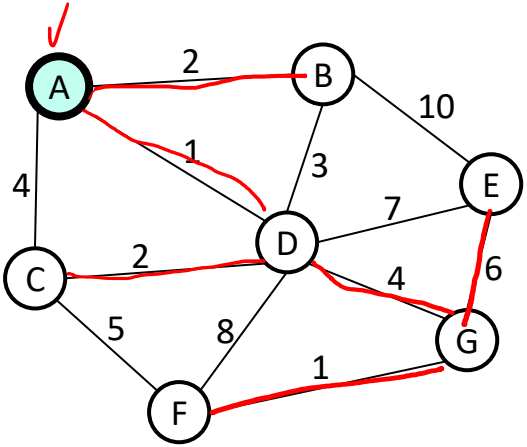
- Dijkstra's Visualization
  - https://www.youtube.com/watch?v=1oiQ0hrVwJk
  - Dijkstra's proceeds radially from its source, because it chooses edges by *path length from source*
- Prim's Visualization
  - https://www.youtube.com/watch?v=6uq0cQZOyoY
  - Prim's jumps around the MST-under-construction (the fringe), because it chooses edges by *edge weight* (there's no source)
- Demo:
  - https://docs.google.com/presentation/d/1GPizbySYMsUhnXSXKvbqV4UhPCvrt750MiqPPgU-eCY/present?ueb=true&slide=id.g9a60b2f52_0_205

Prim's Algorithm: Analysis

- Correctness:
  - A bit tricky to prove, but intuitively similar to Dijkstra
  - Might return to this time permitting (unlikely)

- Run-time:
  - Same as Dijkstra's! $O(|E|\log|V| + |V|\log|V|)$ using a priority queue
  - But since $E \in O(|V|^2)$, can also state as $O(|E|\log|V|)$

Prim's Algorithm: Student Activity



Order Added to Known Set:

Vertex	Known?	Distance	Previous
A	Y	0	—
B	Y	∞ 2	A
C	Y	∞ 2	D
D	Y	∞ 1	A
E	Y	∞ 6	G
F	Y	∞ 1	G
G	Y	∞ 4	D

Lecture Outline

- Minimum Spanning Tree
  - Prim's Algorithm
  - **Kruskal's Algorithm**
- Disjoint Sets ADT
- Up-Tree Data Structure
  - Representation
  - Optimization: Weighted Union
  - Optimization: Path Compression

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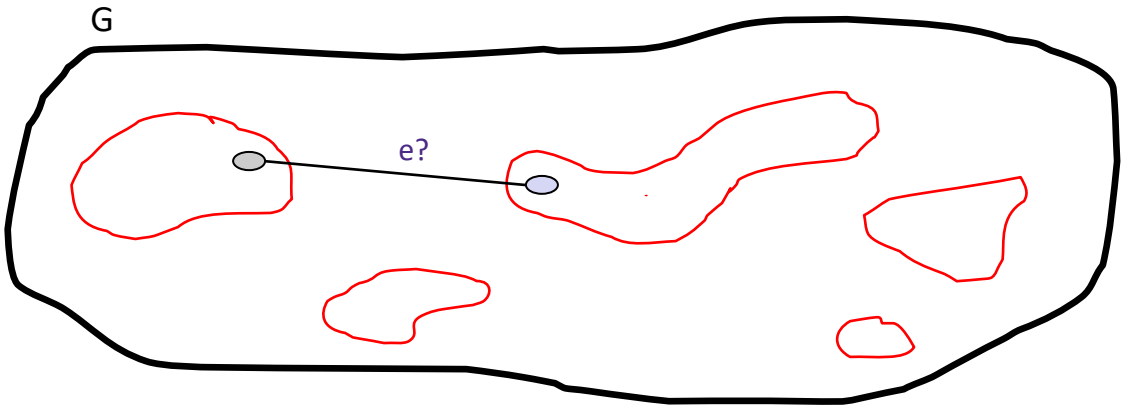
Kruskal's Algorithm: A Different Approach

- Prim's thinks vertex by vertex
  - Eg, add the closest vertex to the currently reachable set
- What if you think edge by edge instead?
  - Eg, start from the lightest edge; add it if it connects new things to each other (don't add it if it would create a cycle)

Kruskal's Algorithm

- *Intuition*: an edge-based greedy algorithm
  - Builds MST by greedily adding edges

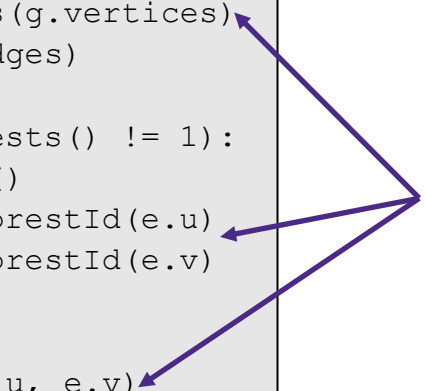
- *Summary*: Start with a ***forest*** of MSTs, and successively connect them by adding edges; do not create a cycle



Kruskal's Algorithm: Pseudo-pseudocode

```
kruskals(Graph g) {
  mst = {}
  forests = buildForests(g.vertices)
  edges = buildHeap(g.edges)

  while (forests.numForests() != 1):
    e = edges.deleteMin()
    u_id = forests.getForestId(e.u)
    v_id = forests.getForestId(e.v)
    if (u_id != v_id):
      mst.addEdge(e)
      forests.connect(e.u, e.v)
}
```



Does this fit our 5-step pattern for a graph traversal?

What data structure is this?!?!

Aside: Disjoint Sets ADT (1 of 2)

Disjoint Sets ADT. A collection of elements and sets of those elements.

- An element can only belong to a single set.
- Each set is identified by a unique id.
- Sets can be combined/ connected/ unioned.

- The Disjoint Sets ADT has two operations:
  - find(e): gets the id of the element's set
  - union(e1, e2): combines the set containing e1 with the set containing e2
- Example: ability to travel to drive to a country
  - union(france, germany)
  - union(spain, france)
  - find(spain) == find(germany)?
  - union(england, france)

Aside: Disjoint Sets ADT (2 of 2)

- The Disjoint Sets ADT has two operations:
  - find(e): gets the id of the element's set
  - union(e1, e2): combines the set containing e1 with the set containing e2

- Applications include percolation theory (computational chemistry) and …. Kruskal's algorithm

- Simplifying assumptions
  - We can map elements to indices quickly
  - We know all the items in advance; they're all disconnected initially

- In the this/next lecture, we'll see:
  - We can do union() in constant time O(1)
  - We can get find() to be ***amortized*** constant time O(1)
    - Worst case O(log n) for an individual find operation

Kruskal's Algorithm: Pseudocode

```
kruskals(Graph g) {
  mst = {}
  forests = buildDisjointSets(g.vertices)
  numforests = g.vertices
  edges = buildHeap(g.edges)

  while (numforests > 1):
    e = edges.deleteMin()
    u_id = forests.find(e.u)
    v_id = forests.find(e.v)
    if (u_id != v_id):
      mst.addEdge(e)
      forests.union(e.u, e.v)
      numforests--
}
```

log(n)

v times

} |E| deleteMin()s

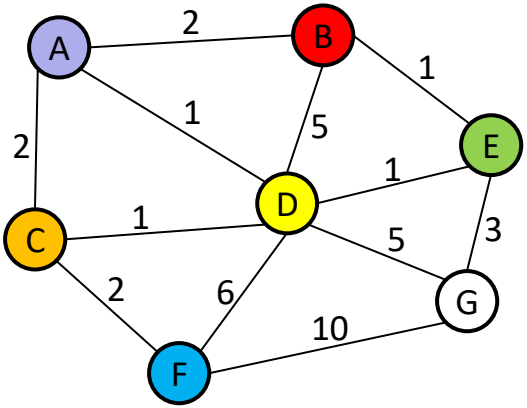
} 2|E| find()s

} |V| union()s

Runtime: $|E|(\log|E| + 2|E|\log|V| + 1) + |V|(1 + 1 + 1) \in O(|E|\log|V| + |V|\log|V|)$

However, since we know $E \in O(|V|^2)$, *runtime* $\in O(|E|\log|V|)$

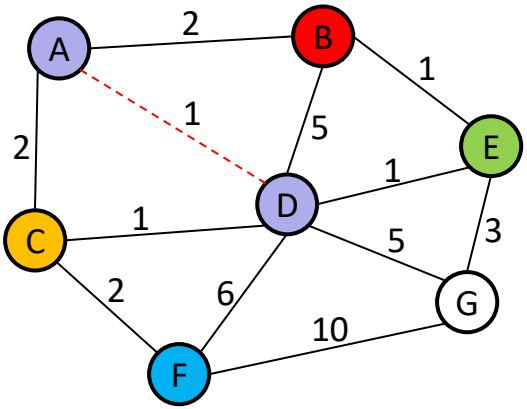
Kruskal's Algorithm: Example



MST:

Weight	Edges
1	(A,D), (C,D), (B,E), (D,E)
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

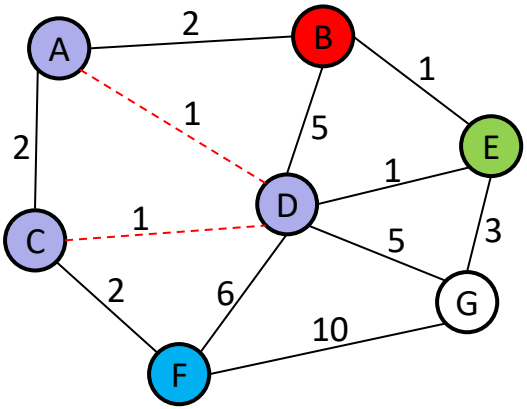
Kruskal's Algorithm: Example



MST:
(A, D)

Weight	Edges
1	*(A,D),* (C,D), (B,E), (D,E)
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

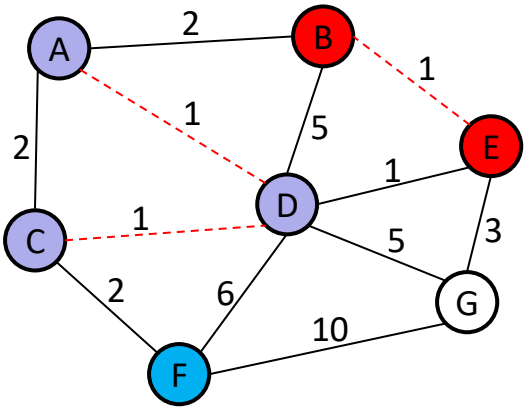
Kruskal's Algorithm: Example



MST:
(A, D), (C, D)

Weight	Edges
1	*(A,D), (C,D),* (B,E), (D,E)
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

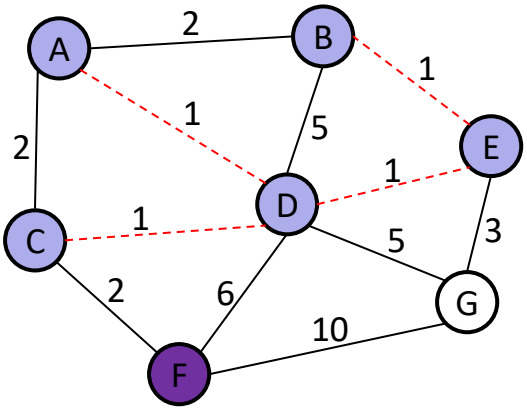
Kruskal's Algorithm: Example



MST:
(A, D), (C, D), (B, E)

Weight	Edges
1	*(A,D), (C,D), (B,E),* (D,E)
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

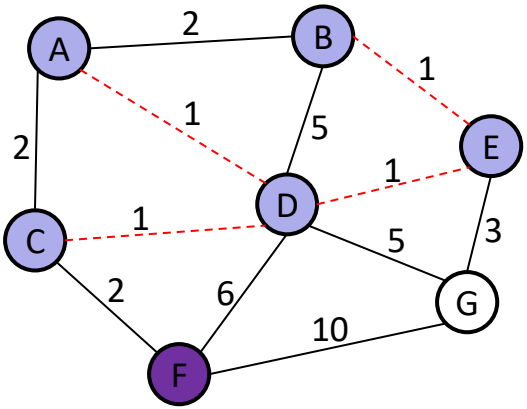
Kruskal's Algorithm: Example



MST:
(A, D), (C, D), (B, E), (D, E)

Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	(A,B), (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

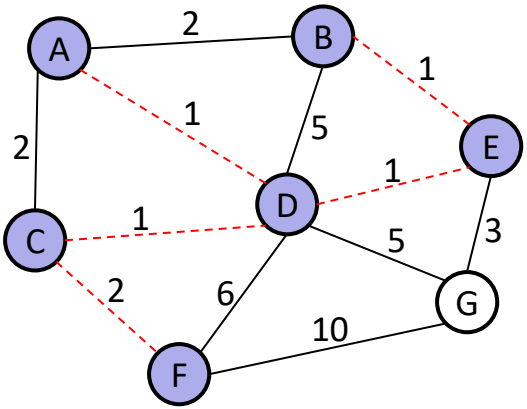
Kruskal's Algorithm: Example



MST:
(A, D), (C, D), (B, E), (D, E)

Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	*(A,B),* (C,F), (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

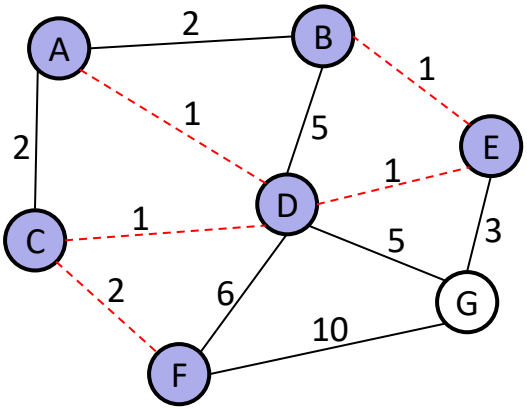
Kruskal's Algorithm: Example



MST:
(A, D), (C, D), (B, E), (D, E), (C, F)

Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	*(A,B), (C,F),* (A,C)
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

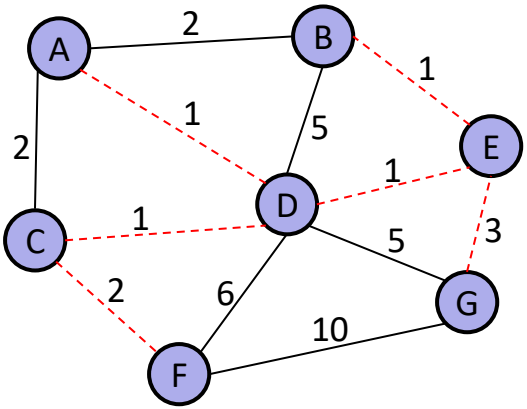
Kruskal's Algorithm: Example



MST:
(A, D), (C, D), (B, E), (D, E), (C, F)

Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	*(A,B), (C,F), (A,C)*
3	(E,G)
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

Kruskal's Algorithm: Example



An MST!!!

Total Cost: 9

Weight	Edges
1	*(A,D), (C,D), (B,E), (D,E)*
2	*(A,B), (C,F), (A,C)*
3	*(E,G)*
5	(D,G), (B,D)
6	(D,F)
10	(F,G)

MST:
(A, D), (C, D), (B, E), (D, E), (C, F), (E, G)

Kruskal's Algorithm: Demos and Visualizations

- Prim's Visualization
  - https://www.youtube.com/watch?v=6uq0cQZOyoY
  - Prim's jumps around the fringe, adding edges by edge weight
- Kruskal's Visualization:
  - https://www.youtube.com/watch?v=ggLyKfBTABo
  - Kruskal's jumps around the graph – not just the fringe – because it chooses edges by edge weight independent of the "tree under construction"
- Conceptual demo:
  - https://docs.google.com/presentation/d/1RhRSYs9Jbc335P24p7vR-6PLXZUl-1EmeDtqieL9ad8/present?ueb=true&slide=id.g375bbf9ace_0_645

Kruskal's Algorithm: Correctness

- Kruskal's algorithm is clever, simple, and efficient
  - But does it generate a minimum spanning tree?

- *First:* prove it is a spanning tree
  - Use **Contradiction!**
  - *Intuition*: Original graph was connected; we kept edges that didn't create a cycle

- *Second*: there is no spanning tree with lower total cost
  - A more complex proof that uses **Induction** and **Contradiction**
  - Take CSE 421!

Summary

- Minimum Spanning Trees are a subset of the edges in an undirected connected graph

- Prim's looks a lot like the vertex-based graph traversals we've seen so far, except it uses *edge weight* instead of *path weight*
  - And since edge weights don't change during the algorithm's execution, we don't need a decreaseKey() operation

- Kruskal's is an edge-based graph traversal (which we haven't seen so far), but still uses *edge weight* to choose edges
  - Doesn't need decreaseKey() for the same reason
  - Needs an auxiliary ADT – the Disjoint Sets ADT – to speed up execution

Lecture Outline

- Minimum Spanning Tree
  - Prim's Algorithm
  - Kruskal's Algorithm

- **Disjoint Sets ADT**

- Up-Tree Data Structure
  - Representation
  - Optimization: Weighted Union
  - Optimization: Path Compression

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Implementing the Disjoint Sets ADT (1 of 2)

- If we have *n* elements:
  - What is the total cost of m find()s + ≤ n-1 union()s?

Can we have >n union()s?

- Goal: O(m+n)
  - i.e. O(1) amortized
  - Can get O(1) worst-case union()
  - Would be nice if we could get O(1) worst-case find(), but…
  - *Known result*: both find() and union() can't have worst-case O(1)

Implementing the Disjoint Sets ADT (2 of 2)

- *Observation*:
  - Trees let us find many elements given a single root
- *Idea*:
  - If we reverse the pointers (ie, point up from child to parent), we can find a single root from many elements
- *Decision*:
  - One up-tree for each set
  - The ID of the set is (hash of) the tree root

Lecture Outline

- Minimum Spanning Tree
  - Prim's Algorithm
  - Kruskal's Algorithm
- Disjoint Sets ADT
- Up-Tree Data Structure
  - **Representation**
  - Optimization: Weighted Union
  - Optimization: Path Compression

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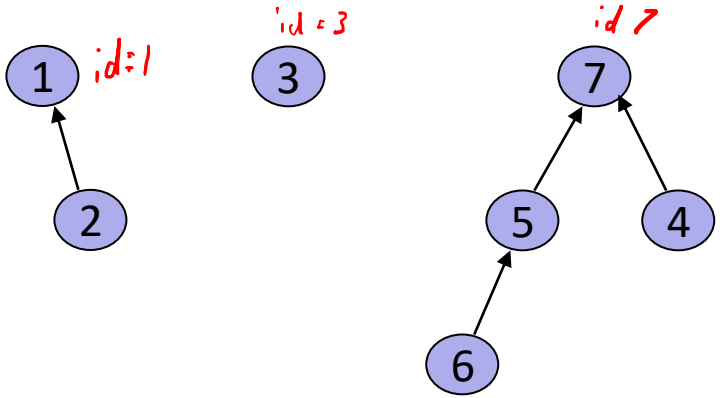
Up-Tree Data Structure for Disjoint Sets ADT

- Initial State:

- After several union()s:

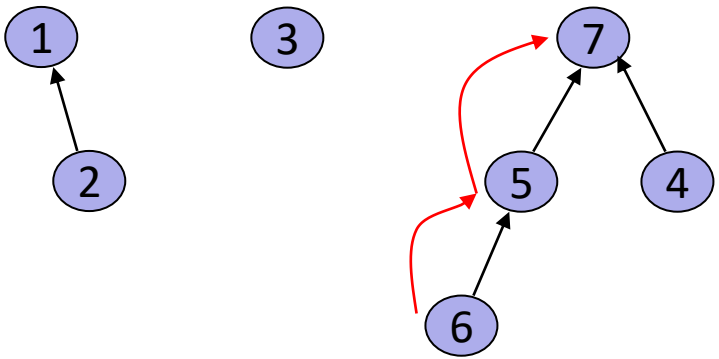
union (2, 1)
union (5, 4)
union (6, 7)
union (4, 7)



- Roots are the IDs for each set

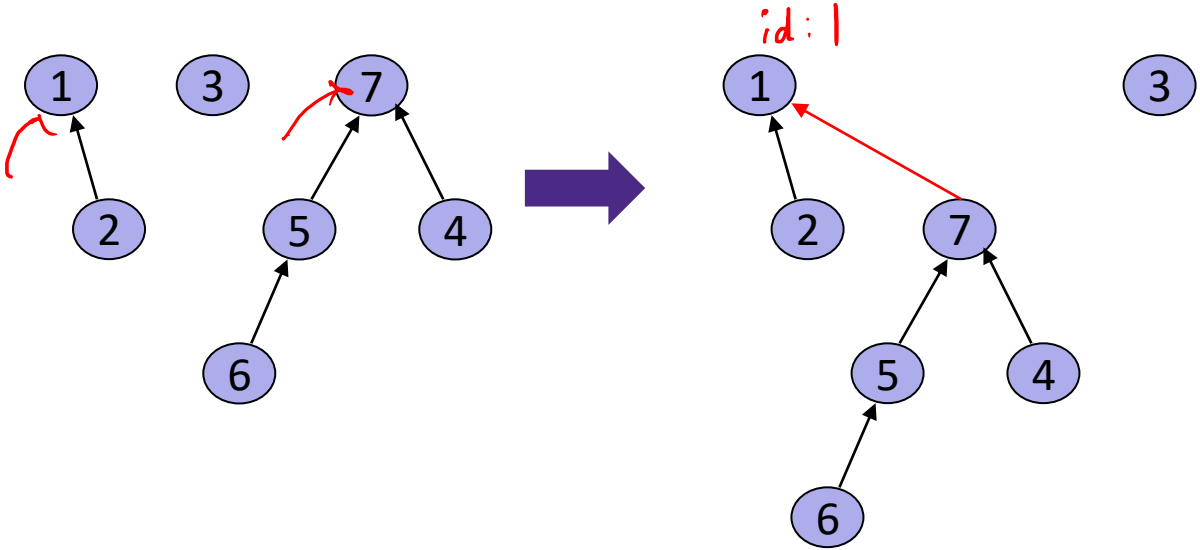
Up-Tree Find

- find(x): follow x to the root and return the root
  - Eg: find(6) = 7



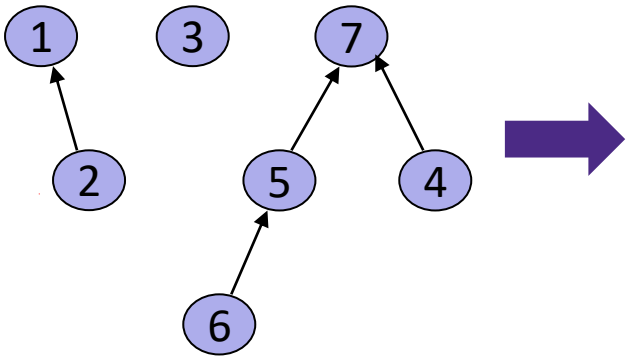
Up-Tree Union

- union(x): assuming x and y are roots, point y to x
  - If x or y are not roots, can require caller to call find() first
  - Eg: union(1, 7)



Up-Tree Representation (1 of 2)

- Up-Trees can be represented as an array of indices
  - up[x] = 0 means x is a root
  - Note: array is 1-indexed

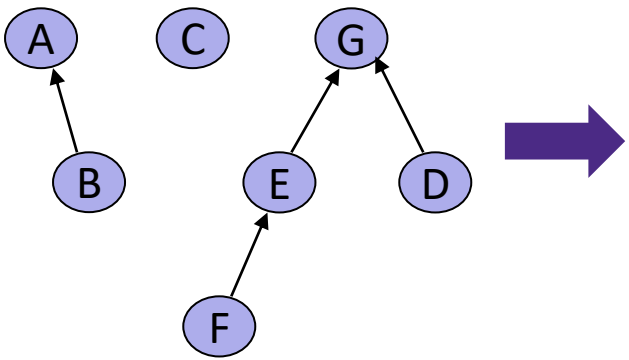


int[] up

0	1	0	7	7	5	0
1	2	3	4	5	6	7

Up-Tree Representation (2 of 2)

- Up-Trees can be represented as an array of indices
  - Can contain non-integer values using a hash table to map to indices



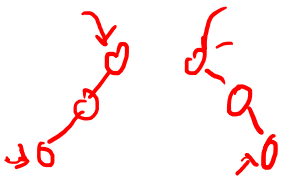
A	1
B	2
C	3
D	4
E	5
F	6
G	7

int[] up	0	1	0	7	7	5	0
	1	2	3	4	5	6	7

Up-Tree Implementation

```
void union(int x, int y) {
  up[y] = x;
}
```

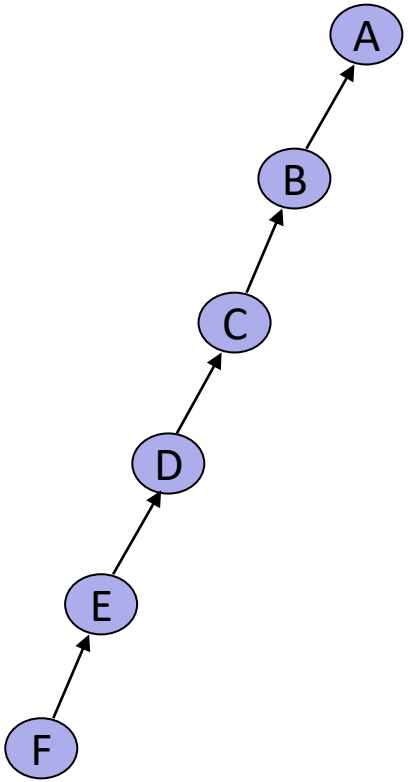
```
int find(int x) {
  while (up[x] != 0) {
    x = up[x];
  }
  return x;
}
```



- Worst-case runtime for union():
  - accepts non root nodes $O(\log n)$
  - doesn't accept non root nodes $O(1)$
- Worst-case runtime for find(): $O(\log n)$
- Total runtime for n-1 union()s and m find()s: $O(n + m \log n)$

Worst-case Union

```
union(A, B)
union(B, C)
union(C, D)
union(D, E)
union(E, F)
```



🤔 *If only I could keep these trees (semi-?)balanced*

Lecture Outline

- Minimum Spanning Tree
  - Prim's Algorithm
  - Kruskal's Algorithm
- Disjoint Sets ADT
- Up-Tree Data Structure
  - Representation
  - **Optimization: Weighted Union**
  - Optimization: Path Compression

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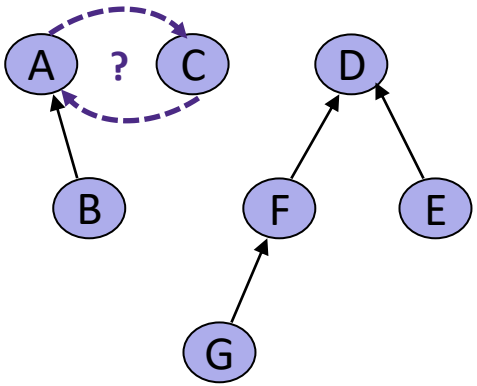
Weighted Union (1 of 2)

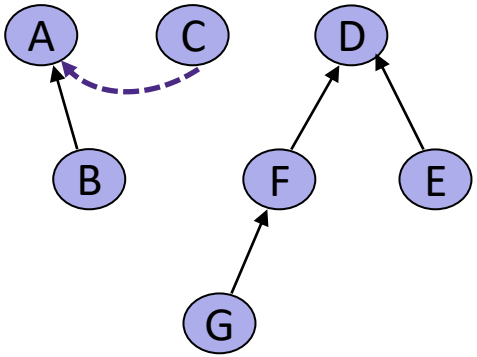
- Our naïve union() always picked the same argument (the second argument) to become the child in the unioned structure




```
union(A, B)
union(A, C)
union(A, D)
…
```

Weighted Union (2 of 2)

- Our naïve union() always picked the same argument (the second argument) to become the child in the unioned structure

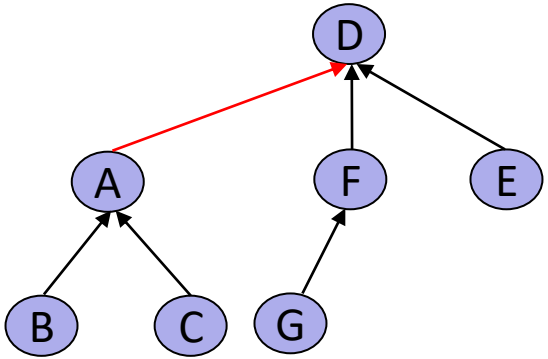


- Instead:
  - Pick the smaller tree (ie, tree with fewer nodes) to be the new child
    - Let "weight" = "num nodes"
  - Add the new child to the root

```
union(A, B)
union(A, C)
union(A, D)
…
```

Weighted Union (2 of 2)

- Our naïve union() always picked the same argument (the second argument) to become the child in the unioned structure

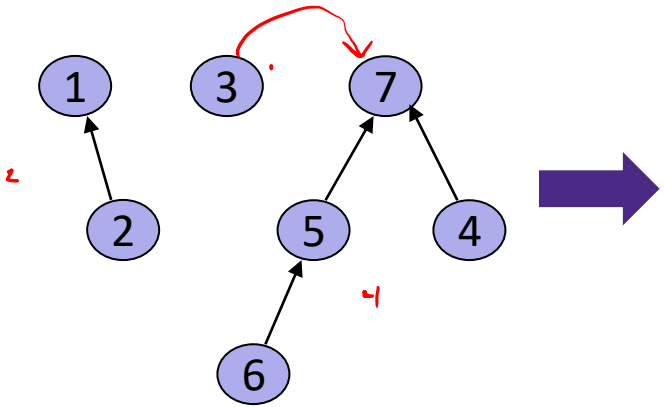


- Instead:
  - Pick the smaller tree (ie, tree with fewer nodes) to be the new child
    - Let "weight" = "num nodes"
  - Add the new child to the root

```
union(A, B)
union(B, C)
union(A, D)
…
```

Weighted Union: Representation

- Need to store *number of nodes* (or "weight") of each tree
- Instead of '0', we can store the root's weight instead!
  - Use negative values to indicate they're not indices
  - See Weiss, 8.4



int[] up

-2	1	-1	7	7	5	-4
1	2	3	4	5	6	7

Weighted Union: Implementation

```
void union(int x, int y) {
  up[y] = x;
}
```

```
weightedUnion(int x, int y) {
  wx = weight[x];
  wy = weight[y];
  if (wx < wy) {
    up[x] = y;
    weight[y] = wx + wy;
  } else {
    up[y] = x;
    weight[x] = wx +wy;
  }
}
```

union()'s runtime is still O(1)!

Does this (slightly) added complexity help us balance the up-trees and improve find()?

Weighted Union: Performance

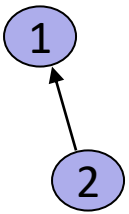
- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are "spindly"

N	H
1	0

Weighted Union: Performance

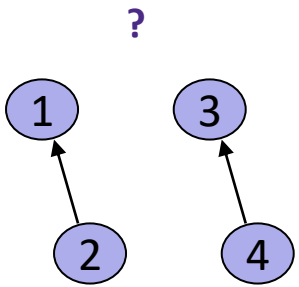
- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are "spindly"

N	H
1	0
2	1



Weighted Union: Performance

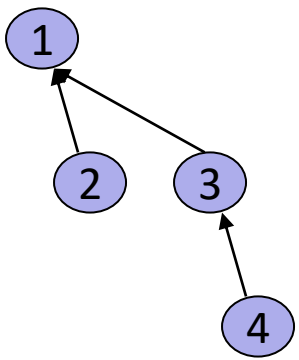
- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are "spindly"



N	H
1	0
2	1
4	?

Weighted Union: Performance

- Consider the worst case: tree height grows as fast as possible

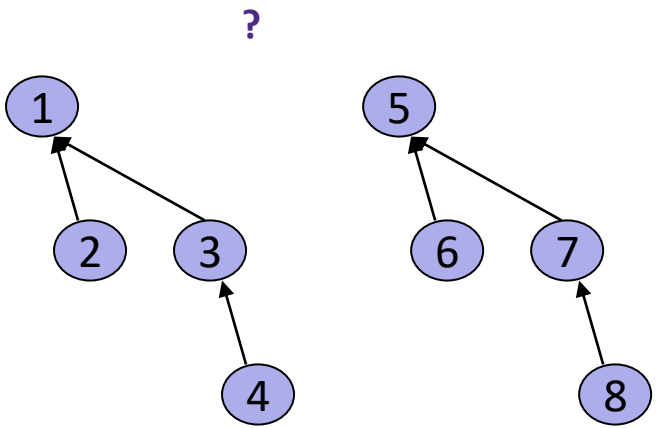


N	H
1	0
2	1
4	2

Weighted Union: Performance

- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are "spindly"

N	H
1	0
2	1
4	2
8	?

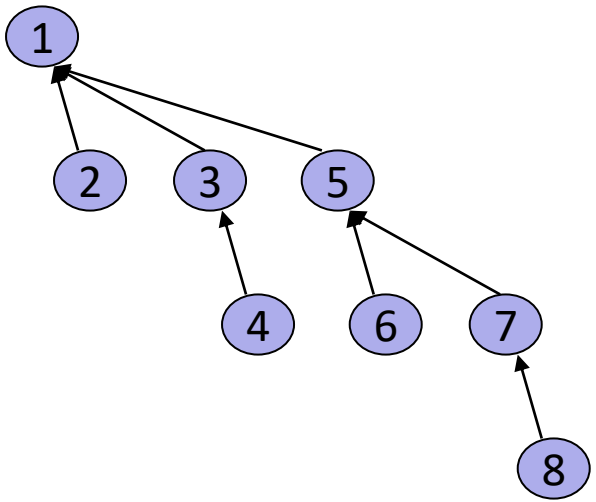


Weighted Union: Performance

- Consider the worst case: tree height grows as fast as possible
  - ie, up-tree and up-subtrees are "spindly"

N	H
1	0
2	1
4	2
8	3
2^n	n

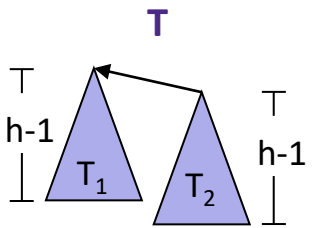
- Worst-case height and worst-case find() is Θ(log N)



Weighted Union Performance: Proof

- An up-tree with height h using weighted union has weight at least 2^h

- Proof by induction
  - *Base-case*: h = 0. The up-tree has one node and $2^0 = 1$
  - *Inductive step*: Assume true for all h' < h



Minimum weight up-tree of height h formed by weighted unions

We know:

$W(T_1) \geq 2^{h-1}$
$W(T_2) \geq 2^{h-1}$
} *Induction hypothesis*

$W(T_1) \geq W(T_2)$
} *Definition of weighted union*

Since $W(\mathbf{T}) = W(T_1) + W(T_2)$, we know that

$W(\mathbf{T}) \geq W(T_1) + W(T_2)$
$= 2^{h-1} + 2^{h-1}$
$= 2^h$

Therefore $W(\mathbf{T}) \geq 2^h$

Poll Everywhere

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- What is the runtime for …
  - weighted union(), worst-case
  - find(), worst-case
  - n-1 union()s + m find()s

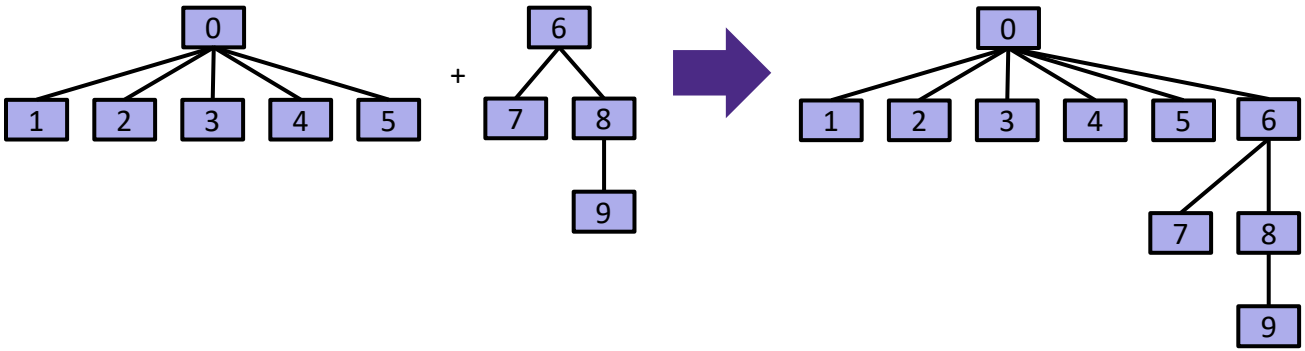
A. $\Theta(1)$ / $\Theta(1)$ / $O(n + m)$
B. $\Theta(1)$ / $\Theta(n)$ / $O(n + m^2)$
C. $\Theta(1)$ / $\Theta(\log n)$ / $O(n + m^2)$
D. $\Theta(1)$ / $\Theta(\log n)$ / $O(n + m \log n)$
E. I'm not sure …

```
weightedUnion(int x, int y) {
  wx = weight[x];
  wy = weight[y];
  if (wx < wy) {
    up[x] = y;
    weight[y] = wx + wy;
  } else {
    up[y] = x;
    weight[x] = wx +wy;
  }
}
```

```
int find(int x) {
  while (up[x] > 0) {
    x = up[x];
  }
  return x;
}
```

Why Weights Instead of Heights?

- We used the *number of items* in a tree to decide upon the root
- Why not use the *height* of the tree?
  - Heighted Union's runtime is asymptotically the same: Θ(log(N))
    - Proof is left as an exercise to the reader ;)



  - Easier to track weights than heights, and heighted union doesn't combine very well with the next optimization technique for find()

Lecture Outline

- Minimum Spanning Tree
  - Prim's Algorithm
  - Kruskal's Algorithm
- Disjoint Sets ADT
- Up-Tree Data Structure
  - Representation
  - Optimization: Weighted Union
  - **Optimization: Path Compression**

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Modifying Data Structures To Preserve Invariants

- Thus far, the modifications we've studied are designed to *preserve invariants* (aka "repair the data structure")
  - **Tree rotations**: preserve AVL tree invariants
  - **Promoting keys / splitting leaves**: preserve B-tree invariants (eg, L+1 keys stored in a leaf node)

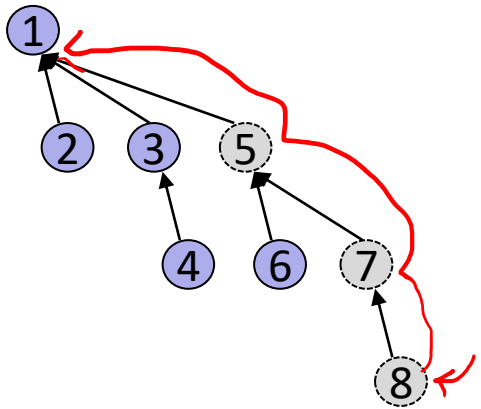
- Notably, the modifications don't improve runtime *between identical method calls*
  - If avl.find(x) takes 2 µs, we expect future calls to take ~2 µs
  - If we call avl.find(x) m times, the total runtime should be ~2m µs

Modifying Data Structures for Future Gains

- Path compression is entirely different: we are modifying the up-tree to *improve future performance*
  - If uptree.find(x) takes 2 µs, we expect future calls to take *<2 µs*
  - If we call uptree.find(x) m times, the total runtime should be <2m µs
    - … and possibly even << 2m µs

Path Compression: Idea

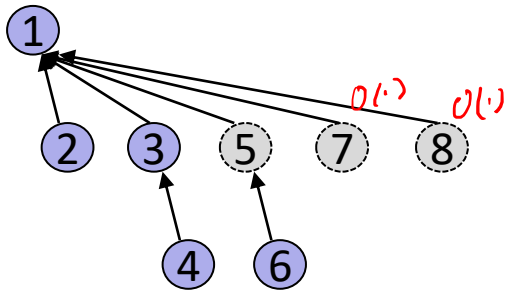
- Recall the worst-case structure if we use weighted union:



- *Idea*: When we find(8), move all visited nodes under the root
  - Additional cost is insignificant (same order of growth) , so run path compression on every find()

Path Compression: Example

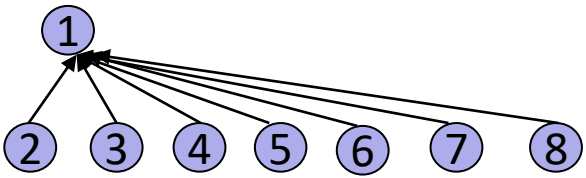
- Recall the worst-case structure if we use weighted union



- *Idea*: When we find(8), move all visited nodes under the root
  - Additional cost is insignificant (same order of growth), so run path compression on every find()
  - Doesn't meaningfully change runtime for *this* invocation of find(8), but *subsequent* find(8)s (and subsequent find(7)s and find(5)s and …) will be faster!

Path Compression: Details and Runtime

- With "enough" path compression aka "enough" find()s …



  - How much is "enough"? Probably m>n

m calls where m > # of elements

Path Compression: Implementation

```
int find(int x) {
  while (up[x] != 0) {
    x = up[x];
  }
  return x;
}
```

```
int pathCompressionFind(int x) {
  while (up[x] > 0) {
    x = up[x];
  }
  int root = x;

  // Change the parent for all
  // nodes along this path
  while (up[x] > 0) {
    x = up[x];
    up[x] = root;
  }
  return root;
}
```

find()'s worst-case runtime is still O(log n)!

Does this (slightly) added complexity help us make the up-trees shallower and improve sequences of find()?

Path Compression: Runtime

- In total, m>1 find()s will take O(m log*n) time
  - log*n is the "iterated log": the number of times you need to apply log to n before the result is <=1
  - See Weiss for proof

- find()s and union()s are now amortized constant time
  - log*n < 5 for any realistic n
  - If log*n ≈ 5, then m find()/union()s amortizes to O(1)! 

n	log* n
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

2^{16}

Number of atoms in the known universe is 2^{256}ish

Interlude: A Really Slow Function

- Ackermann's function is a really big function A(x, y) with inverse $\alpha(x, y)$ which is really small

- α shows up in:
  - Computation Geometry (surface complexity)
  - Combinatorics of sequences

- How fast does $\alpha(x, y)$ grow?
  - Even slower than iterated log!
  - For all practical purposes, $\alpha(x, y) < 4$

Path Compression: Tighter Runtime

- A sequence of p union()s + find()s on a set of n elements has worst-case time of $O(p \cdot \alpha(p, n))$
  - Assumes weighted union and path compression
  - Proved by Robert Tarjan in 1984 (see also Fibonacci heaps and splay trees)

Union O(1)

- So find()'s amortized runtime is O(1)
  - Since $O(p \cdot 4)$ for p operations!

Find amortized O(1)
O(log n)

- Complex analysis, but inverse-Ackermann's is a tighter bound than iterated-log

Summary

- Worst-case runtime for up-trees with weighted union and path compression:
  - union is O(1)
  - find is O(log n)

- Total time for $m \geq n$ operations on n elements is $O(m \cdot \alpha(m, n))$
  - An individual operation can be costly, but over time the "average" cost per operation is not
  - Logic is similar to array-resizing: an individual insert() can be costly *(due to resizing)*, but over time the "average" insert() is not

- Using "ranked union" gives an even better bound theoretically